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Interpolation approach to H^∞ estimation and its interconnection to loop transfer recovery

Minyue Fu

Department of Electrical and Computer Engineering, University of Newcastle, N.S.W. 2308, Australia

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Abstract: This paper proposes to solve the H^∞ optimal estimation problem via interpolation theory. The advantages of this approach are the following. (i) Direct and simple solutions can be obtained for the optimal estimators; (ii) frequency weightings on estimation error and disturbance can be easily incorporated; and (iii) unnecessary high observer gains can be avoided. We also show the interconnection between the H^∞ optimal estimation problem and the optimal loop transfer recovery problem. The interpolation approach applies to both continuous-time and discrete-time systems.

Keywords: Optimal estimation; H^∞ control; interpolation theory; loop transfer recovery.

1. Introduction

Consider the following system:

$$\dot{x}(t) = Ax(t) + Bw(t), \quad x(0) = 0, \tag{1a}$$

$$y(t) = Cx(t) + Dw(t), \tag{1b}$$

$$z(t) = Fx(t), \tag{1c}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^m$ is the *disturbance* satisfying $\| w \|_2 \le 1$, $y(t) \in \mathbf{R}^r$ is the *measured output*, $z(t) \in \mathbf{R}^p$ is the *linear combination of x(t) to be estimated*, and A, B, C, D, F are given matrices with appropriate dimensions. The *H^∞ optimal estimation problem* can be stated as follows: Find an estimate $\hat{z}(t)$ of $z(t)$ based on $y(t)$ and its history such that

$$\max_{\| w \|_2 \le 1} \| z - \hat{z} \|_2 = \min. \tag{2}$$

Similarly, for any given $\lambda > 0$, the H^∞ *suboptimal estimation problem* is to find $\hat{z}(t)$ such that

$$\max_{\| w \|_2 \le 1} \| z - \hat{z} \|_2 < \lambda. \tag{3}$$

It is well known [1,2,3] that for any $\lambda > 0$, if the H^∞ suboptimal estimation problem is solvable, then it must be solvable with an estimator of the following form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = 0,$$
$$\hat{z}(t) = F\hat{x}(t), \tag{4}$$

where $K \in \mathbf{R}^{n \times r}$ is the observer gain matrix. The suboptimal K which guarantees (3) can be determined via an algebraic Riccati equation; see [1,2,3], for example.

Although the observer (4) given by the Riccati equation method appears simple and that the usage of algebraic Riccati equations has become very popular in H^∞ control [4], this approach to estimation has three limitations. First, in order to determine the minimal λ, the suboptimal problem needs to be solved repeatedly for successively smaller λ's. This is usually computationally demanding. Secondly, it is difficult to handle frequency weightings on estimation error and disturbance. Although the frequency weightings can be embedded in the system equations (1), the system dimension will increase. Furthermore, changing the weighting functions will cause structural variations in the system (both dimension and the system matrices), which is rather inconvenient. Finally, the observer gain is usually very high especially when λ approaches its minimum due to the restriction that the observer (4) is strictly proper. Consequently, very small stability margin can be assured.

We propose in this paper to solve the H^∞ optimal estimation problem using the interpolation theory. This is done by allowing $\hat{z}(t)$ to be a general linear dynamic function of $y(t)$ rather than in the form of (4). Using this approach, a more direct and simpler solution can be obtained for the optimal problem; frequency weightings on estimation error and disturbance are easy to handle; the estimator obtained is of minimum order; and unnecessary high observer gain can be avoided by allowing nonstrictly proper estimators. We also show that the H^∞ optimal estimation problem and the so-called *optimal loop transfer recovery problem* proposed by the author [6] are essentially the same, thus offering more insights to the both problems. In fact, the interpolation approach given in this paper was initially used in [6] for solving the optimal loop transfer recovery problem.

2. Problem description

For the system (1), we denote the transfer functions from w to y and z by $G(s)$ and $L(s)$, respectively, i.e.,

$$G(s) = D + C(sI - A)^{-1}B, \tag{5}$$

$$L(s) = F(sI - A)^{-1}B. \tag{6}$$

It is assumed that $G(s)$ is left invertible, with its left inverse denoted by $G^\dagger(s)$. The transfer functions $G(s)$ and $L(s)$ are minimal and rational, but not necessarily proper (causal).

