

A LATTICE STRUCTURE FOR PERFECT RECONSTRUCTION LINEAR TIME VARYING FILTER BANKS WITH ALL PASS ANALYSIS BANKS

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ABSTRACT

We consider a multi-input, multi-output lattice realization for linear time-varying analysis banks which are all pass. Such a realization was given for LTI systems in [4]; and under certain conditions generalizes to the LTV case. Moreover, our implementation is simpler than the one presented in [4]. Finally, we describe the anticausal inverse of a lattice realization which is used in the synthesis bank.

1. INTRODUCTION

In recent years there has been considerable interest in the theory and design of linear time varying (LTV) filter banks (FB) [2]. Two requirements often imposed on FB design are that the analysis bank (AB) be collectively all pass (individual analysis filters of course will not be all pass), and that the overall FB have the perfect reconstruction (PR) property.

More precisely [1], it is known that a LTVFB can be represented as in figure 1, where q^{-1} is the unit delay operator, k is the time index, $u(k)$ is the FB input, $\hat{u}(k)$ is the output, $E(k, q^{-1})$ is the M -input M -output type I polyphase matrix of the AB, and $R(q^{-1}, k)$ is the $M \times M$ type II polyphase matrix of the synthesis bank (SB). For the purposes of this paper, $E(k, q^{-1})$ and $R(q^{-1}, k)$ can be treated as two $M \times M$ matrix LTV systems, the presence of k indicating their time varying nature. Then the all pass requirement on the AB boils down to the requirement that with $E(k, q^{-1})$ at initial rest and '†' the hermitian transpose, for all square summable $u_i(k)$,

$$\sum_{k=-\infty}^{\infty} \sum_{i=0}^{M-1} u_i^\dagger(k) u_i(k) = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{M-1} y_i^\dagger(k) y_i(k). \quad (1.1)$$

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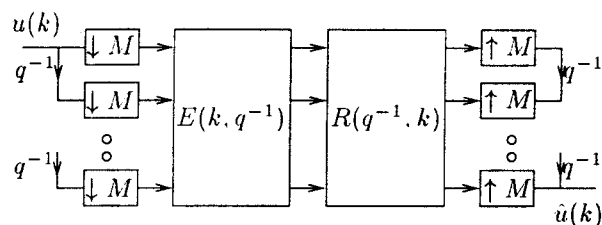


Figure 1: A PR filter bank

The PR requirement boils down to

$$R(q^{-1}, k) E(k, q^{-1}) = I. \quad (1.2)$$

Generally $E(k, q^{-1})$ is causal, and $R(q^{-1}, k)$ anticausal [3]. See [3] for details on how an anticausal $R(q^{-1}, k)$ can be implemented through the transmission of judiciously chosen samples of the states of the AB.

It is known that a limited class of FIR, LTV, all pass $E(k, q^{-1})$ admit a dyadic implementation [1]. In [6], we have shown that a wide class of LTV, IIR, all pass $E(k, q^{-1})$ also admits a more general dyadic implementation. This implementation also covers certain LTV, FIR all pass $E(k, q^{-1})$ not covered by [1].

Equally, it is known that should the FB be LTI then the all pass requirement in 1.1 allows $E(k, q^{-1})$ to have a lattice implementation. This lattice structure, proposed by Vaidyanathan and Mitra [4], generalizes the normalized Gray and Markel lattice to multiple input, multiple output (MIMO) systems. We will call this lattice the VM lattice after its enunciators.

The questions addressed here are twofold. First under what conditions does the VM lattice generalize to the LTV case? Second, what is its corresponding anticausal inverse? Our principle result is to show that all LTV, IIR, all pass $E(k, q^{-1})$ covered by our result referred to in the foregoing, admit a lattice based implementation. Since the implementation presented here, even in its LTI specialization, is simpler than

its counterpart in [4], we label it as the Simplified Vaidyanathan Mitra (SVM) lattice.

Section 2 gives some necessary preliminaries. Section 3 presents the SVM lattice together with a comparison with the VM lattice. Section 4 contains the main results. Section 5 is the conclusion.

