Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Robust Relative Stability of Time-Invariant and Time-Varying Lattice Filters

Soura Dasgupta* Minyue Fu† Chris Schwarz‡

Abstract

We consider the relative stability of time invariant and time varying unnormalized lattice filters. First, we consider a set of lattice filters whose reflection parameters α_i, obey $|\alpha_i| \le \delta_i$, and provide necessary and sufficient conditions on the δ_i that guarantee that each time invariant Lattice in the set has poles inside a circle of prescribed radius $1/\rho < 1$, i.e. they are relatively stable with degree of stability $\ln\rho$. We also show that the relative stability of the whole family is equivalent to the relative stability of a single filter obtained by fixing each α_i to δ_i, and can be checked with only the real poles of this filter. Counterexamples are given to show that a number of properties that hold for stability of LTI Lattices do not apply to relative stability verification. Second, we give a diagonal Lyapunov matrix that is useful in checking the above pole condition. Finally, we consider the time varying problem where the reflection coefficients vary in a region where the frozen transfer functions have poles with magnitude less than $1/\rho$, and provide bounds on their rate of variations that ensure that the zero input state solution of the time varying Lattice decays exponentially at a rate faster than $1/\rho' > 1/\rho$.

1 Introduction

This paper explores the relative stability of Linear Time Invariant (LTI) and Linear Time Varying (LTV) Lattice filters. Lattice filters have been studied extensively in the last two decades. They bear a direct relationship to the celebrated Levinson-Durbin algorithm [1], and have been applied in speech processing and linear predictive coding [2].

*Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grants ECS-9350346 and ECS-9211593.

†Department of Electrical and Computer Engineering, The University of Newcastle, 2308, Australia. On leave to The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grant ECS-9350346.

‡Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grants ECS-9350346 and ECS-9211593.

Of particular interest is their stability property. To elaborate consider the n-th order unnormalized Lattice filter, the subject of this paper, depicted in figure 1, below.

In it $u(k)$ and $y(k)$ represent the input and output of the lattice, k, is the time index, and each block F_i, $1 \le i \le n$, is described as below.

$$F_i: \begin{bmatrix} y_i(k) \\ w_i(k) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_i(k) \\ -\alpha_i(k) & 1-\alpha_i^2(k) \end{bmatrix} \begin{bmatrix} y_{i+1}(k) \\ u_i(k) \end{bmatrix} \quad (1.1)$$

Here, $\alpha_i(k)$ are the so called reflection coefficients assumed to be real in this paper. The time dependence of the $\alpha_i(k)$ recognizes our intention to study the time varying unnormalized lattice. It is well known that in the LTI case, i.e. when $\alpha_i(k) = \alpha_i$, for all k, the unnormalized Lattice is asymptotically stable if and only if its reflection coefficients obey

$$|\alpha_i| < 1 \quad \forall\, 1 \le i \le n. \quad (1.2)$$

Furthermore, the Lattice transfer function

$$G(z) = \frac{Y(z)}{U(z)}, \quad (1.3)$$

is all-pass, i.e. obeys for all $\omega \in [0, 2\pi)$,

$$|G(e^{j\omega})| = 1. \quad (1.4)$$

There are, however, two outstanding open issues in the understanding of Lattice filters. The first of these concerns the issue of relative stability. Simply put, what are the conditions on the reflection coefficients for the defined Lattice to have roots inside a circle of radius $1/\rho$, $\rho > 1$, i.e. when is $G(z/\rho)$ stable? In such a case we say that $\ln\rho$ is the degree of stability of the filter. Such a property, as opposed to mere stability, is important in most practical applications as it reduces the likelihood of quantization induced limit cycles. Further, as will become evident in the sequel, the relative stability of the unnormalized LTV Lattice is also critical to the stability of the LTV Lattice.

