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On stability robustness with respect to LTV uncertainties

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Abstract

It is shown that the well-known (D, G)-scaling upper bound of the structured singular value is a nonconservative test for robust stability with respect to certain linear time-varying uncertainties.

1 Introduction

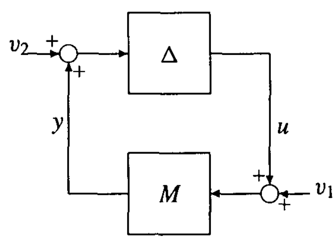

Figure 1: The closed loop.

Is the above closed loop stable for all Δ's in a given set of stable operators $\mathcal{B}$? That, roughly, is the fundamental robust stability problem.

There is an intriguing result by Megretski and Treil [3] and Shamma [8] which says, loosely speaking, that if M is a stable LTI operator and the set of Δ's is the set of contractive linear time-varying operators of some fixed block diagonal structure

$$\Delta = \operatorname{diag}(\Delta_1, \Delta_2, \dots, \Delta_{m_F}), \tag{1}$$

that then the closed loop is robustly stable—that is, stable for all such Δ's—if and only if the $\mathcal{H}_\infty$-norm of DMD^{-1} is less than one for some constant diagonal matrix D that commutes with the Δ's. The problem can be decided in polynomial time, and it is a problem that has since long been associated with an *upper bound* of the structured singular value. The intriguing part is that the result holds for any number of LTV blocks Δ_i, which is in stark contrast with the case that the Δ_i's are assumed time-invariant.

Paganini [6] extended this result by allowing for the more general block diagonal structure

$$\Delta = \operatorname{diag}(\delta_1 I, \dots, \delta_{m_c} I, \Delta_1, \dots, \Delta_{m_F}). \tag{2}$$

A precise definition is given in Section 2. Paganini's result is an exact generalization and leads, again, to a convex optimization problem over the constant matrices D that commute with Δ.

In view of the connection of these results with the upper bounds of the structured singular it is natural to ask if the well known (D, G)-scaling upper bound of the *mixed* structured singular value also has a similar interpretation. In this note we show that that is indeed the case.

We show that the (D, G)-scaling condition is both necessary and sufficient for robust stability for arbitrary LTI plants M with respect to the contractive LTV operators Δ of the form

$$\Delta = \operatorname{diag}(\bar{\delta}_1 I, \dots, \bar{\delta}_{m_r} I, \delta_1 I, \dots, \delta_{m_c} I, \Delta_1, \dots, \Delta_{m_F}), \tag{3}$$

with $\bar{\delta}_i$ denoting linear time-varying *self-adjoint* operators on ℓ_2. A precise definition follows. The condition holds for any number of blocks, while it is known that for LTI Δ's and constant M the (D, G)-scaling condition is necessary and sufficient if and only if

$$2(m_r + m_c) + m_F \leq 3,$$

see [5]. Paganini [7] has gone through considerable trouble to show that for his structure (2) one may assume causality of Δ without changing the condition. In the extended structure (3) with self-adjoint $\bar{\delta}_i$ this is no longer possible.

2 Notation and preliminaries

$\ell_2 := \{x : \mathbb{Z} \mapsto \mathbb{R} : \sum_{k \in \mathbb{Z}} x^2(k) < \infty\}$. The norm $\|v\|_2$ of $v \in \ell_2$ is the usual norm on ℓ_2 and for vector-valued signals $v \in \ell_2^n$ the norm $\|v\|_2$ is defined as $(\|v_1\|_2^2 + \dots + \|v_n\|_2^2)^{1/2}$. The induced norm is denoted by $\|\cdot\|$. So, for $F : \ell_2^n \mapsto \ell_2^n$ it is defined as $\|F\| := \sup_{u \in \ell_2^n} \|Fu\|_2 / \|u\|_2$. For matrices $F \in \mathbb{C}^{n \times m}$ the induced norm will be the spectral norm, and for vectors this reduces to the Euclidean norm. F^{H} denotes the complex conjugate transpose of F, and $\operatorname{He} F := \frac{1}{2}(F + F^{\mathrm{H}})$. An operator $\Delta : \ell_2^n \mapsto \ell_2^n$ is said to be *contractive* if $\|\Delta v\|_2 \leq \|v\|_2$ for every $v \in \ell_2^n$. Lower case δ's always denote operators from ℓ_2^1 to ℓ_2^1. Then for $u, y \in \ell_2^n$ the expression $y = \delta I_n u$ is defined to mean that the entries y_k of y satisfy $y_k = \delta u_k$. An operator $\delta : \ell_2 \mapsto \ell_2$ is *self-adjoint* if $\langle u, \delta v \rangle = \langle \delta u, v \rangle$ for all $u, v \in \ell_2$.