Let the estimator of z to be in the following form:

$$\hat{z}(s) = M(s)y(s), \tag{7}$$

where $M(s)$ is a stable rational matrix with appropriate dimensions, but not necessarily proper. (For the purpose of this paper, we define a stable transfer matrix to be a transfer matrix void of *finite* unstable poles in the closed right half plane.) Then, the transfer function from w to $\hat{z}$ is simply given by

$$\hat{z}(s) = M(s)G(s)w(s). \tag{8}$$

Obviously, the estimation error reads

$$z(s) - \hat{z}(s) = (M(s)G(s) - L(s))w(s) = N(s)w(s) \tag{9}$$

where

$$N(s) = M(s)G(s) - L(s) \tag{10}$$

is the transfer function from w to the estimator error. It is easy to verify that for the observer (4), the functions $M(s)$ and $N(s)$ are given by

$$M_{\text{obs}}(s) = F(sI - A + KC)^{-1}K, \tag{11}$$

$$N_{\text{obs}}(s) = F(sI - A + KC)^{-1}B, \tag{12}$$

which are both stable. For a general estimator (7), the equations (11) and (12) do not hold. However, constraints on M and N are needed to guarantee the *internal stability* of the estimation error dynamics, as shown below.

Lemma 1. *Suppose $M(s)$ is stable. Then, $N(s)$ in* (10) *is stable* only if *$L(s)G^\dagger(s)$ is void of all unstable poles of $L(s)$ and $G(s)$ for all $G^\dagger(s)$.*

Proof. For any $G^\dagger(s)$, rewrite

$$N(s) = \left[M(s) - L(s)G^\dagger(s)\right]G(s).$$

If $LG^\dagger$ has an unstable pole of G, so will be $M - LG^\dagger$. Due to the minimality of G, this unstable pole can not be cancelled by any zero of $G(s)$, leading to the contradiction that N will be unstable. On the other hand, suppose $LG^\dagger$ has an unstable pole p of L. Since M is stable, this unstable pole must be cancelled by G in order for N to be stable. That is, $LG^\dagger G$ must be void of the pole p, which contradicts the assumption that p is a pole of L. □

Based on Lemma 1, it is essential to have the following assumption in order to assure the stability of the error dynamics (9):

Assumption 1. $L(s)G^\dagger(s)$ is void of all unstable poles of $L(s)$ and $G(s)$ for all $G^\dagger(s)$.

The H^∞ optimal estimation problem becomes to find a stable $M(s)$ such that

$$\lambda_m = \| MG - L \|_\infty = \min. \tag{13}$$

Remark. Since we do not restrict the estimator to be proper, the solution to (13) will be unique, achievable by an improper $M(s)$. However, a proper or even a strictly proper estimator can be obtained by cascading with $M(s)$ a low pass filter with sufficiently large bandwidth and sufficiently high relative degree (see Example 4.1). The corresponding function $N(s)$ will also be proper when

$L(s)$ and $G(s)$ are proper. In this case, the maximum singular value $\sigma_{\max}[N(j\omega)]$ will be slightly higher than λ_m, but the difference vanishes as the bandwidth of the low-pass filter approaches to infinity.

If frequency weightings on estimation error and disturbance are desired, the *weighted H^∞ optimal estimation problem* is to find a stable M such that

$$\bar{\lambda}_m = \| W_1(MG - L)W_2 \|_\infty = \min. \tag{14}$$

where $W_1(s)$ and $W_2(s)$ are (left and right) weighting rational matrices which are stable and of minimum phase, representing frequency weightings on estimation error and disturbance, respectively. Note, however, that the weighted problem can be converted to an unweighted problem by defining

$$\bar{G} = GW_2, \quad \bar{L} = W_1 L W_2, \quad \bar{M} = W_1 M. \tag{15}$$

With the above definitions, the weighted problem (14) becomes

$$\| \bar{M}\bar{G} - \bar{L} \|_\infty = \min. \tag{16}$$

Therefore, we will only address the unweighted problem (13) in the sequel. A careful examination of the algorithms given in the next section will indicate that the weighting matrices do not significantly effect the computation, it only increases the dimension of the estimator which is unavoidable. However, if the Riccati equation approach is used, a weighted problem will require much more computation on higher dimensional matrices.