The second concerns the relative stability of the LTV Lattice. It is known that the normalized Lattice [3, 4] is stable under arbitrary time variations in the reflection coefficients as long as they obey

$$|\alpha_i(k)| < 1 \quad \forall\, i \in \{1, \ldots, n\}, k. \quad (1.5)$$

0-7803-3590-2/96 $5.00

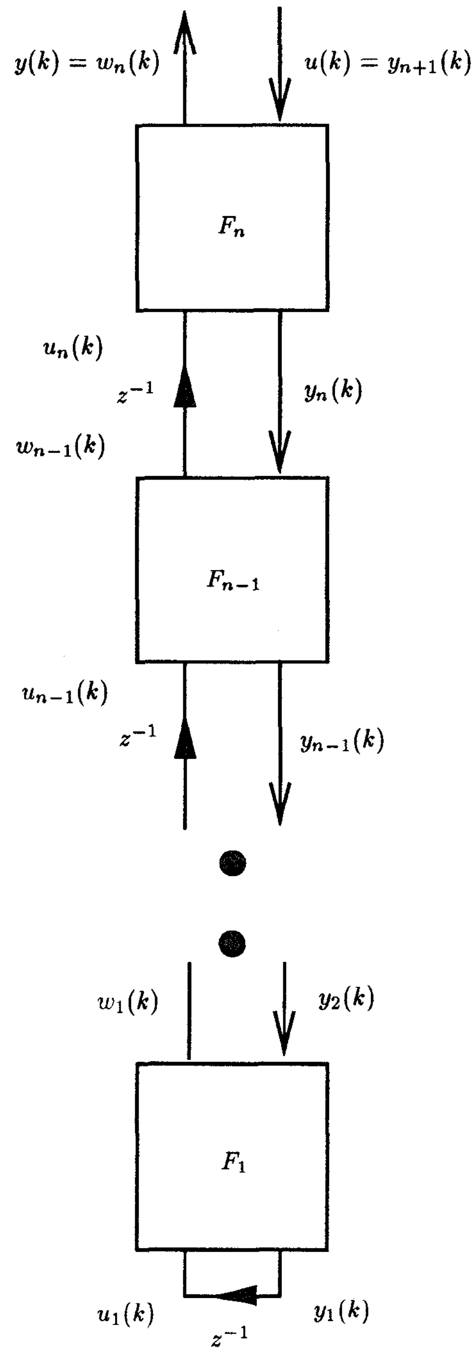

Figure 1: The Unnormalized Lattice Structure

However, no comparable result exists for the unnormalized Lattice. In fact it is well known that the unnormalized LTV Lattice could be unstable despite the satisfaction of (1.5). Moreover, to our knowledge, no nontrivial conditions exist that guarantee the stability, let alone the relative stability of the LTV, unnormalized Lattice.

Equally, in recent years there have been a number of developments in the stability of LTV systems. One such result is that of [5] which considers digital filters in the direct form, and adopts a Lyapunov, [6], approach to analyze stability. It gives bounds on the logarithmic rate of variation of the filter coefficients that guarantee the relative stability of the underlying LTV systems. These bounds provide a natural tradeoff between the relative stability of the frozen LTI systems and the rate of time variations that sustain a prescribed relative stability or stability of the LTV systems.

More specifically, suppose that the frozen digital filters in direct form representation have transfer function $G(z)$, and denominator coefficients a_i, and that $G(z/\rho)$ is stable for all frozen systems. Suppose also that the denominator coefficients vary in the intervals

$$a_i^- < a_i < a_i^+ \quad 1 \leq i \leq n. \tag{1.6}$$

Then with

$$\gamma_i(k) = \frac{a_i(k) - a_i^-}{a_i^+ - a_i(k)}, \tag{1.7}$$

the LTV filter is shown in [5] to be stable with degree of stability $\ln \rho'$, $1 < \rho' < \rho$ if there exist $N > 0$ and $0 < \beta < 1$, such that

$$\sup_{k \geq 0} \frac{1}{N} \sum_{l=k}^{k+N-1} \sum_{i=1}^{n} \left[\ln \frac{\gamma_i(l+1)}{\gamma_i(l)} \right]^+ < 2 \ln \left(\frac{\rho\beta}{\rho'} \right). \tag{1.8}$$

Here

$$[a]^+ = \begin{cases} a & a \geq 0 \\ 0 & \text{else} \end{cases} \tag{1.9}$$

Observe, first there is a tradeoff between the frozen system relative stability $\ln \rho$, the LTV filter degree of stability $\ln \rho'$ and the *average* rate of variation in the parameters γ_i, directly related to the filter coefficients a_i; the γ_i monotonically increase with the a_i. Further, only increases in $\gamma_i(k)$, and hence $a_i(k)$ are of concern. Diminishing $a_i(k)$ carry no destabilizing influence.

