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Linear quadratic regulation for discrete-time systems with multiplicative noise and multiple input delays[‡]

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SUMMARY

This paper is concerned with the optimal linear quadratic regulation problem for discrete-time systems with state and control dependent noises and multiple delays in the input. We show that the problem admits a unique solution if and only if a sequence of matrices, which are determined by coupled difference equations developed in this paper, are positive definite. Under this condition, the optimal feedback controller and the optimal cost are presented via some coupled difference equations. Our approach is based on the stochastic maximum principle. The key technique is to establish relations between the costate and the state.

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1. INTRODUCTION

Time delays exist in many real processes in economics, finance, networked control systems, population dynamics, and so on. Its wide applications have inspired long-term interest and active research on time-delay systems. The control problems for time-delay systems have been extensively studied since 1960s [1–7]. For example, [2] considers the linear quadratic regulation (LQR) problem for systems with constant delays in both state and control variables and characterizes the feedback gain of the optimal controller by a three-argument matrix function which obeys a set of partial differential equations. The presence of a delay in the state (resp. control) variable of the system makes the optimal controller dependent on the past state (resp. past control). This property creates serious technical difficulties for the optimal control problem for time-delay systems.

On the other hand, systems with stochastic uncertainties have received much attention as well [8–13]. Reference [10] is concerned with H^∞ control problem for discrete-time systems with multiplicative noises. Necessary and sufficient conditions for the existence of a stabilizing controller making the norm of a perturbation operator less than a specified value are derived by means of coupled nonlinear matrix inequalities. The authors of [12] investigate the finite-horizon LQR problem for systems with state and control dependent noises subject to an indefinite performance cost function. The existence of an optimal controller is shown to be equivalent to the solvability of a generalized difference Riccati equation.

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Systems involving both time delay and stochastic uncertainties are often used to model financial quantities [14], networked control systems with random delay [15, 16] or with both delay and packet dropout [17], and so on. Their great significance in application is one of the motivations for us to consider the control problem for such systems. In addition, optimal control problems for stochastic systems with delays have attracted much attention. Dynamic programming principle and maximum principle for systems described by stochastic differential equations with delays are established in [18] and [19], respectively. Both the finite-horizon LQR problem and the stabilization problem are solved in [20] for discrete-time systems with state and control dependent noises and a single input delay, which derives the condition for the existence of a unique optimal controller in terms of a Riccati-ZXL difference equation and the stabilizing condition of an algebraic Riccati-ZXL equation. The authors of [21] investigate the finite-horizon LQR problem for continuous-time stochastic linear systems with multiple delays in both state and control variables. It presents the optimal feedback controller in terms of a different type of Riccati equation. However, it also points out that the solvability of this type of Riccati equation is not easy to obtain. By examining the existing literature, we find that conditions for the existence of an optimal feedback controller for stochastic systems with multiple input delays are not available. Motivated by this, we will study the conditions for the existence of an optimal feedback controller for such a system.

The purpose of this paper is to extend the finite-horizon LQR problem in our early work [20] to discrete-time systems with state and control dependent noises and multiple input delays. The approach is based on the maximum principle, which is given by delayed forward backward stochastic difference equations composing of a state equation (delayed and forward), a costate equation (backward) and an equilibrium condition. The key technique is to establish relationships between the costate λ and the state x. Difficulties encountered in the generalization from the single-delay case [20] to the multiple delay case include two aspects. First, only one relation between λ_{k-1} and x_k is enough to derive the optimal controller in [20]. However, another one between λ_{k+d-1} (d is the maximal time delay) and x_k is necessary for the multiple delay case. Because of the time lag between λ_{k+d-1} and x_k, the establishment of this relation is complicated, and the associated coefficient matrices can not be expressed directly but only recursively. Secondly, the optimal controller in the multiple delay case no longer admits a simple predictor form as in [20]. This further leads to the complexity of the coupled difference equations which yield the gain matrices of the optimal controller compared with the simple Riccati-ZXL difference equation in the single-delay case.

The contribution of this paper is as follows. First, a necessary and sufficient condition for the LQR problem to admit a unique solution is proposed. Most papers in the literature on the LQR problem impose the condition that the control weight in the quadratic cost function is positive definite to guarantee that the optimal controller is unique, see, for example, [1–6, 8], and [9]. But this condition is too strong in general. Here, we only require that the control weight is positive semi-definite. Secondly, we present explicitly the optimal feedback controller and the optimal cost via coupled difference equations. Finally, our approach to finding relations between the costate and the state can be used to solve more general delayed forward backward stochastic difference equations such as equations involving delays in both the forward equation and the backward equation.