Incidentally, the H^∞ optimal estimation problem is found to be the same as the so-called optimal loop transfer recovery problem [6] to which a complete solution is established via the interpolation theory; see Section 5. Hence, solution to the H^∞ optimal estimation problem is derived. This is discussed in the next section.

3. Solution

To show the simplicity of the formulation of the H^∞ optimal estimation problem (13), we first examine the special case for which the minimal estimation error can be made to be zero. The condition for this case is simply given as follows:

Theorem 1. *Consider system* (1). *There exists a stable estimator in the form of* (7) *such that* $\lambda_m = 0$ *if and only if* $L(s)G^\dagger(s)$ *is stable for some* $G^\dagger(s)$. *In this case,* $M(s) = L(s)G^\dagger(s)$ *and* $N(s) = 0$.

Proof. This is obvious to verify and therefore omitted. □

If a proper estimator is required, a slight modification is needed for the result above:

Theorem 2. *Consider system* (1). *There exists a stable and* proper *estimator in the form of* (7) *such that* $\lambda_m = 0$ *if and only* $L(s)G^\dagger(s)$ *is stable and* proper *for some* $G^\dagger(s)$. *In this case,* $M(s) = L(s)G^\dagger(s)$ *and* $N(s) = 0$. *Moreover, there exists a stable and* proper *estimator in the form of* (7) *such that*

$$\sigma_{\max}[N(j\omega)] < \varepsilon, \quad 0 \leqslant \omega \leqslant \Omega, \tag{17}$$

for any given (fixed) $\varepsilon > 0$, $\Omega > 0$ *if and only if* $L(s)G^\dagger(s)$ *is stable* (but not necessarily proper) *for some* $G^\dagger(s)$. *In this case, we can choose*

$$M(s) = P_{\text{low}}(s)L(s)G^\dagger(s) \tag{18}$$

and $N(s)$ *by* (10), *where* $P_{\text{low}}(s)$ *is a unity gain low-pass filter with sufficiently large bandwidth and sufficiently high relative degree.*

Now we consider the solution to the general situation where λ_m is nonzero. If G and L are stable, then the problem (13) is a standard H^∞ problem which can be solved by using various methods. See the conjugation method in [12] and references thereof and a method in [13], for example. Due to the fact that G and L may be unstable, the problem (13) is not a standard H^∞ optimization problem. Fortunately, because L and G share the identical unstable poles, this problem can be solved using the interpolation theory. The solution is mainly due to [6,10,11]. For simplicity, we further require the following:

Assumption 2. The unstable zeros $\alpha_1, \ldots, \alpha_l$ of $G(s)$ are distinct and simple, satisfying $\text{Re}[\alpha_i] > 0$.

The requirement of distinct unstable zeros is for simplicity. For treatment of unstable zeros with multiplicities, see [15].

Assumption 2 implies that, for each α_i, there exists a unique $\xi_i \in \mathbf{C}^m$ with $\|\xi_i\| = 1$ such that

$$G(\alpha_i)\xi_i = 0, \quad i = 1, 2, \ldots, l. \tag{19}$$

Note that $\xi_i = 1$ for the single input case ($m = 1$). Let

$$\eta_i := L(\alpha_i)\xi_i \in \mathbf{C}^m, \quad i = 1, 2, \ldots, l. \tag{20}$$

Then, the constraint (10) is equivalent to

$$N(\alpha_i)\xi_i = \eta_i, \quad i = 1, 2, \ldots, l, \tag{21}$$

i.e., $L - N$ contains all unstable zeros of G. Hence, we have the following result; see [6] for proof.