Unfortunately, the result of [5] does not readily extend to the Lattice framework, for two reasons. First there is no simple characterization of the relative stability of LTI Lattice filters in terms of the reflection coefficients, an issue that goes directly to the first subject of this paper. Secondly, the LTV analysis of [5] is founded on a Lyapunov based methodology. While there are diagonal Lyapunov matrices obtained by exploring the all pass property of Lattice filters that address the stability of LTI Lattice filters [7, 8], these matrices are not useful for the relative stability problem. More precisely, with $\rho > 1$, and A the state matrix of the LTI Lattice, we need a diagonal positive definite Lyapunov matrix $P = P' > 0$, which obeys

$$\rho^2 A'PA - P \leq 0. \tag{1.10}$$

The Lyapunov matrix of [7, 8] works only for $\rho = 1$.

Accordingly, the structure and contributions of this paper are as follows. In section 2, we present certain preliminary results that provide recursive relationships defining the state space and transfer function descriptions of Lattice filters.

Section 3 addresses the LTI relative stability issue within the following context. Consider the set of reflection coefficients that obey

$$|\alpha_i| \leq \delta_i < 1 \quad \forall\, i \in \{1, \ldots, n\}. \tag{1.11}$$

Then we provide a necessary and sufficient condition on the δ_i for all the corresponding $G(z/\rho)$ to be stable, with $\rho > 1$. This result exploits certain Bounded Real (BR) property ideas associated with $G(z/\rho)$.

It is known that the stability of all members of the Lattice filter set obtained via (1.11), can be verified by checking the stability of just one member, namely $\alpha_i = \delta_i$, $\forall\ 1 \leq i \leq n$. The second result of this Section shows that the same conclusion applies concerning the verification of relative stability of the members of this set. At the same time a counterexample is presented to disprove the conjecture that Lattice filter sets characterized by more general variations, e.g.

$$-1 < \alpha_i^- \leq \alpha_i \leq \alpha_i^+ < 1,$$

do not have corner verifiable relative stability properties. We also give a series of counterexamples to demonstrate that the relative stability verification of a *single* Lattice filter, as opposed to those of all members of sets such as in (1.11), does not have a number of nice properties that characterize the issue of stability invariance verification.

Section 4 derives a Lyapunov matrix that obeys (1.10) whenever $G(z/\rho)$ is stable for all α_i as in (1.11). Section 5 uses the results of Sections 3 and 4 to give a logarithmic rate of variation condition similar to [5] that assures that the LTV Lattice has a prescribed degree of stability, as long as the frozen LTI values assumed by the LTV Lattice has a larger degree of stability, and the reflection coefficients obey

$$|\alpha_i(k)| < \delta_i < 1, \quad \forall\, i \in \{1, \ldots, n\}, k. \tag{1.12}$$

Section 6 concludes.

2 Preliminaries

This Section derives a number of preliminary definitions. We begin with the definition of relative stability of an LTV system.

An LTV, SISO system with state variable representation (SVR) $\{A(k), B(k), C(k), D(k)\}$, the $n \times 1$ state $x(k)$, and $u(k)$ and $y(k)$ the input and output respectively.

Then we say that this system has degree of stability $\ln \rho$, $\rho > 1$ if there exist constants $\beta_1 > 0$, $0 \leq \beta_2 < 1$ such that the zero input state solution obeys for all k, and initial time k_0,

$$\rho^{k-k_0}\|x(k)\| \;\leq\; \beta_1\|x(k_0)\|\beta_2^{k-k_0}. \tag{2.1}$$

Henceforth $\|\cdot\|$ will denote the 2-norm. Observe in the LTI case (2.1) ensures that the transfer function

$$G(z) = d + c(zI - A)^{-1}b \tag{2.2}$$

obeys: $G(z/\rho)$ is stable.

We next recall the Lyapunov approach to stability analysis. As is well known, (2.1) holds iff there exists a symmetric $n \times n$ Lyapunov matrix,

$$\mu_2 I \geq P(k) = P'(k) \geq \mu_1 I > 0 \quad \forall\, k \tag{2.3}$$

for which

$$\rho^2 A'(k)P(k+1)A(k) - P(k) \leq -Q'(k)Q(k), \tag{2.4}$$

$Q(k)$ is real, and $[\rho A(k), Q(k)]$ is uniformly completely observable (u.c.o.), i.e. there exist $\mu_3, \mu_4 > 0$ and integer N such that for all k

$$\begin{aligned} \mu_3 I \;&\leq\; \sum_{i=k}^{k+N-1} \left(\prod_{l=k}^{i} \rho A(l)\right)' Q'(i)Q(i) \left(\prod_{l=k}^{i} \rho A(l)\right) \\ &\leq\; \mu_4 I. \end{aligned} \tag{2.5}$$