2. PRELIMINARIES

Throughout we are concerned with $M \times M$ IIR all pass systems. We call a matrix $P(k)$ uniformly positive definite (upd) if $\exists \beta_1, \beta_2 > 0$ such that $\forall k$,

$$\beta_1 I \leq P(k) \leq \beta_2 I. \quad (2.1)$$

We then say that an $M \times M$ system has *McMillan degree* n , if there exist time varying matrices $A(k)$, $B(k)$, $C(k)$, and $D(k)$ with dimensions $n \times n$, $n \times M$, $M \times n$, and $M \times M$ respectively, if with

$$\Phi(k, k_0) = A(k-1)A(k-2) \cdots A(k_0) \quad (2.2)$$

the following hold:

(i) The system has a state variable realization

$$x(k+1) = A(k)x(k) + B(k)U(k) \quad (2.3)$$

$$Y(k) = C(k)x(k) + D(k)U(k) \quad (2.4)$$

(ii) $[A(k), C(k)]$ is a uniformly completely observable (UCO) pair, i.e. there exist positive N_1 such that the matrix $\sum_{i=k}^{k+N_1} \Phi^\dagger(i, k)C^\dagger(i)C(i)\Phi(i, k)$ is upd. and

(iii) $[A(k), B(k)]$ is a uniformly completely controllable (UCC) pair, i.e. there exist positive N_2 such that the matrix $\sum_{i=k}^{k+N_2} \Phi(k+N_2, i)B(i-1)B^\dagger(i-1)\Phi^\dagger(k+N_2, i)$ is upd.

Note that frozen versions of an LTV system having a McMillan degree may have time varying degrees. Further, we will assume that (2.3,2.4) is exponentially asymptotically stable (eas), i.e., $B(k)$, $C(k)$ and $D(k)$ are bounded and there exist some constants $c, 0 < \delta < 1$ such that for all k_0 and $k \geq k_0$

$$\|x(k)\| \leq c\|x(k_0)\|\delta^{k-k_0}. \quad (2.5)$$

One can write (2.3,2.4) as

$$\begin{bmatrix} x(k+1) \\ Y(k) \end{bmatrix} = \Sigma(k) \begin{bmatrix} x(k) \\ U(k) \end{bmatrix} \quad (2.6)$$

where the realization matrix

$$\Sigma(k) = \left[\begin{array}{c|c} A(k) & B(k) \\ \hline C(k) & D(k) \end{array} \right]. \quad (2.7)$$

We will call $\Sigma(k)$ compatible if the uco, ucc, and eas assumptions apply.

Our goal is to treat only all pass $E(k, q^{-1})$, for which an eas lattice implementation exists. As noted in [6], this often requires that $E(k, q^{-1})$ itself have a compatible realization. Further, for the sake of brevity, only real implementations will be considered, although as in [6], all results easily extend to complex cases. Thus the following will be a standing assumption on the $E(k, q^{-1})$ considered here.

Assumption 2.1 *The $M \times M$ causal system $E(k, q^{-1})$, with input $U(k)$ and output $Y(k)$ is all pass. i.e. it obeys for all $U(k) \in l_2$*

$$\sum_{k=-\infty}^{\infty} U'(k)U(k) = \sum_{k=-\infty}^{\infty} Y'(k)Y(k) \quad (2.8)$$

whenever it is at initial rest. Further, $E(k, q^{-1})$ has a real compatible realization, as in (2.3,2.4), and has McMillan degree n .

We next state the following result from [6]

Theorem 2.1 *Suppose $E(k, q^{-1})$ obeys Assumption 2.1. Then it has a real compatible SVR such that the realization matrix $\Sigma(k)$ obeys for all k*

$$\Sigma'(k)\Sigma(k) = I \quad (2.9)$$

i.e. $\Sigma(k)$ is unitary. Further if $\Sigma(k)$ is a unitary realization matrix of $E(k, q^{-1})$, then $E(k, q^{-1})$ is all pass.

Finally, we note that every symmetric positive semidefinite (psd) $N \times N$ matrix G has a *unique*, real, symmetric, psd square root [5, pp. 180-181] $S = S' \geq 0$ for which $S^2 = G$. We denote the square root by $S = G^{1/2}$.