Theorem 3. *The optimization problem* (13) *is equivalent to finding*

$$\lambda_m = \inf\{ \|N(s)\|_\infty : N(\alpha_i)\xi_i = \eta_i, \\ i = 1, 2, \ldots, l\} \tag{22}$$

and all such $N(s)$. The corresponding stable $M(s)$ is given by

$$M(s) = (L(s) - N(s))G^+(s). \tag{23}$$

Now we discuss two cases: (i) Single input case ($m = 1$) and (ii) multi-input case ($m > 1$).

Single input case

In the single input case ($m = 1$), the equivalent problem (22) is a classical Nevanlinna–Pick interpolation problem which has a *unique* solution [10,14], solved in the following four steps. The first step is to compute λ_m. Let $N_\lambda(s) = N(s)/\lambda$; then $\|N(s)\|_\infty \le \lambda$ if and only if $\|N_\lambda(s)\|_\infty \le 1$. According to [10,14], there exists $N_\lambda(s)$ interpolating $N_\lambda(\alpha_i) = \eta_i/\lambda$, $i = 1, 2, \ldots, l$, if and only if the following $l \times l$ *Pick matrix* is nonnegative definite:

$$P_\lambda = P_0 - \lambda^{-2}P_1, \tag{24}$$

where

$$P_0 = \left\{ \frac{1}{\alpha_i + \bar{\alpha}_j} \right\}, \qquad P_1 = \left\{ \frac{\bar{\eta}_i \eta_j}{\alpha_i + \bar{\alpha}_j} \right\}. \tag{25}$$

and the overbar $^-$ denotes the complex conjugate. Then, λ_m is given by

$$\lambda_m = \sup\{\lambda: \det P_\lambda = 0\}.$$

Equivalently, due to the positive-definiteness of P_0, we have

$$\lambda_m = \sup\{\lambda: \det[\lambda^2 I - P_0^{-1}P_1] = 0\}$$

or

$$\lambda_m = \sqrt{\lambda_{\max}[P_0^{-1}P_1]} \tag{26}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue. The second step is to scale $N(s)$ and η_i: set $N(s) := N(s)/\lambda_m$ and $\eta_i := \eta_i/\lambda_m$, $i = 1, 2, \ldots, l$. Then the problem (22) becomes to find all the stable $N(s)$ with $\|N(s)\|_\infty \le 1$ subject to $N(\alpha_i) = \eta_i$, $i = 1, 2, \ldots, l$. Step 3 is to solve this scaled problem. The final step is to reverse the scaling done in step 2. The complete procedure, which is summarized from [10,14], is given below.

Step 0.1. Compute α_i, $i = 1, 2, \ldots, l$.
Step 0.2. Compute η_i, $i = 1, 2, \ldots, l$, according to (20).
Step 1.1. Compute P_0 and P_1 in (25).
Step 1.2. Compute λ_m according to (26).
Step 2. Set $\eta_i := \eta_i/\lambda_m$, $i = 1, 2, \ldots, l$.
Step 3.1. Form the so-called *Fenyves array* $\beta_{i,j}$ as follows:

$$\beta_{i,1} := \eta_i, \quad i = 1, 2, \ldots, l;$$

$$\beta_{i,j+1} := \frac{(\alpha_i + \bar{\alpha}_j)(\beta_{i,j} - \beta_{j,j})}{(\alpha_i - \alpha_j)(1 - \bar{\beta}_{j,j}\beta_{i,j})}, \\ 1 \le j \le i - 1 \le l - 1. \tag{27}$$

Step 3.2. Find $k < l$ (which must exist) such that

$$|\beta_{i,i}| < 1, \; i = 1, \ldots, k, \quad |\beta_{k+1,k+1}| = 1. \tag{28}$$