The rest of paper is organized as follows. Section 2 describes the optimal control problem to be addressed. Section 3 presents the solution to the problem. Section 4 shows the derivation of the solution. Section 5 gives two numerical examples. Section 6 gives conclusions. Appendix supplements some details of the proof.

The following notations will be used in this paper: R^p stands for an $p-$dimensional Euclidean space; I denotes an identity matrix with appropriate dimension. For a matrix X, X' is its transpose; A symmetric matrix $M > 0$ (reps. $\geqslant 0$) means that it is strictly positive definite (reps. positive semi-definite). For a random variable ξ and a σ-algebra $\mathcal{G}$, $E[\xi]$ and $E[\xi|\mathcal{G}]$ represent the expectation of ξ and the conditional expectation of ξ with regard to $\mathcal{G}$, respectively; $\theta_{i,j}$ is the usual Kronecker function, that is, $\theta_{i,j} = 1$ if $i = j$ and $\theta_{i,j} = 0$ otherwise.

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2. PROBLEM FORMULATION

Consider the following discrete-time stochastic system with multiple input delays:

$$x_{k+1} = \left[A + \sum_{j=1}^{r} \omega_k(j)\bar{A}_j \right] x_k + \sum_{i=0}^{d} \left[B_i + \sum_{j=1}^{r} \omega_k(j)\bar{B}_{i,j} \right] u_{k-i},\ k = 0, \ldots, N. \quad (1)$$

Here, $x_k \in R^p$ and $u_k \in R^q$ are the state and input control, respectively; d is a constant delay; x_0 and u_{-i}, $i = 1, \ldots, d$, are known deterministic initial values; A, $\bar{A}_j$, B_i and $\bar{B}_{i,j}$ with $j = 1, \ldots, r$ and $i = 0, \ldots, d$ are constant matrices with compatible dimensions; $\omega_k \doteq \left(\omega_k(1) \cdots \omega_k(r) \right)'$ is a r-dimensional zero-mean white noise defined on a complete probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with covariance

$$E[\omega_k \omega_t'] = \begin{cases} \Sigma, & k = t, \\ 0, & k \neq t, \end{cases}$$

where $\Sigma \geqslant 0$. Let $\{\mathcal{F}_k\}_{k=0,\ldots,N}$ be the natural filtration generated by $\{\omega_k\}_{k=0,\ldots,N}$, that is, $\mathcal{F}_k \subseteq \mathcal{F}$ is the σ-subalgebra generated by $\{\omega_0, \ldots, \omega_k\}$, and $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ be the smallest σ-subalgebra of $\mathcal{F}$.

Consider the cost function

$$J = E\left[\sum_{k=0}^{N} \left(x_k' Q x_k + u_k' R u_k \right) + x_{N+1}' P_{N+1} x_{N+1} \right], \quad (2)$$

where $Q \geqslant 0$, $R \geqslant 0$ and $P_{N+1} \geqslant 0$. The optimal control problem under consideration is formulated as follows:

Problem 1
Find $\mathcal{F}_{k-1}$-measurable u_k, $k = 0, \ldots, N$, to minimize the cost function (2) subject to system (1).

The aim of this paper is twofold: (i) Establish a necessary and sufficient condition for Problem 1 to admit a unique optimal controller; and (ii) If this condition is satisfied, give explicitly the optimal feedback controller and the optimal cost.

Remark 1
System (1) studied in this paper is with one input u_k and with multiple delays in this input, while systems considered in the literature [22–25] contain several inputs $u_k^1, \ldots, u_k^h$ where each input admits a single (but different) delay. If we define $u_k \doteq \left(\left(u_k^1\right)' \cdots \left(u_k^h\right)' \right)'$, then systems in [22–25] can be changed into system (1), and the LQR problem studied in [22–25] can be equivalently converted into Problem 1. However, the converse is not possible in general. In addition, if the inputs u_k^i, $i = 1, \ldots, h$, admit the same delay, the LQR problem in [22–25] reduces to the LQR problem for single-delay system investigated in [20].

Remark 2
The differences between this paper and [26] are as follows. Reference [26] focuses on the stabilization problem for discrete-time systems with control dependent noise and multiple input delays, while this paper studies the finite-horizon LQR problem. Also, the multiplicative noise in [26] is required to satisfy additional independence assumptions.