3. SIMPLIFIED VAIDYANATHAN MITRA LATTICE

We present two versions. Recall, we are concerned with $M \times M$, LTV, IIR systems with McMillan degree n . Define

$$l = n \operatorname{div} M \quad (3.1)$$

$$m = n \operatorname{mod} M. \quad (3.2)$$

Then the first form, labelled SVM1, is as in figure 2 with the following definitions. The operator q^{-1} is the delay operator. If $m = 0$, then $F(k, q^{-1})$ is a zeroth order matrix, $G(k)$, otherwise it is as in figure 3. $U_i(k)$, $Y_i(k)$, $1 \leq i \leq l$ are all $M \times 1$, as are $U_{l+1}(k)$ and $Y_{l+1}(k)$. In figure 3, $w_1(k)$, $w_2(k)$, and $w_4(k)$, are $m \times 1$ and $w_3(k)$ is $(M-m) \times 1$. Each $H_i(k)$ is as in figure 4, where $A_i(k)$, $B_i(k)$, $C_i(k)$, $D_i(k)$ are real and $M \times M$

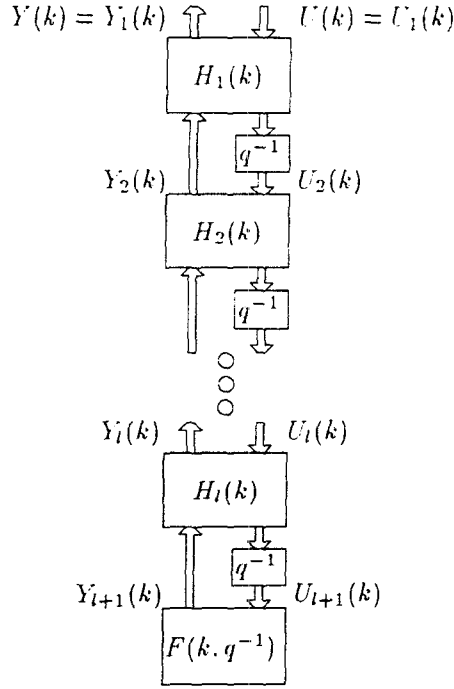


Figure 2: Simplified Vaidyanathan Mitra Lattice I

for $1 \leq i \leq l$ and $m \times m$ for $i = l + 1$. $D_i(k)$ obeys $I - D_i'(k)D_i(k) \geq 0$ and hence $I - D_i(k)D_i'(k) \geq 0$. Then

$$A_i(k) = -D_i'(k), \quad (3.3)$$

$$B_i(k) = [I - D_i'(k)D_i(k)]^{1/2}, \quad (3.4)$$

$$C_i(k) = [I - D_i(k)D_i'(k)]^{1/2}. \quad (3.5)$$

Finally, $G(k)$, $G_1(k)$, $G_2(k)$, and $G_3(k)$ are real unitary matrices respectively having dimensions $M \times M$, $M \times M$, $M \times M$, and $m \times m$.

The second SVM lattice (SVM2) is depicted in figure 5 with $H_i(k)$ precisely as in the SVM1 case, but with $F(k, q^{-1})$ for $m > 0$ as shown in figure 6. Observe that for the SISO LTI case with $M = 1$, figure 5 reduces to the celebrated Gray-Markel lattice.

It is clear that matrices such as

$$\Sigma_D = \begin{bmatrix} -D' & [I - D'D]^{1/2} \\ [I - DD']^{1/2} & D \end{bmatrix} \quad (3.6)$$

play an important role in this structure. The following Lemma makes this role clear.

Lemma 3.1 Suppose D is an $N \times N$ real matrix such that

$$\Sigma_D \Sigma_D' = \Sigma_D' \Sigma_D = I. \quad (3.7)$$

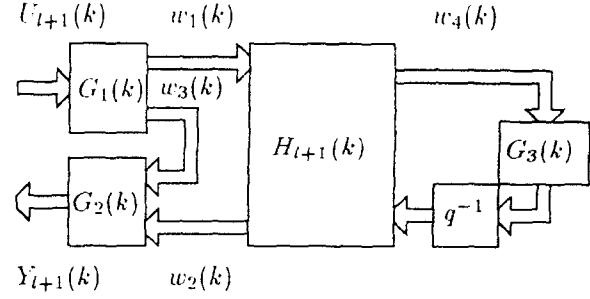


Figure 3: $F(k, q^{-1})$, $m > 0$

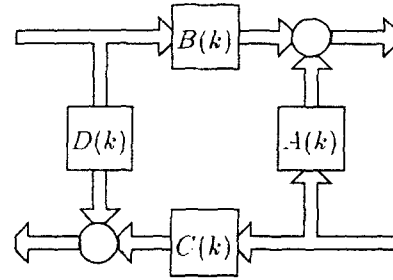


Figure 4: Structure of $H_i(k)$

Thus Σ_D is unitary. To understand why we use the term SVM lattice for these structures, first note that the last block in figures 2 and 5 are more complicated than the rest. In the VM lattice in the LTI case, Σ_D above is replaced by Σ_K given by the matrix