Step 3.3. Set $N^{(k+1)}(s) = \beta_{k+1,k+1}$.
Step 3.4. For $i = k, k-1, \ldots, 1$, do

$$N^{(i)} = \frac{(s - \alpha_i)N^{(i+1)}(s) + \beta_{i,i}(s + \bar{\alpha}_i)}{(s + \bar{\alpha}_i) + \bar{\beta}_{i,i}(s - \alpha_i)N^{(i+1)}(s)} \tag{29}$$

and set $N(s) = N^{(1)}(s)$.
Step 4. Set $N(s) = \lambda_m N(s)$ and compute

$$M(s) = [L(s) - N(s)]G^{-1}(s).$$

Multi-input case

In the multi-input case ($m > 1$), the problem (22) is known as the *directional interpolation prob-*

lem (DIP) [11] or the Nevanlinna–Pick tangent problem [15]. Thanks to a recent paper by Kimura [11], this can be solved by using an extension of the so-called *Schur–Nevanlinna algorithm*. As in the single input case, the solution involves four steps. The first step is to compute λ_m given by (26) but with P_0 and P_1 defined by

$$P_0 = \left\{ \frac{\tilde{\xi}_i \xi_j}{\alpha_i + \bar{\alpha}_j} \right\}, \qquad P_1 = \left\{ \frac{\tilde{\eta}_i \eta_j}{\alpha_i + \bar{\alpha}_j} \right\} \tag{30}$$

where the tilde $\tilde{}$ denotes the Hermitian transpose. The second step is to scale $N(s)$ and η_i, same as in the single input case. Step 3 is to solve the scaled DIP problem using an extension of the Schur–Nevanlinna algorithm. The final step is to reverse the scaling done in Step 2. The complete procedure is given below.

Step 0.1. Compute α_i, $i = 1, 2, \ldots, l$.
Step 0.2. Compute ξ_i and η_i, $i = 1, 2, \ldots, l$, according to (19) and (20).
Step 1.1. Compute P_0 and P_1 in (30).
Step 1.2. Compute λ_m according to (26).
Step 2. Set $\eta_i := \eta_i / \lambda_m$, $i = 1, 2, \ldots, l$.
Step 3.1. Initialize $\nu_1 = 0$ and

$$\psi_j^{(1)} = \begin{pmatrix} \xi_j^{(1)} \\ \eta_j^{(1)} \end{pmatrix} = \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}, \quad j = 1, \ldots, n. \tag{31}$$

Step 3.2. For $i = 1, 2, \ldots, l$, do the following. Given the i-th interpolation vectors

$$\psi_j^{(i)} = \begin{pmatrix} \xi_j^{(i)} \\ \eta_j^{(i)} \end{pmatrix}, \quad \xi_j^{(i)}, \eta_j^{(i)} \in \mathbf{C}^{m-\nu_i},$$
$$j = i, i+1, \ldots, l,$$

and the integer ν_i, compute

$$\rho_i = \left\| \xi_i^{(i)} \right\|^2 - \left\| \eta_i^{(i)} \right\|^2. \tag{32}$$

If $\rho_i = 0$, take $U_i, V_i \in \mathbf{C}^{(m-\nu_i)\times(m-\nu_i-1)}$ such that

$$I_{m-\nu_i} - \frac{\xi_i^{(i)} \tilde{\xi}_i^{(i)}}{\tilde{\xi}_i^{(i)} \xi_i^{(i)}} = U_i \tilde{U}_i, \tag{33}$$

$$I_{m-\nu_i} - \frac{\eta_i^{(i)} \tilde{\eta}_i^{(i)}}{\tilde{\eta}_i^{(i)} \eta_i^{(i)}} = V_i \tilde{V}_i, \tag{34}$$

$$v_{i+1} = v_i + 1, \tag{35}$$

$$\Theta_i = \begin{bmatrix} U_i & 0 \\ 0 & V_i \end{bmatrix}, \tag{36}$$

and

$$\tilde{\psi}_j^{(i+1)} = \tilde{\psi}_j^{(i)} J \Theta_i J, \quad j = i+1, \ldots, l, \tag{37}$$

where

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}.$$

If $\rho_i > 0$, take

$$B_i(s) = \frac{s - \alpha_i}{s + \bar{\alpha}_i}, \tag{38}$$

$$E_i = \frac{\xi_i^{(i)} \tilde{\eta}_i^{(i)}}{\tilde{\xi}_i^{(i)} \xi_i^{(i)}}, \tag{39}$$

$$l_i = \left(I_{m-\nu_i} - E_i \tilde{E}_i \right)^{-1/2} \left(\xi_i^{(i)} - E_i \eta_i^{(i)} \right), \tag{40}$$