3. MAIN RESULTS

For simplicity, we will first consider a special case of system (1) with $r = 1$ and $B_i = \bar{B}_{i,1} = 0$ for $i = 1, \ldots, d-1$, that is,

$$x_{k+1} = \left[A + \omega_k(1)\bar{A}_1 \right] x_k + \left[B_0 + \omega_k(1)\bar{B}_{0,1} \right] u_k + \left[B_d + \omega_k(1)\bar{B}_{d,1} \right] u_{k-d}. \quad (3)$$

In this case, the white noise $\omega_k = \omega_k(1)$ becomes one dimensional, and its variance Σ is a nonnegative scalar; $\{\mathcal{F}_k\}_{k\geqslant 0}$ still represents the natural filtration generated by $\{\omega_k\}_{k\geqslant 0}$. Notations $\bar{A}_1$, $\bar{B}_{0,1}$ $\bar{B}_{d,1}$ will be re-denoted by $\bar{A}$, $\bar{B}_0$, and $\bar{B}_d$. Then, the LQR problem is restated as follows:

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Problem 2
Find $\mathcal{F}_{k-1}$-measurable u_k, $k = 0, \ldots, N$, to minimize the cost function (2) subject to system (3).

Following the results in [22], we give the maximum principle of Problem 2 which will play a key role in this paper:

$$x_{k+1} = A(k)x_k + B_0(k)u_k + B_d(k)u_{k-d}, \tag{4}$$

$$\lambda_N = P_{N+1}x_{N+1}, \tag{5}$$

$$\lambda_{k-1} = E\left[A'(k)\lambda_k|\mathcal{F}_{k-1}\right] + Qx_k, \tag{6}$$

$$0 = E\left[B_0'(k)\lambda_k + B_d'(k+d)\lambda_{k+d}|\mathcal{F}_{k-1}\right] + Ru_k, \;\; k = 0, \ldots, N, \tag{7}$$

where λ_k is the costate with $\lambda_k \doteq 0$ for $k > N$ and

$$A(k) \doteq A + \omega_k\bar{A}, \;\; B_0(k) \doteq B_0 + \omega_k\bar{B}_0, \;\; B_d(k) \doteq B_d + \omega_k\bar{B}_d.$$

Define the coupled difference equations as

$$P_k = A'P_{k+1}A + \Sigma\bar{A}'P_{k+1}\bar{A} - T_k'R_k^{-1}T_k + Q, \tag{8}$$

where

$$\begin{aligned} R_k =& R + B_0'P_{k+1}B_0 + \Sigma\bar{B}_0'P_{k+1}\bar{B}_0 + B_d'P_{k+d+1}B_d + \Sigma\bar{B}_d'P_{k+d+1}\bar{B}_d \\ &+ B_0'P_{k+1}^{d-1} + \left(P_{k+1}^{d-1}\right)'B_0 - \sum_{i=1}^{d}\left(T_{k+i}^{d-i}\right)'R_{k+i}^{-1}T_{k+i}^{d-i}, \end{aligned} \tag{9}$$

$$T_k = B_0'P_{k+1}A + \Sigma\bar{B}_0'P_{k+1}\bar{A} + \left(P_{k+1}^{d-1}\right)'A, \tag{10}$$

with

$$T_k^0 = B_0'P_{k+1}B_d + \Sigma\bar{B}_0'P_{k+1}\bar{B}_d + \left(P_{k+1}^{d-1}\right)'B_d, \tag{11}$$

$$T_k^j = B_0'P_{k+1}^{j-1} + \left(P_{k+j+1}^{d-j-1}\right)'B_d - \sum_{i=1}^{j}\left(T_{k+i}^{d-i}\right)'R_{k+i}^{-1}T_{k+i}^{j-i}, \tag{12}$$

$$P_k^0 = A'P_{k+1}B_d + \Sigma\bar{A}'P_{k+1}\bar{B}_d - T_k'R_k^{-1}T_k^0, \tag{13}$$

$$P_k^j = A'P_{k+1}^{j-1} - T_k'R_k^{-1}T_k^j, \;\; j = 1, \ldots, d-1. \tag{14}$$

The terminal value is given by

$$P_{N+1}, \;\; P_{N+i+1} = 0, \;\; P_{N+i}^j = 0, \;\; T_{N+i}^j = 0, \;\; R_{N+i} = I, \; i \geqslant 1, \;\; j = 0, \ldots, d-1. \tag{15}$$

The solution to Problem 2 is stated in the following theorem.