Here, the products are identity should the lower index exceed the upper. Further, the order of operation for the products is exemplified by

$$\prod_{l=k}^{i-1} \rho(A(k)) = (\rho A(i-1))(\rho A(i-2)) \cdots (\rho A(k)).$$

Observe in the LTI case this reduces to: [6], $P = P' > 0$

$$\rho^2 A'PA - P \leq -QQ' \tag{2.6}$$

and

$$\sum_{i=0}^{n} (\rho A')^i Q'Q(\rho A)^i > 0. \tag{2.7}$$

We next present a recursive formula for determining the transfer function of a Lattice filter

In the sequel (see figure 1) we will define

$$G_i(z,\alpha_1,\ldots,\alpha_i)=\frac{W_i(z)}{Y_{i+1}(z)}. \qquad (2.8)$$

Thus

$$G_n(z,\alpha_1,\ldots,\alpha_n)=G(z,\alpha_1,\ldots,\alpha_n), \qquad (2.9)$$

the overall transfer function of the Lattice. Further, we will define the transfer function sets, $1\le i\le n$,

$$\mathcal{G}_i(z)=\{G_i(z,\alpha_1,\ldots,\alpha_i)\mid |\alpha_j|\le\delta_j<1, 1\le j\le i\}, \qquad (2.10)$$

for all $0\le i\le n-1$

$$G_{i+1}(z,\alpha_1,\ldots,\alpha_{i+1})=\frac{z^{-1}G_i(z,\alpha_1,\ldots,\alpha_i)-\alpha_{i+1}}{1-z^{-1}\alpha_{i+1}G_i(z,\alpha_1,\ldots,\alpha_i)}, \qquad (2.11)$$

with

$$\mathcal{G}_0(z)=\{1\}. \qquad (2.12)$$

Finally we will call the SVR of $G_i(z)$ $\{A_i(k),b_i(k),c_i(k),d_i(k)\}$.

3 Robust Relative Stability of the LTI Lattice

We call a set of transfer functions *stable invariant* if all its members are asymptotically stable. In this Section we provide a necessary and sufficient condition for $\mathcal{G}_n(z/\rho)$ to be stable invariant, given $\rho>1$. Thus, this solves the problem of determining whether each member of $\mathcal{G}_n(z/\rho)$ has degree of stability $\ln\rho$.

Henceforth we consider the stable invariance of all the $\mathcal{G}_i(z/\rho)$. In order to state the main result of this Section, we must consider the following sequence:

$$f_0 = 1, \qquad (3.1)$$

$$f_i = \frac{\rho f_{i-1}-\delta_i}{1-\rho\delta_i f_{i-1}}, \quad i=1,\ldots,n. \qquad (3.2)$$

Then the necessary and sufficient condition for the stable invariance of the $\mathcal{G}_i(z/\rho)$, $1\le i\le n$, is as follows.

Theorem 3.1 *Consider the sets $\mathcal{G}_i(z)$, $1\le i\le n$, as defined in (2.8-2.10). Then with $\rho>1$, $\mathcal{G}_n(z/\rho)$ is stable invariant iff the f_i defined in (3.1,3.2) exist and obey, for all $1\le i\le n$*

$$0<\rho f_{i-1}\delta_i<1. \qquad (3.3)$$

Further, under (3.3), for all $1\le i\le n$

$$f_i>\rho f_{i-1}. \qquad (3.4)$$

Note that with $\rho=1$, the recursion in (3.1-3.2) gives $f_i=1$, for all $1\le i\le n$, and condition (3.3) boils down to

$$\delta_i<1, \qquad (3.5)$$

a fact well known about Lattice filters. Note, however, that (3.5) is necessary and sufficient for stability of any $G_n(z)$, while (3.3) is not necessary for the stability of $G_n(z/\rho)$.

Observe, (3.4) implies that

$$f_i>\rho^i, \qquad (3.6)$$

whence we have that a necessary, though not sufficient condition for stable invariance of $\mathcal{G}_n(z/\rho)$ is

$$\delta_i<\frac{1}{\rho^i}, \quad \forall\, 1\le i\le n. \qquad (3.7)$$

Finally, observe that the number of computations needed to check the condition in question grows linearly with n.

We conclude this Section with two results of independent interest.