$$X_i(s) = I_{m-\nu_i} - \left(B_i(s) - 1 \right) \frac{l_i \tilde{l}_i}{\tilde{l}_i l_i}, \tag{41}$$

$$H_i = \begin{bmatrix} \left(I_{m-\nu_i} - E_i \tilde{E}_i \right)^{-1/2} & 0 \\ 0 & \left(I_{m-\nu_i} - \tilde{E}_i E_i \right)^{-1/2} \end{bmatrix} \cdot \begin{bmatrix} I_{m-\nu_i} & E_i \\ \tilde{E}_i & I_{m-\nu_i} \end{bmatrix}, \tag{42}$$

$$\Theta_i(s) = H_i \begin{bmatrix} X_i(s) & 0 \\ 0 & I_{m-\nu_i} \end{bmatrix}, \tag{43}$$

$$\tilde{\psi}_j^{(i+1)} = \tilde{\psi}_j^{(i)} J \Theta_i(\alpha_j) J, \quad j = i+1, \ldots, l, \tag{44}$$

$$\nu_{i+1} = \nu_i. \tag{45}$$

Step 3.3. Choose any stable matrix $\Psi^{(l+1)} \in \mathbf{R}(s)^{(m-\nu_{l+1})\times(m-\nu_{l+1})}$ with $\| \Psi^{(l+1)} \|_\infty \le 1$. For $i = l, l-1, \ldots, 1$, do: if $\rho_i > 0$, set

$$\Psi^{(i)} = T_1 T_2^{-1} \tag{46}$$

where

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \Theta_i \begin{bmatrix} \Psi^{(i+1)} \\ I \end{bmatrix};$$

if $\rho_i = 0$, set

$$\Psi^{(i)} = E_i + U_i \Psi^{(i+1)} \tilde{V}_i. \tag{47}$$

Then, assign $N(s) := \tilde{\Psi}^{(1)}(s)$.
Step 4. Set $N(s) := \lambda_m N(s)$ and compute

$$M(s) = \left[L(s) - N(s) \right] G^\dagger(s).$$

4. Examples

In this section, we illustrate the results in Section 3 by two examples.

Example 4.1. The system (1) is given by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2x_1 - x_2 + w,$$
$$y = x_1,$$
$$z = [x_1 \quad x_2]^{\mathrm{T}}. \tag{48}$$

It follows that

$$G(s) = \frac{1}{(s+1)^2},$$

$$L(s) = \left[\frac{1}{(s+1)^2} \quad \frac{s}{(s+1)^2}\right]^{\mathrm{T}}.$$

Since G and L do not have any unstable poles and that G is of minimum phase, zero estimation error is possible (Theorem 1). Indeed, the optimal solution to M is given by

$$M(s) = L(s)G^{-1}(s) = [1 \quad s]^{\mathrm{T}}.$$

Because this optimal solution is not proper, we can modify it to be

$$M(s) = \left[1 \quad \frac{s}{\varepsilon s+1}\right]^{\mathrm{T}},$$

where $\varepsilon > 0$ is sufficiently small. The corresponding transfer function for the estimation error is then given by

$$N(s) = MG - L = \left[0 \quad -\frac{\varepsilon s^2}{(\varepsilon s+1)(s+1)^2}\right]^{\mathrm{T}}.$$

Note that $N(s) \to 0$ as $\varepsilon \to 0$.

Example 4.2. This example illustrates the design procedure of the optimal estimator for the single input case. Suppose

$$G(s) = \frac{(1-2s)(1-4s)}{s(s+2)}, \tag{49}$$

$$L(s) = \frac{2-3s}{4s}. \tag{50}$$

It is easily verified that G and L have the same unstable pole $s = 0$ and that λ_m can not be zero because L does not contain the unstable zeros of G. Therefore, we apply the first algorithm described in Section 3 to solve for λ_m and the corresponding estimator.