$$\Sigma_K = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (3.8)$$

where

$$A = -((I - K'K)^{-1/2})' K' ((I - KK')^{-1/2})'$$

$$B = (I - KK')^{1/2}$$

$$C = ((I - K'K)^{1/2})'$$

$$D = K$$

for some K . Two points are noteworthy. Evidently, in the above, the unique square root is not employed. Consequently to preserve the unitary nature of Σ_K , the inversion noted above is needed. This is clearly impossible if $I - K'K$ or $I - KK'$ are not positive definite. Whenever that happens, some of the intermediate stages must also be replaced by the more complicated structure exemplified by $F(k, q^{-1})$.

In the SVM lattice, however, this is not an issue. The last stage is there only to ensure minimality when

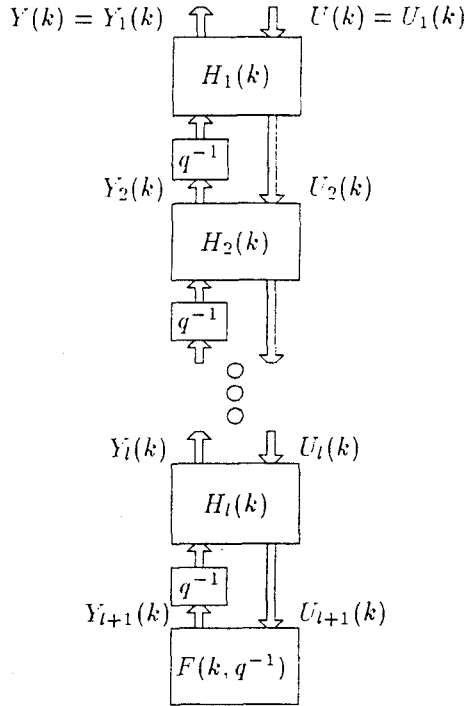


Figure 5: Simplified Vaidyanathan Mitra Lattice 2

n is not a multiple of M . Observe the total number of delays in both SVM1 and SVM2 is n , the assumed McMillan degree.

4. MAIN RESULTS

We present the following Theorems. The first concerns SVM1.

Theorem 4.1 Consider SVM1 defined in Section 3. Then the system relating $U(k)$ to $Y(k)$ is all pass. Further suppose a system, $E(k, q^{-1})$ relating $U(k)$ to $Y(k)$ is $M \times M$, all pass and has McMillan degree n , with a real compatible realization of degree n . Then $E(k, q^{-1})$ has a realization as in figure 2 with the assumptions stated in Section 3.

The second concerns SVM2.

Theorem 4.2 Consider SVM2 defined in Section 3. Then the system relating $U(k)$ to $Y(k)$ is all pass. Further suppose a system $E(k, q^{-1})$ relating $U(k)$ to $Y(k)$ is $M \times M$, all pass and has McMillan degree n , with a real compatible realization of degree n . Then $E(k, q^{-1})$ has a realization as in figure 5 with the assumptions stated in Section 3.

Finally, we describe the anticausal inverse of SVM1.

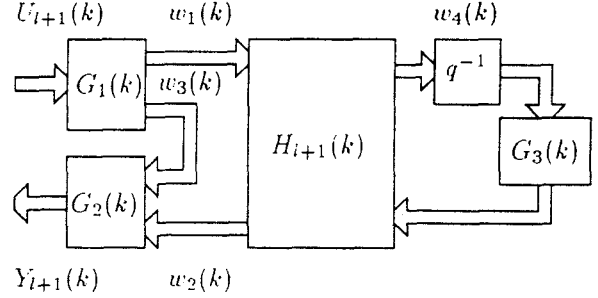


Figure 6: $F(k, q^{-1})$, $m > 0$

Theorem 4.3 The anticausal inverse of SVM1 (respectively SVM2) is the same as SVM2 (SVM1) with q^{-1} replaced by q , $B_i(k)$ by $C'_i(k)$, $C_i(k)$ by $B'_i(k)$, $G_1(k)$ by $G'_2(k)$, $G_2(k)$ by $G'_1(k)$ and the matrices $G(k)$, $A_i(k)$, $D_i(k)$ by their transposes.

5. CONCLUSION

We have shown that all all pass square LTV systems with a well defined McMillan degree and uco, ucc, eas realizations admit two lattice based realizations. We have also characterized the anticausal inverse of such systems.

6. REFERENCES

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