Theorem 1
Problem 2 has a unique optimal controller if and only if $R_k > 0$, $k = 0, \ldots, N$. In this context, the optimal controller u_k is given by

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$$u_k = -R_k^{-1} T_k x_k - R_k^{-1} \sum_{j=0}^{d-1} T_k^j u_{j+k-d}. \tag{16}$$

The associated optimal value of (2) is given by

$$\begin{aligned} J^\star &= x_0' P_0 x_0 + 2x_0' \sum_{j=0}^{d-1} P_0^j u_{j-d} + \sum_{j=0}^{d-1} u_{j-d}' \left(B_d' P_{j+1} B_d + \Sigma \bar{B}_d' P_{j+1} \bar{B}_d \right) u_{j-d} \\ &\quad + 2 \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} u_{j-d}' B_d' P_{j+1}^{i-j-1} u_{i-d} - \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} \sum_{f=0}^{d-1} u_{j-d}' \left(T_f^{j-f} \right)' R_f^{-1} T_f^{i-f} u_{i-d}. \end{aligned} \tag{17}$$

In addition, the optimal costate λ_{k-1} and state x_k satisfy the following non-homogeneous relationship

$$\lambda_{k-1} = P_k x_k + \sum_{j=0}^{d-1} P_k^j u_{j+k-d}. \tag{18}$$

The proof of Theorem 1 will be provided in the next section.

Remark 3

P_k^j and T_k^j with $j = 0, \ldots, d-1$ are well defined by (8)–(15). For simplicity, we have used the notations P_k^j and T_k^j for $j < 0$ in (17). Throughout the paper, we set $P_k^j = 0$ and $T_k^j = 0$ for $j < 0$.

Remark 4

For a delay-free system, that is, $B_d = \bar{B}_d = 0$, (11) and (13) become

$$T_k^0 = 0, \; P_k^0 = -T_k' R_k^{-1} T_k^0 = 0.$$

According to (12) and (14), it yields

$$\begin{aligned} T_k^1 &= B_0' P_{k+1}^0 - \left(T_{k+1}^{d-1} \right)' R_{k+1}^{-1} T_{k+1}^0 = 0, \\ P_k^1 &= A' P_{k+1}^0 - T_k' R_k^{-1} T_k^1 = 0. \end{aligned}$$

Inductively, it can be derived that $T_k^j = 0$ and $P_k^j = 0$ for $j = 0, \ldots, d-1$ and any k. Then, (10) and (9) reduce to

$$R_k = R + B_0' P_{k+1} B_0 + \Sigma \bar{B}_0' P_{k+1} \bar{B}_0, \tag{19}$$

$$T_k = B_0' P_{k+1} A + \Sigma \bar{B}_0' P_{k+1} \bar{A}. \tag{20}$$

It can be easily observed that (8), (19), and (20) are the generalized Riccati equation arising in the standard stochastic LQR problem [12]. In addition, (16) and (17) degenerate to

$$\begin{aligned} u_k &= -R_k^{-1} T_k x_k, \\ J^\star &= x_0' P_0 x_0. \end{aligned}$$

Therefore, Theorem 1 contains the standard stochastic LQR problem as a special case.

Now, we extend Theorem 1 to system (1). First, we generalize (8)–(14) as follows:

$$P_k = A' P_{k+1} A + \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{A}_f' P_{k+1} \bar{A}_l - T_k' R_k^{-1} T_k + Q, \tag{21}$$

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$$R_k = R + \sum_{i=0}^{d} \left(B_i' P_{k+i+1} B_i + \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{B}_{i,f}' P_{k+i+1} \bar{B}_{i,l} \right) + \sum_{i=0}^{d-1} B_i' P_{k+i+1}^{d-i-1}$$

$$+ \sum_{i=0}^{d-1} \left(P_{k+i+1}^{d-i-1} \right)' B_i - \sum_{i=1}^{d} \left(T_{k+i}^{d-i} \right)' R_{k+i}^{-1} T_{k+i}^{d-i}, \tag{22}$$

$$T_k = B_0' P_{k+1} A + \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{B}_{0,f}' P_{k+1} \bar{A}_l + \left(P_{k+1}^{d-1} \right)' A, \tag{23}$$