Step 0.1. $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$.
Step 0.2. $\eta_1 = L_{\mathrm{d}}(\alpha_1) = \frac{1}{4}$; $\eta_2 = L_{\mathrm{d}}(\alpha_2) = \frac{5}{4}$.
Step 1.1.

$$P_0 = \begin{bmatrix} 1 & \frac{4}{3} \\ \frac{4}{3} & 2 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} \frac{1}{16} & \frac{5}{12} \\ \frac{5}{12} & \frac{25}{8} \end{bmatrix}.$$

Step 1.2. $\lambda_m = \sqrt{\lambda_{\max}[P_0^{-1}P_1]} = 3.1008$.
Step 2. $\eta_1 = \eta_1/\lambda_m = 0.0806$, $\eta_2 = \eta_2/\lambda_m = 0.4031$.
Step 3.1.

$$\beta_{1,1} = \eta_1 = 0.0806, \qquad \beta_{2,1} = \eta_2 = 0.4031,$$
$$\beta_{2,2} = (\alpha_2 + \bar{\alpha}_1)(\beta_{2,1} - \beta_{1,1}) /(\alpha_2 - \alpha_1)(1 - \bar{\beta}_{1,1}\beta_{2,1}) = -1.$$

Step 3.2. $k = 1$ because $|\beta_{1,1}| < 1$ and $|\beta_{2,2}| = 1$.
Step 3.3. Set $N^{(2)}(s) = \beta_{2,2} = -1$.
Step 3.4.

$$N(s) = N^{(1)}(s) = \frac{(s-\alpha_1)N^{(2)}(s) + \beta_{1,1}(s+\bar{\alpha}_1)}{(s+\bar{\alpha}_1) + \bar{\beta}_{1,1}(s-\alpha_1)N^{(2)}(s)} = \frac{1-1.7016s}{1+1.7016s}.$$

Step 4.

$$N(s) := \lambda_m N(s) = 3.1008\frac{1-1.7016s}{1+1.7016s},$$

$$M(s) = [L(s) - N(s)]G^{-1}(s) = 4\frac{1+s}{1+1.7016s}.$$

Remark. Note in the above that the optimal $M(s)$ is not strictly proper. Therefore, if the observer (4) is used to approximate the optimal solution, high observer gain is necessary due to the fact the observer (4) is strictly proper. This shows that the general structure of estimator (7) obviates the unnecessary high observer gains.

5. Interconnection between estimation and loop transfer recovery

The theory of loop transfer recovery (LTR) has been developed following the nominal work of

Doyle and Stein [7] as a means of designing a robust observer. (For an introduction to the LTR theory, the reader is referred to [7,8,9,6] and the references thereof.) However, the research on LTR has been focused mainly on the issues of the so-called *exact* LTR and *asymptotic* LTR where the difference between a target loop transfer and the achievable loop transfer is required to be zero or arbitrarily small. Consequently, the applicability of the theory is more or less limited to minimum phase systems; see [6] for more discussion on this point. The issue of optimal LTR has been recently addressed by the author to deal with nonminimum phase systems [6]. In this section, we briefly describe the optimal LTR problem formulated in [6] and show that this problem and the H^∞ optimal estimation problem are essentially the same, thus bridging the two theories together.

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + Bw(t), \tag{51a}$$

$$y(t) = Cx(t) + Du(t), \tag{51b}$$

where x, y, w, A, B, C, D are as in system (1), and $u(t) \in \mathbf{R}^m$ is the *control*. We denote the open loop transfer function of the system by $G(s)$, i.e.,

$$G(s) = D + C(sI - A)^{-1}B. \tag{52}$$

Let the *desired control law* be described by

$$u(t) = r(t) - z(t), \tag{53}$$

$$z(t) = Fx(t), \tag{54}$$

where $r(t)$ is the *input reference* and $z(t)$ is the *feedback signal*. It follows that the desired open loop transfer function from w to z is given by

$$L(s) = F(sI - A)^{-1}B \tag{55}$$

and the output is

$$y(s) = G(s)[I + L(s)]^{-1}[r(s) + w(s)]. \tag{56}$$

Now consider the following dynamic output feedback law:

$$u(s) = r(s) - \hat{z}(s), \tag{57}$$

$$\hat{z}(s) = M(s)y(s) + N(s)u(s), \tag{58}$$

where $M(s)$ and $N(s)$ are stable rational matrices. If the observer (4) were employed as in many standard LTR techniques, then $M(s)$ and $N(s)$ would be given by (11) and (12). But here, we allow M and N to be general dynamic compensators. Corresponding to (57) and (58), the open loop transfer function from w to $\hat{z}$ becomes

$$L_o(s) = (I + N(s))^{-1}M(s)G(s) \tag{59}$$

and the output is

$$y(s) = G[I + N + MG]^{-1}r(s) + G\left[I + (I + N)^{-1}MG\right]^{-1}w(s). \tag{60}$$

In order for the control law (57)–(58) to have the same closed-loop input–output transfer function as the sate feedback law (54) does so that the separation principle is assured, we must have

$$N(s) + M(s)G(s) = L(s). \tag{61}$$

However, this constraint may cause the closed-loop transfer function from w to y differ from that given by the state feedback due to a possible nonzero N. Therefore, the following optimal LTR problem arises: design M and N such that $\| N \|_\infty = \min$, subject to (61) and a stable M. Equivalently, we need to find a stable M such that

$$\lambda_m = \| MG - L \|_\infty = \min. \tag{62}$$

When $\lambda_m = 0$, then $L_o = L_s$ and

$$y = G[I + L_s]^{-1}(r + w)$$

as in the state feedback case. When the resulting $L(s)G^\dagger(s)$ is proper, this situation is called *exact LTR* and the corresponding $M(s) = L(s)G^\dagger(s)$ and $N(s) = 0$. When the resulting $L(s)G^\dagger(s)$ is improper, we can cascade with it a high bandwidth low-pass filter to achieve properness. Consequently, $N(s)$ can be made arbitrarily small. This situation is referred to as *asymptotic LTR* and the corresponding $M(s)$ and $N(s)$ can be given by (18) and (10) for some small ε; see [6]. In general, however, $\lambda_m \neq 0$. From the analysis in section 2, we immediately conclude that the optimization problem in optimal LTR for system (51) and the that in H^∞ optimal estimation for system (1) are exactly the same.

Example 5.1. This example is adopted from [7]. The system is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + w), \tag{63a}$$

$$y(t) = [2 \quad 1]x. \tag{63b}$$

(The white noises in [7] are neglected because we are not involved in LQG designs.) The plant transfer function is

$$G(s) = \frac{s+2}{(s+1)(s+3)}. \tag{64}$$

The desired control law derived from an optimal full state designs assigns the closed loop poles at $s = -7.0 \pm j2.0$, i.e.,

$$G(s)[I + L(s)]^{-1} = \frac{s+2}{(s+7.0+j2.0)(s+7.0-j2.0)}$$

which gives

$$L(s) = 10\frac{s+5}{(s+1)(s+3)} \tag{65}$$

or $F = [50 \quad 10]$. Since

$$L(s)G^{-1}(s) = 10\frac{s+5}{s+2}$$

is stable and proper, we can achieve the exact LTR by making $M(s) = L(s)G^{-1}(s)$ and $N(s) = 0$. That is, the desired dynamic output feedback law is given by

$$u(s) = r(s) - \hat{z}(s), \tag{66a}$$

$$\hat{z}(s) = 10\frac{s+5}{s+2}y(s). \tag{66b}$$

It is straightforward to check that this control law gives the desired LTR result: $L_0(s) = L(s)$. Note that if an strictly proper observer of the form (4) is used, the exact LTR is not achievable, and the approximation of the output feedback control law (66) must employ a high observer gain.

6. Conclusion

In this paper, we have shown that the H^∞ optimal estimation problem can be solved via the interpolation theory. Among several advantages, this approach offers a more direct and simpler solution than the Riccati equation method and an easy treatment for frequency weightings on estimation error and disturbance. The interconnection between the H^∞ optimal estimation problem and the so-called optimal loop transfer recovery problem is hoped to provide better understanding of the both problems. We finally emphasize that the interpolation approach given in this paper applies to both continuous-time and discrete-time systems. This can be accomplished by the means of bilinear transformation, for example.

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