C_T and substituting for $\underrightarrow{\tilde{\mathbf{y}}}, \Delta \underrightarrow{\tilde{\mathbf{u}}}$ from (37b) and (37c) gives

$$\begin{aligned} C_A \underrightarrow{\mathbf{y}} - C_A H_T \underleftarrow{\tilde{\mathbf{y}}} &= C_b \Delta \underrightarrow{\mathbf{u}} - C_b H_T \Delta \underleftarrow{\tilde{\mathbf{u}}} + C_T \check{\mathbf{p}} \\ \Rightarrow \quad \underrightarrow{\mathbf{y}} &= C_A^{-1}[C_b \Delta \underrightarrow{\mathbf{u}} + \mathbf{p}'] \\ \mathbf{p}' &= -C_b H_T \Delta \underleftarrow{\tilde{\mathbf{u}}} + C_T \check{\mathbf{p}} + C_A H_T \underleftarrow{\tilde{\mathbf{y}}}. \end{aligned} \tag{38}$$

This is exactly the same as (5) except that $\mathbf{p}$ has been replaced by $\mathbf{p}'$. From this point on the treatment is identical to that presented in the main text of the paper.

Appendix B

Here we discuss the general set-point trajectory $r_{t+1}, \cdots, r_{t+n_y-1}, r, r, \cdots$ whose z-transform is

$$r(z^{-1}) = r_{t+1} + r_{t+2} z^{-1} + \cdots + r_{t+n_y-1} z^{-n_y+1} + \frac{r z^{-n_y}}{1 - z^{-1}}. \tag{39}$$

In this case (7) must be replaced by

$$\begin{aligned} e(z^{-1}) &= \frac{r}{1 - z^{-1}} - y(z^{-1}) + \rho(z^{-1}) \\ &= \frac{q(z^{-1}) - b(z^{-1}) \Delta u(z^{-1})}{A(z^{-1})} + \rho(z^{-1}) \\ q(z^{-1}) &= a(z^{-1}) r - p(z^{-1}) \end{aligned} \tag{40}$$

where $\rho(z^{-1}) = (r_{t+1} - r) + (r_{t+2} - r) z^{-1} + \cdots + (r_{t+n_y-1} - r) z^{-n_y+1}$. The definitions of $\phi(z^{-1})$ and $\psi(z^{-1})$ are exactly the same, but now (16) becomes

$$\begin{aligned} \underrightarrow{\mathbf{e}} &= \Gamma_{\frac{1}{a^-}} [-\Gamma_{b^+} \mathbf{c} + P_1 \mathbf{q}] + \rho \\ \Delta \underrightarrow{\mathbf{u}} &= \Gamma_{\frac{1}{b^-}} [\Gamma_{A^+} \mathbf{c} + P_2 \mathbf{q}] \\ \boldsymbol{\rho} &= [(r_{t+1} - r), \cdots, (r_{t+n_y-1} - r), 0, \cdots]^T. \end{aligned} \tag{41}$$

The optimal solution for the future values of c given in (26) must be replaced by

$$\mathbf{c} = -P^{-1} [R\mathbf{q} - 2\Gamma_{b+}^T \Gamma_{\frac{1}{a_-}}^T \boldsymbol{\rho}]. \tag{42}$$

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A Revisit to the Gain and Phase Margins of Linear Quadratic Regulators

Cishen Zhang and Minyue Fu

***Abstract*—In this paper, we revisit the well-known robustness properties of the linear quadratic regulator (LQR), namely, the guaranteed gain margin of -6 to $+\infty$ dB and phase margin of $-60°$ to $+60°$ for single-input systems. We caution that these guaranteed margins need to be carefully interpreted. More specifically, we show via examples that an LQR may have a very small margin with respect to the variations of the gain and/or phase of the open-loop plant. Such a situation occurs in most practical systems, where the set of measurable state variables cannot be arbitrarily selected. Therefore the lack of robustness of the LQR can be very popular and deserves attention.**

I. Introduction

The robustness properties of the linear quadratic regulators (LQR) have been known for many years. That is, an LQR for a single-input plant possesses a guaranteed gain margin of -6 to $+\infty$ dB and phase margin of $-60°$ to $60°$; see [6], [1], and [2]. This result is extended in [9] and [7] to the multivariable case, where the weighting matrix for the control is diagonal.