Theorem 3.2 *The set $\mathcal{G}_n(z/\rho)$ is stable invariant iff for all $1\le i\le n$ and $|\alpha_i|\le\delta_i$,*

$$\frac{1}{f_i}G_i(z/\rho,\alpha_1,\ldots,\alpha_i) \qquad (3.8)$$

and

$$\frac{1}{g_i(|\alpha_1|,\ldots,|\alpha_i|)}G_i(z/\rho,\alpha_1,\ldots,\alpha_i) \qquad (3.9)$$

are BR. Further, for all $\omega\in[0,2\pi)$

$$1\le|G_i(e^{j\omega}/\rho,\alpha_1,\ldots,\alpha_i)|. \qquad (3.10)$$

Compare this to the all pass property when $\rho=1$.

The next Theorem relates the stable invariance of $\mathcal{G}_n(z/\rho)$ to the stability, in fact the real poles, of a 'worst' member.

Theorem 3.3 *Given $\rho>1, 0<\delta_i<1, i=1,\ldots,n$. The following are equivalent:*
(i) The set $\mathcal{G}_n(z/\rho)$ is stable invariant.
(ii) $G_n(z/\rho,\delta_1,\ldots,\delta_n)$ is stable.
(iii) $G_n(z/\rho,\delta_1,\ldots,\delta_n)$ has no poles on $z\in[1,\rho)$.

Thus, Theorem 3.3 shows that the stability invariance of the whole set $\mathcal{G}_n(z/\rho)$ boils down to the stability of a single corner Lattice filter. Recall that when $\rho=1$, the set of $G(z/\rho,\delta_1,\ldots,\delta_n)$ stability preserving Lattice coefficients form a convex set ($|\delta_i|<1$). Therefore, it is intuitive to conjecture that the result in Theorem 3.3 can be generalized to the case where the set of reflection coefficients lie in a non-symmetric interval, i.e.

$$\alpha_i^-\le\alpha_i\le\alpha_i^+.$$

We show via the following example that when the parameter set becomes non-symmetric, relative stability of corner filters will not imply the relative stability of the whole set.

Example 3.1
$n = 5$, $(\alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-0.5, 0.1, 0, -0.1)$, $\alpha_1 \in [-0.8, 0.8]$, $\rho = 1.25$. *It is straightforward to verify that $G_n(z/\rho)$ is stable at $\alpha_1 = \pm 0.8$ but unstable at $\alpha_1 = -0.45$. Note from this example that α_1 even lies in a symmetric interval, although $\alpha_2, \dots, \alpha_5$ don't.*

Remark 3.1 *The condition (iii) in Theorem 3.3 offers a simple way of determining the maximum ρ, ρ_{max}, for which relative stability of $G_n(z/\rho)$ is guaranteed for all $1 \le \rho < \rho_{max}$. Indeed, ρ_{max}^{-1} is the smallest pole of $G_n(z, \delta_1, \dots, \delta_n)$ on the positive real axis, which can be checked easily by solving the real eigenvalues of A_n.*

4 Lyapunov Matrix for Relatively Stable LTI Lattices

In order to address the LTV stability problem considered in Section 5, we need to determine a Lyapunov matrix that proves the stable invariance of $\mathcal{G}_n(z/\rho)$. It is known [7] that with

$$\begin{aligned} P &= \text{diag}\,\{(1-\alpha_1^2)\cdots(1-\alpha_{n-1}^2), \\ &\quad (1-\alpha_2^2)\cdots(1-\alpha_{n-1}^2), \dots, 1\}, \end{aligned} \quad (4.1)$$

and $\{A_n, b_n, c_n, d_n\}$ the SVR of $G_n(z, \alpha_1, \dots, \alpha_n)$,

$$A_n' P A_n - P = -(1-\alpha_n^2) e_n e_n'. \quad (4.2)$$

However for the stable invariance of $\mathcal{G}_n(z/\rho)$, we need to find a positive definite symmetric Π_n that obeys

$$\rho^2 A_n' \Pi_n A_n - \Pi_n \le -Q_n' Q_n \quad (4.3)$$

for $|\alpha_i| \le \delta_i$, $1 \le i \le n$ with $[A_n, Q_n]$ obeying (2.7). The main result of this section, presented below, solves this problem by employing the sequence

$$g_0 = 1, \quad (4.4)$$

$$g_i(|\alpha_1|, \dots, |\alpha_i|) = \frac{\rho g_{i-1}(|\alpha_1|, \dots, |\alpha_{i-1}|) - |\alpha_i|}{1 - \rho g_{i-1}(|\alpha_1|, \dots, |\alpha_{i-1}|)|\alpha_i|}. \quad (4.5)$$