Bounded operators on ℓ_2^n are called *stable*. Hats denote Z-transforms, so if $y \in \ell_2$ then $\hat{y}(z)$ is defined as $\hat{y}(z) = \sum_{k \in \mathbb{Z}} y(k) z^{-k}$. To avoid clutter we shall use for functions $\hat{f}$ of frequency the notation

$$\hat{f}_\omega := \hat{f}(e^{i\omega}).$$

The closed loop depicted in Figure 1 is called *uniformly robustly stable* with respect to some set $\mathcal{B}$ of stable LTV operators Δ if there is a $\gamma > 0$ such that $\left\|\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right\|_2 \leq \gamma \left\|\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]\right\|_2$ for all $\Delta \in \mathcal{B}$, $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \ell_2^{2n}$. We only consider Δ's with norm at most one and stable M. In that case the closed loop is uniformly robustly stable if and only if there is an $\epsilon > 0$ such that $\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2 \; \forall \Delta \in \mathcal{B}, \; u \in \ell_2^n$.

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Throughout we assume that $\Delta : \ell_2^n \mapsto \ell_2^n$ is of the form (3) with

$$\left\{ \begin{array}{lcll} \tilde{\delta}_i & : & \ell_2 \mapsto \ell_2 & \text{LTV, self-adjoint and } \|\tilde{\delta}_i\| \leq 1, \\ \delta_i & : & \ell_2 \mapsto \ell_2 & \text{LTV and } \|\delta_i\| \leq 1, \\ \Delta_i & : & \ell_2^{q_i} \mapsto \ell_2^{q_i} & \text{LTV and } \|\Delta_i\| \leq 1. \end{array} \right. \tag{4}$$

The dimensions of the various identity matrices and Δ_i blocks are fixed, but otherwise Δ may vary over all possible $n \times n$ LTV operators of the form (3),(4). The sets $\mathcal{D}$ and $\mathcal{G}$ are defined as

$$\mathcal{D} = \{D : D = D^{\mathrm{T}} > 0, D \in \mathbb{R}^{n \times n}, D = \operatorname{diag}(\tilde{D}_1, .., \tilde{D}_{m_r}, D_1, .., D_{m_c}, d_1 I_{q_1}, .., d_{m_F} I_{q_{m_F}})\}$$

and

$$\mathcal{G} = \{G : G = G^{\mathrm{H}}, G \in j\mathbb{R}^{n \times n}, G = \operatorname{diag}(\tilde{G}_1, \ldots, \tilde{G}_{m_r}, 0, \ldots, 0, 0, \ldots, 0)\}$$

Note that the D-scales are assumed real-valued and that the G-scales are taken to be purely imaginary. As it turns out there is no need to consider a wider class of D and G-scales.

3 The result

Theorem 3.1. *The discrete time closed-loop in Figure 1 with stable LTI plant with transfer matrix M is uniformly robustly stable with respect to Δ's of the form (3, 4) if and only if there is a constant matrix $D \in \mathcal{D}$ and a constant matrix $G \in \mathcal{G}$ such that*

$$M_\omega^{\mathrm{H}} D M_\omega + j(G M_\omega - M_\omega^{\mathrm{H}} G) - D < 0 \quad \forall \omega \in [0, 2\pi]. \tag{5}$$

Megretski [2] showed this for the full block case (1); Paganini [6] derived this result for the case that the Δ's are of the form (2) and with Δ causal. The proof of the general case (3) follows the same lines as that of [6] and [5], but now the Δ's must be allowed to be non-causal; for causal Δ's the condition (5) is generally only sufficient for uniform robust stability. A key idea is to replace the condition of the contractive Δ-blocks with an integral quadratic condition independent of Δ:

Lemma 3.2. *Let $u, y \in \ell_2^q$ and consider the quadratic integral*

$$\Sigma(u, y) := \int_0^{2\pi} (\hat{y}_\omega - \hat{u}_\omega)(\hat{y}_\omega + \hat{u}_\omega)^{\mathrm{H}} \, d\omega \;\in\; \mathbb{R}^{q \times q}. \tag{6}$$

The following holds.