In this paper, we point out that the aforementioned robustness properties of LQR's should be carefully interpreted.

Consider the following single-input/single-output (SISO) plant

$$G(s) = K G_0(s) \tag{1}$$

where $G_0(s)$ is a fixed transfer function, and K, having a nominal value of one, is a complex parameter representing gain and phase variations of the plant. Suppose a set of state variables is measurable and an LQR is designed.

The basic robustness question is: do the guaranteed gain and phase margins apply to the gain and phase variations of the open-loop plant? We show via examples that the answer is negative, in general. It turns

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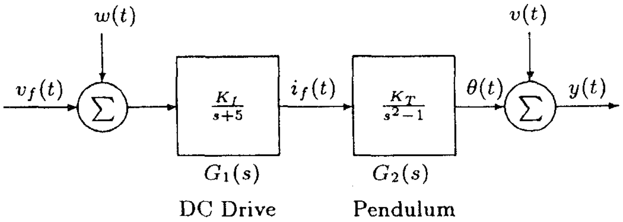

Fig. 1. Inverted pendulum system.

out that the gain and phase margins for K can be arbitrarily small for specially constructed examples of (1).

On the surface, the conclusion above appears contradictory to the well-known robustness properties of the LQR. However, as we will show, the guaranteed margins hold only when a very unique set of state variables is available for the feedback control. When this set of state variables is not used, the guaranteed gain and phase margins cannot adequately account for the variations of K in (1).

A related question arises: can we achieve the guaranteed gain and phase margins by suitably choosing the weighting matrices in the cost function? Our example shows that if the set of measurable state variables cannot be arbitrarily chosen, it may even be impossible to find any weighting matrices for the LQR to have the guaranteed margins.

We further ask the following question: given a system in (1), is it practical to find a unique set of state variables for the feedback control such that the guaranteed gain and phase margins can be achieved? Unfortunately, we argue that the answer is usually negative, due to the physical constraints of the system.

The robustness of LQR is also compared with linear quadratic gaussian regulators (LQG's) which use the observed state variables for the feedback control. We provide an example for which an LQR fails to have the guaranteed margins with respect to K, yet an LQG regulator surpasses it.

The comparison between LQR and LQG leads us to question the theory of loop-transfer recovery (LTR). As we know, the original motivation of LTR is to recover the guaranteed margins of the LQR or of a similar state feedback controller [5], [12]. Because these margins may be very small in practice, the question is how to reinterpret LTR. We point out that the use of LTR is to transfer a nice robustness property in the state feedback loop to the output feedback loop for which the LQR does not have the guaranteed margins. In other words, LTR is used not to "recover" the margins of LQR (because there may be none with respect to the open-loop variations), but to design a dynamic output feedback controller which is more robust to the gain and phase variations of the plant than a state feedback one. So, it is "loop transfer," not "recovery." Indeed, LTR does provide the guaranteed margins, provided that the so-called asymptotic LTR is achievable.

In summary, the guaranteed margins of the LQR cannot be assumed in practical applications, and its robustness deserves careful analysis.

II. Gain and Phase Margins of LQR

In this section, we consider the SISO plant (1) and show via an example that an LQR may not provide the guaranteed margins with respect to the gain and phase variations of the open-loop plant.

A. Example 1: Smaller Margins than Expected

To best understand this phenomenon, we consider the control problem of an inverted pendulum, depicted in Fig. 1. The system is controlled through a DC drive. There are three sensors for the field current $i_f(t)$ of the drive, angular position $\theta(t)$, and angular velocity $\omega(t)$ of the pendulum, respectively. The control input $u(t)$ is the field voltage of the drive, and the controlled output is $\theta(t)$. In this example, we assume that the input disturbance $w(t)$ and the output measurement noise $v(t)$ are zero. The transfer function of a linearized model from $u(t)$ to $\theta(t)$ is given by

$$G(s) = G_2(s)G_1(s) = \frac{K_I}{(s+5)} \cdot \frac{K_T}{(s^2-1)} \tag{2}$$

where $G_1(s)$ is the transfer functions of the DC drive from $u(t)$ to $i_f(t)$, and $G_2(s)$ of the inverted pendulum from $i_f(t)$ to $\theta(t)$. The current gain K_I and torque gain K_T are normalized so that their nominal values are equal to one. Defining $K = K_I K_T$, we can rewrite $G(s)$ as

$$G(s) = \frac{K}{(s+5)(s^2-1)}. \tag{3}$$

The measured state of the plant is naturally chosen to be

$$x(t) = \begin{bmatrix} \theta(t) \\ \omega(t) \\ i_f(t) \end{bmatrix}. \tag{4}$$