Theorem 4.1 *Suppose $\mathcal{G}_n(z/\rho)$ is stable invariant with $\rho > 1$. Then with $\{A_n, b_n, c_n, d_n\}$ the SVR of $G_n(z, \alpha_1, \dots, \alpha_n) \in \mathcal{G}_n(z)$, and Π_n defined by*

$$\begin{aligned} \Pi_n &= \text{diag}\,\{\rho^{n-1}(1-\alpha_{n-1}^2)\cdots(1-\alpha_1^2) g_n/g_0, \\ &\quad \rho^{n-2}(1-\alpha_{n-1}^2)\cdots(1-\alpha_2^2) g_{n-1}/g_1, \\ &\quad \dots, \rho(1-\alpha_{n-1}^2) g_{n-1}/g_{n-2}, 1\}, \end{aligned} \quad (4.6)$$

for all $|\alpha_i| \le \delta_i$, $1 \le i \le n$, we have $1 - \rho^2 g_{n-1}^2 \alpha_n^2 > 0$ and

$$\rho^2 A_n' \Pi_n A_n - \Pi_n \le -(1 - \rho^2 g_{n-1}^2 \alpha_n^2) e_n e_n'. \quad (4.7)$$

Here the g_i are as in (4.4,4.5). We have dropped the arguments $|\alpha_i|$ in g_i, A_n, and Π_n.

The stable invariance of $\mathcal{G}_n(z/\rho)$ ensures that Π_n is positive definite for all $|\alpha_i| \le \delta_i$, $1 \le i \le n$. Further,

$$0 \le \rho g_{n-1} \alpha_n < 1. \quad (4.8)$$

Thus Q_n in (4.3) is

$$Q_n = \sqrt{1 - \rho^2 g_{n-1}^2 \alpha_n^2}\, e_n'. \quad (4.9)$$

Further observe that

$$\sum_{i=0}^{n} (\rho A_n')^i Q_n' Q_n (\rho A_n)^i = W_n' W_n \quad (4.10)$$

where

$$W_n = \begin{bmatrix} Q_n \\ \rho Q_n A_n \\ \vdots \\ \rho^{n-1} Q_n^{n-1} A_n \end{bmatrix}. \quad (4.11)$$

Then it is readily verified that $W_n' W_n$ is positive definite throughout $\mathcal{G}_n(z)$.

Observe, as $g_i = 1$ for all $1 \le i \le n$, whenever $\rho = 1$, we recover the result of [7] when $\rho = 1$. A few further comments on the nature of the derived Lyapunov Matrix are in order. In the setting of [5] involving direct form realization, the Lyapunov Matrix was multiaffine in the coefficients of the transfer function denominator. This fact considerably simplified the LTV analysis conducted in [5]. The Lyapunov Matrix in (4.6) is clearly *not* multiaffine. There is however one vast simplification in the form of (4.6) over its counterpart in [5]: namely that it is diagonal. As will be shown in Section 5, this diagonal nature does aid the LTV analysis conducted there. Two other points to be exploited in Section 5 are as follows. First, Π_n is independent of α_n. Further, because of (4.4,4.5), the Lyapunov Matrix in (4.6) depends only on $|\alpha_i|$, $1 \le i \le n-1$, as opposed to depending on α_i directly.

5 Relative Stability of the LTV Lattice

This Section addresses the relative stability of LTV Lattice filters. We will assume that there exists $\epsilon_i > 0$, arbitrarily small, such that for all $1 \le i \le n$

$$|\alpha_i(k)| \le \delta_i - \epsilon_i \quad (5.1)$$

We will further assume that the f_i in (3.1,3.2) obey (3.3) for all $1 \le i \le n$; i.e. all frozen systems are stable with degree of stability $\ln \rho$. The question is, given

$$1 < \rho' < \rho, \quad (5.2)$$

what rates of time variations can be sustained to ensure that the LTV Lattice has degree of stability $\ln \rho'$?

To this end we present two results. The first is a simple consequence of the comments made at the end of the previous section. The second constitutes the main result of this Section.