1. *There is a contractive self-adjoint LTV $\tilde{\delta} : \ell_2 \mapsto \ell_2$ such that $u = \tilde{\delta} I_q y$ if and only if $\Sigma(u, y)$ is Hermitian and nonnegative definite.*
2. *There is a contractive LTV $\delta : \ell_2 \mapsto \ell_2$ such that $u = \delta I_q y$ if and only if the Hermitian part of $\Sigma(u, y)$ is nonnegative definite.*
3. *There is a contractive LTV $\Delta : \ell_2^q \mapsto \ell_2^q$ such that $u = \Delta y$ if and only if the trace of $\Sigma(u, y)$ is nonnegative.*

Proof. See [4]. ∎

A consequence of this result is the following.

Lemma 3.3. *Let u be a nonzero element of ℓ_2^n. Then $(I - \Delta M)u = 0$ for some Δ of the form (3, 4) if-and-only-if*

$$\Sigma(u, Mu) := \int_0^{2\pi} (M_\omega - I)\hat{u}_\omega \hat{u}_\omega^{\mathrm{H}} (M_\omega + I)^{\mathrm{H}} \, d\omega \tag{7}$$

is of the form

$$\left[\begin{array}{cc|cc|cc} \tilde{Z}_1 & ? & ? & ? & ? & ? \\ ? & \ddots & ? & ? & ? & ? \\ \hline ? & ? & Z_1^c & ? & ? & ? \\ ? & ? & ? & \ddots & ? & ? \\ \hline ? & ? & ? & ? & Z_1 & ? \\ ? & ? & ? & ? & ? & \ddots \end{array}\right] \in \mathbb{R}^{n \times n}, \tag{8}$$

with $\tilde{Z}_i = \tilde{Z}_i^{\mathrm{T}} \geq 0$, $\operatorname{He} Z_i^c \geq 0$, $\operatorname{Tr} Z_i \geq 0$, and with "?" denoting an irrelevant entry. Here the partitioning of (8) is compatible with that of Δ.

Proof (sketch). With appropriate partitionings the expression $(I - \Delta M)u = 0$ can be written row-block by row-block as

$$u_1 - \tilde{\delta}_1 M_1 u = 0, \; u_2 - \tilde{\delta}_2 M_2 u = 0, \; \ldots, \; u_K - \Delta_{m_F} M_K u = 0.$$

By Lemma 3.2 there exist contractive $\tilde{\delta}_i$, δ_i and Δ_i of the form (4) for which the above equalities hold iff certain quadratic integrals Σ_i have certain properties. It is not to difficult to figure out that these quadratic integrals Σ_i are exactly the blocks on the diagonal of $\Sigma(u, Mu)$, and that the conditions on these blocks are that they satisfy $\Sigma_i = \Sigma_i^{\mathrm{T}} \geq 0$, $\operatorname{He} \Sigma_i \geq 0$, or $\operatorname{Tr} \Sigma_i \geq 0$, corresponding to the three types of uncertainties. ∎

Proof of Theorem 3.1 (rough sketch). Lemma 3.3 states that $(I - \Delta M)u = 0$ can occur for some Δ if and only if

$$\mathcal{W} \cap \mathcal{Z} = \emptyset,$$

where $\mathcal{W} := \{\Sigma(u, Mu) \;:\; \|u\|_2 = 1\}$ and $\mathcal{Z} := \{Z : Z$ is of the form (8) with $\tilde{Z}_i = \tilde{Z}_i^{\mathrm{T}} \geq 0$, $\operatorname{He} Z_i^c \geq 0$, $\operatorname{Tr} Z_i \geq 0\}$. For uniform robust stability we need that $\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2$ for some $\epsilon > 0$ independent of u. In view of the above it will be no surprise that uniform stability is equivalent to that $\mathcal{W}$ and $\mathcal{Z}$ are bounded away from each other. Equivalently, uniform robust stability holds if and only if $\overline{\mathcal{W}} \cap \mathcal{Z} = \emptyset$. Here $\overline{\mathcal{W}}$ denotes the closure of $\mathcal{W}$. Now $\mathcal{Z}$ is easily seen to be convex, and remarkably $\overline{\mathcal{W}}$ is convex as well [3]. Then by a standard duality argument $\overline{\mathcal{W}} \cap \mathcal{Z} = \emptyset$ is equivalent to the existence of a seperating hyper-plane. The normal vector of this hyper-plane turns out to be $D + jG$ for some $D \in \mathcal{D}$ and $G \in \mathcal{G}$, and that $\mathcal{W}$ and $\mathcal{Z}$ are on opposite sides of the hyper-plane then reduces to the inequality (5). Details are in [4]. ∎

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