Let the LQ performance index be

$$J = \int_0^\infty (2\theta^2 + 10\omega^2 + u^2)\, dt. \tag{5}$$

A straightforward LQ design yields the optimal control as follows:

$$u(t) = -fx(t) = -(12.28180\theta(t) + 12.6033\omega(t) + 2.0857 i_f(t)). \tag{6}$$

Suppose that the gain and/or phase of K are perturbed due to parametric uncertainty or unmodeled dynamics in the plant. We would like to examine the corresponding robustness of the closed-loop system. It turns out that the gain and phase margins depend on whether the perturbation comes from the DC drive or the pendulum. In the former case, the closed-loop system indeed has the guaranteed margins. In the latter case, however, the gain margin is found to be from 0.576 (-4.76 dB) to $+\infty$, and the phase margin, $\pm 44.5°$ only!

B. Example 2: Arbitrarily Small Margins

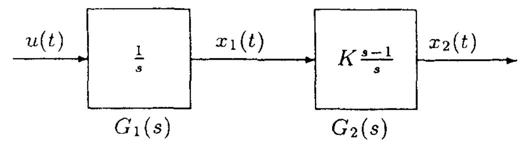

Fig. 2. Cascaded plant.

The purpose here is to show via an example a stronger fact, i.e., that an LQR may not guarantee any gain margin for K in the plant (1).

Consider the plant depicted in Fig. 2. The open-loop input–output transfer function is given by

$$G(s) = G_1(s)G_2(s) = K\,\frac{s-1}{s^2} \tag{7}$$

and the nominal value of K is equal to one.

Let the state $x = (x_1, x_2)'$ be chosen as in Fig. 2 and the LQ performance index be

$$J = \int_0^\infty (x'q'qx + u^2)\, dt \tag{8}$$

where

$$q = [\sqrt{2r} - r \quad r] \tag{9}$$

and $r > 0$ is a tuning parameter to be specified later.

The state-space realization of (7) at $K = 1$ is given by

$$\dot{x}(t) = Ax(t) + bu(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = cx(t) = [0 \quad 1]x(t). \tag{10}$$

The solution of the LQR

$$u(t) = -fx(t) \tag{11}$$

is obtained by solving [6]

$$1 + b^T(-sI - A^T)^{-1}q^T q(sI - A)^{-1}b = (1 + b^T(-sI - A^T)^{-1}f^T)(1 + f(sI - A)^{-1}b). \tag{12}$$

Its analytical solution for the nominal K is given by

$$u(t) = -(2\sqrt{r} + r)x_1(t) + rx_2(t). \tag{13}$$

The corresponding closed-loop characteristic polynomial is written by

$$p(s) = s^2 + (2\sqrt{r} + r - Kr)s + Kr. \tag{14}$$

At $K = 1$, we have

$$p(s) = s^2 + 2\sqrt{r}s + r$$

which means that the nominal closed-loop system is stable for all $r > 0$.

When $K = 1 + \epsilon$, the closed-loop system will lose stability at

$$\epsilon = \epsilon^* = \frac{2}{\sqrt{r}} \to 0, \quad \text{as} \quad r \to \infty. \tag{15}$$

So the conclusion is that the LQR has no guaranteed gain margin with respect to open-loop variations.

In fact, there is no LQR which can provide the guaranteed margins for the plant (7). To show this, we write

$$u(t) = -fx(t) = -f_1x_1(t) - f_2x_2(t). \tag{16}$$

Then the closed-loop characteristic polynomial is given by

$$p(s) = s^2 + (f_1 + Kf_2)s - Kf_2.$$

Obviously, for any given $f_2, p(s)$ becomes unstable when K is sufficiently large. Therefore $+\infty$ gain margin cannot be guaranteed by choosing LQ performance index.