Theorem 5.1 *Consider the Lattice filter depicted in figure 1. Suppose (5.1) holds and that* $\mathcal{G}_n(z/\rho)$ *is stable invariant for some* $\rho > 1$*. Suppose also that there exist* a_i *such that for all* $1 \le i \le n-1$*, and all* k*,*

$$|\alpha_i(k)| = |a_i|. \tag{5.3}$$

Then the LTV Lattice filter is stable with degree of stability $\ln \rho$*.*

Observe, this Theorem states that as long as the frozen LTI systems have degree of stability $\ln \rho$, the LTV filter sustains the same degree of stability for arbitrary rates of variation in $\alpha_n(k)$ as long as the $\alpha_i(k)$, $1 \le i \le n-1$ sustain only changes in sign, and (5.1) holds for all $1 \le i \le n$.

The next Theorem addresses relative stability under simultaneous magnitude variations in multiple reflection coefficients.

Theorem 5.2 *Consider the LTV Lattice in figure 1 that obeys (5.1). Suppose* $\mathcal{G}_n(z/\rho)$ *is stable invariant and* $\rho > 1$*. Then the LTV Lattice is stable with degree of stability* ρ'*, obeying (5.2) if the following holds: there exists an integer* $N > 0$ *and* $0 < \beta < 1$ *such that*

$$\sup_{k \ge 0} \frac{1}{N} \sum_{l=k}^{k+N-1} \nu(l) \le 2 \ln \left[\frac{\rho\beta}{\rho'} \right] \tag{5.4}$$

with

$$\gamma_p(k) = \rho(1 - \alpha_p^2(k)) \frac{g_p(k)}{g_{p-1}(k)}, \tag{5.5}$$

$$\nu(k) = \sup_{i \in \{1,\dots,n-1\}} \left\{ \left[\sum_{p=i}^{n-1} \ln \frac{\gamma_p(k+1)}{\gamma_p(k)} \right]^+ \right\}. \tag{5.6}$$

A few comments concerning (5.4,5.5,5.6) are called for. Essentially, this condition represents a tradeoff between frozen systems and LTV system degree of stability with the rate of variations in the *magnitude* of the α_i. Sign changes are inconsequential.

Equations (5.4,5.5,5.6) essentially quantify the potentially destabilizing time variations as those which increase $\gamma_p(k)$, and limit the average increase in these $\gamma_p(k)$. Declining values of $\gamma_p(k)$ are found not to be destabilizing.

6 Conclusion

We have studied the relative stability of both LTI and LTV Lattice. We have shown that when the LTI set of Lattice filters is defined by bounds on the reflection coefficients, then there is a simple necessary and sufficient condition for all such LTI Lattices to have degree of stability $\ln \rho$. We also show that verification of stable invariance can be effected by checking a single corner of $\mathcal{G}_n(z/\rho)$.

We provide a Lyapunov matrix for checking this degree of stability requirement, and show that it specializes to the matrix of [7]. Finally, we give a logarithmic rate of variation result that suffices for the relative stability of LTV unnormalized Lattices.

References

[1] B.D.O. Anderson and J.B. Moore, *Optimal Filtering*, Prentice Hall, 1979.

[2] J.R. Deller, J.G. Proakis, and J.H.L. Hansen, *Discrete-Time Processing of Speech Signals*, Macmillan Publishing, 1993.

[3] A.H. Gray and J.D. Markel,"A normalized digital filter structure", *IEEE Trans. Acoust. Speech, Signal Processing*, Vol. ASSP-23, pp 268-277, June 1975.

[4] S. Phoong and P.P. Vaidyanathan, "Time-varying filter banks: factorizability of lossless time-varying filters and filter banks", *California Institute of Technology Tech. Report*, April 1995.

[5] S. Dasgupta, G. Chockalingam, B.D.O. Anderson, and M. Fu, "Lyapunov functions for uncertain systems with applications to the stability of time varying systems", *IEEE Transactions on Circuits and Systems*, Vol 41, pp 93-106, Feb. 1994.

[6] J.C. Willems, *Stability Theory of Dynamical Systems*, Nelson, 1970.

[7] M. Mansour, "A note on the stability of linear discrete systems and Lyapunov method", *IEEE Transactions on Automatic Control*, Vol. AC-27, pp 707-708, June 1982.

[8] B.D.O. Anderson, E.I. Jury, and M. Mansour, "Schwarz matrix properties for continuous and discrete time systems", *International Journal of Control*, Vol 23, pp 1-16, 1976.