C. Analysis

To gain more insight into the problem of the gain and phase margins of LQR as demonstrated in the examples above, we consider a state-space realization of (1) given as

$$\dot{x}(t) = Ax(t) + bu(t) \tag{17}$$

$$y(t) = cx(t). \tag{18}$$

Suppose that for a given LQ performance index the optimal state feedback control is

$$u(t) = -fx(t). \tag{19}$$

It is known that the return difference of the LQR is $1 + f(sI - A)^{-1}b$ which satisfies

$$|1 + f(sI - A)^{-1}b| \geq 1.$$

This inequality implies that the Nyquist plot of the transfer function $f(sI - A)^{-1}b$ is away from the $-1 + j0$ point in the complex plane by at least a unity. Following from this, the guaranteed gain and phase margins can be derived which allow a -6 to $+\infty$ dB change in the gain and $-60°$ to $+60°$ change in the phase of the loop transfer function $f(sI - A)^{-1}b$.

However, an apparent point is that the variations in the gain and phase of the loop-transfer function $f(sI - A)^{-1}b$ are, in general, not the same things as that of the plant transfer function $KG_0(s) = c(sI - A)^{-1}b$. This is the reason that the guaranteed gain and phase margins cannot appropriately account for the gain and phase variations of the plant. In fact, the guaranteed gain and phase margins for the loop-transfer function $f(sI - A)^{-1}b$ are meaningful for all possible variations of K in the transfer function of the plant (1) only when the measured set of state variables is very unique. Namely, the state matrix A must be independent of K and the input vector b proportional to K. Such a set of state variables is given by $[y, \dot{y}, \cdots, y^{n-1}]$ and those transformable from it by a constant (K-independent) transformation matrix. Here we assume that K is nondynamic for simplicity.

In light of the above analysis, the reduction of margins in the inverted pendulum example can be simply understood by considering the state-space realization of the plant (2)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & K_T \\ 0 & 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ K_I \end{bmatrix} u(t). \tag{20}$$

Note that K_I is in the input matrix and thus attracts the guaranteed margins. However, K_T appears in the system matrix!

Of course, it is possible to measure a different set of state variables so that both K_T and K_I are lumped together in the input matrix. It is easy to see that the only sets of such state variables are given by $[\theta, \omega, \dot{\omega}]'$ and those transformable from it by a constant transformation matrix.

Since a direct measurement of $\dot{\omega}$ is usually not available due to noise problems, guaranteed margins cannot be achieved by an LQR in reality.

Indeed, the nonrobustness problem of the LQR has been known for a long time in some different context. In particular, we refer to an example given in [13] where the following system is considered:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 + \epsilon_1 \\ 1 + \epsilon_2 \end{bmatrix} u. \tag{21}$$

A particular quadratic cost function can be chosen such that an LQR designed at $\epsilon_1 = \epsilon_2 = 0$ has an arbitrarily small gain margin with respect to the variation in either ϵ_1 (with $\epsilon_2 = 0$) or ϵ_2 (with $\epsilon_1 = 0$). A similar example is given in [11]. An interesting point involving our example in (7) is that it reveals the nonrobustness of LQR with respect to the phase and gain variations of the open-loop plant.

III. Margins of LQG Regulators

It is also well known that linear-quadratic Gaussian (LQG) regulators do not have guaranteed margins, in general [3]. This gives the general belief that LQG regulators are not as robust as LQR's. This is actually the original motivation for the LTR theory; see [4], [12], and [10]. More specifically, an LTR design involves two steps. First, an LQR controller is designed to achieve the required performance and robustness margins. Then, a dynamic output feedback controller takes over, and it is so designed that the guaranteed margins are recovered. In this section, we point out that it is misleading to conclude that LQR is more robust than LQG. In particular, we demonstrate via an example that it is possible an LQG regulator offers gain and phase margins larger than the "guaranteed" values, but an LQR for the same LQ performance index cannot.

Example 3: We return to Example 1 and add the input disturbance $w(t)$ and measurement noise $v(t)$. It is assumed that $w(t)$ and $v(t)$ are zero mean white noises with intensities 10^7 and 1, respectively. The corresponding LQG regulator (obtained using the Control Toolbox on Matlab) $u(s) = G_c(s)y(s)$ is given by

$$G_c(s) = -\frac{2.0657s^2 + 12.4751s + 10.097}{4.2475 \times 10^{-5}s^2 + 0.0092s + 1}. \tag{22}$$

It is verified that the gain and phase margins with respect to the open-loop variations are $(0.498, +\infty)$ (or $(-6.05, +\infty)$ dB), and $\pm 60.8°$, respectively, which slightly exceed the "guaranteed" margins.

It is known [8] that there are cases where an LQG regulator gives better margins than its LQR counterpart. What is different in our example is that the LQR counterpart fails to provide the guaranteed margins as far as the gain and phase variations in the open-loop plant are concerned.

IV. Improving Margins of LQR

We have already seen in Section III that LQG regulators may be more robust than their LQ counterparts. This is possible because in the LQG case, dynamic (rather than static) feedback is used. Although it is known that the optimal LQR is always achievable by static state feedback [6], we emphasize that better robustness may be obtained by using dynamic state feedback. This point is illustrated in the following.

Let us return to Example 2 and consider the use of the dynamic state feedback controller below

$$u(s) = -(2\sqrt{r} + r + \frac{s-1}{s}Q(s))x_1(s) + (r + Q(s))x_2 \tag{23}$$

where $Q(s)$ is a stable transfer function to be determined. Note that when $Q(s) = 0$, (23) reduces to (13). Also, adding $Q(s)$ in the controller does not change the system return difference and the closed-loop transfer function for the nominal plant.

We claim that an appropriate choice of $Q(s)$ may greatly improve the robust stability of the closed-loop system with respect to the variations in K. Indeed, the new characteristic equation is given by

$$s^2 + (2\sqrt{r} - \epsilon(r + Q(s))s + ((1+\epsilon)r + \epsilon Q(s)) = 0. \tag{24}$$

Choosing $Q(s) = -r + \delta$ with some $\delta > 0$, the above becomes

$$s^2 + (2\sqrt{r} - \epsilon\delta)s + (r + \epsilon\delta) = 0. \tag{25}$$

Therefore, given any $r > 0$ and bounding set $[0, \epsilon^*]$ for ϵ, we can choose $\delta > 0$ sufficiently small such that (25) is robustly Hurwitz. Hence, the gain margin with respect to K can be arbitrarily large.

Note that if $\delta = 0$, we have an infinite gain margin. In this situation, $x_2(s)$ disappears from the feedback. Intuitively, we can expect to have a poor robustness in LQ performance. In practice, a tradeoff between robustness in LQ performance and gain/phase margins needs to be considered.

Another approach to the improvement of the margins is to use LTR. The good news about LTR is that the recovered system indeed possesses the guaranteed margins with respect to open-loop variations, provided that asymptotic LTR can be achievable. This is an important property of LTR. As we mentioned in Section I, the use of LTR is to transfer a nice robustness property in the state feedback loop to the output feedback loop for which the LQR does not guarantee margins. However, when asymptotic LTR is not achieved, which is the case for most nonminimum-phase plants, one might be better off with dynamic state feedback, provided a set of state variables can be measured. We must also realize another possible disadvantage of LTR, i.e., the use of high gain feedback (for achieving asymptotic LTR or separation of time-scales; see [10]) in the presence of measurement noise. This problem is illustrated in Example 4 when the LQG controller is indeed designed using the LQG/LTR approach suggested in [5].

V. Conclusions

In this paper, we have analyzed the robustness properties of the LQR and have shown that the guaranteed gain/phase margins of LQR need to be carefully interpreted. We have demonstrated that the guaranteed margins usually do not apply to practical systems due to the constraints in the selection of measurable state.

We have also discussed the possible use of dynamic state feedback for improving the robustness of LQR. In this regard, the LTR method becomes handy because it can "transfer" the guaranteed margins in the state feedback loop to the output feedback loop, provided that asymptotic LTR is possible. A more general problem is how to use dynamic state (or partial state) feedback to optimize performance while guaranteeing a certain robustness margin. This issue deserves further research.

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Continuous Least-Squares Observers with Applications

Alexander Medvedev

***Abstract*— A wide class of continuous least-squares (LS) observers is treated in a common framework provided by the pseudodifferential operator paradigm. It is shown that for the operators whose symbols satisfy certain conditions, the continuous LS observer always exists, provided observability of the plant. The general result is illustrated by an LS observer stemmed from a sliding-window convolution operator. Applications to state feedback control and fault detection are discussed.**

I. Motivation and Background

Traditionally, the deterministic state vector observation (reconstruction) problem in linear systems is solved by means of the

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