$$T_k^j = \sum_{i=0}^{j} \left(B_i' P_{k+i+1} B_{i-j+d} + \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{B}_{i,f}' P_{k+i+1} \bar{B}_{i-j+d,l} \right) + \sum_{i=0}^{j-1} B_i' P_{k+i+1}^{j-i-1}$$

$$+ \sum_{i=0}^{j} \left(P_{k+i+1}^{d-i-1} \right)' B_{i-j+d} - \sum_{i=1}^{j} \left(T_{k+i}^{d-i} \right)' R_{k+i}^{-1} T_{k+i}^{j-i}, \tag{24}$$

$$P_k^j = A' P_{k+1} B_{d-j} + \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{A}_f' P_{k+1} \bar{B}_{d-j,l} + A' P_{k+1}^{j-1} - T_k' R_k^{-1} T_k^j, \ j = 0, \ldots, d-1, \tag{25}$$

where the terminal value is given by (15) and $\sigma_{f,l}, \ f = 1, \ldots, r, l = 1, \ldots, r$, are elements of the variance matrix Σ, that is,

$$\Sigma = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,r} \\ \vdots & \vdots & \vdots \\ \sigma_{r,1} & \cdots & \sigma_{r,r} \end{pmatrix}.$$

Theorem 2
Problem 1 has a unique optimal controller if and only if $R_k > 0, k = N, \ldots, 0$. In this case, the optimal controller is

$$u_k = -R_k^{-1} T_k x_k - R_k^{-1} \sum_{j=0}^{d-1} T_k^j u_{j+k-d}, \tag{26}$$

and the optimal cost is

$$\begin{aligned} J^\star =& x_0' P_0 x_0 + 2x_0' \sum_{j=0}^{d-1} P_0^j u_{j-d} + \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} \sum_{m=0}^{d-1} u_{j-d}' \Big[B_{m+d-j}' P_{m+1} B_{m+d-i} \\ &+ \sum_{f=1}^{r} \sum_{l=1}^{r} \sigma_{f,l} \bar{B}_{m+d-j,f}' P_{m+1} \bar{B}_{m+d-i,l} + B_{m+d-j}' P_{m+1}^{i-m-1} \\ &+ \left(P_{m+1}^{j-m-1} \right)' B_{m+d-i} - \left(T_m^{j-m} \right)' R_m^{-1} T_m^{i-m} \Big] u_{i-d}, \end{aligned} \tag{27}$$

where B_i and $\bar{B}_{i,j}$ are defined to be zero for $i > d$.

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Remark 5

Theorems 1 and 2 present solutions to the LQR problem for system (3) and (1), respectively. Differences between systems (3) and (1) are as follows. First, the multiplicative noise of system (3) is $\omega_k(1)$, which is a scalar, while that of system (1) is $\left(\omega_k(1) \cdots \omega_k(r)\right)'$, which is high dimensional. Secondly, system (3) contains one delayed input term u_{k-d}. However, system (1) includes d delayed input terms $\{u_{k-1}, \ldots, u_{k-d}\}$. In addition, by setting $r = 1$ and $B_i = \bar{B}_{i,1} = 0$ for $i = 1, \ldots, d-1$, system (1) reduces to system (3). In this case, it can be easily verified that (21)–(27) naturally becomes (8)–(17). Hence, Theorem 1 is a special case of Theorem 2.

4. PROOF OF THE MAIN RESULTS

In this section, we will first give a detailed proof for Theorem 1. Then, some comments on the derivation of Theorem 2 will be made.

4.1. Proof of Theorem 1

The proof consists of two parts: the necessity is based on the maximum principle (4)–(7); the sufficiency is deduced by constructing a value function.

4.1.1. Necessity. Suppose Problem 2 admits a unique optimal controller. The aim is to prove that $R_k, k = 0, \ldots, N$, defined by (8)–(14), is positive definite and the optimal u_k is as (16).

The proof will be divided into two steps. First, two linear relations will be established: both λ_{k-1} and λ_{k+d-1} are to be expressed as linear combinations of $\{x_k, u_{k-1}, \ldots, u_{k-d}\}$. The feedback gains of the optimal controller will be given by means of the coefficient matrices of these linear relations. The coefficient matrices in the first relation obey some backward recursion equations, but those of the second relation possess complicated expressions and are not easy to compute. To overcome this obstacle, in the second step, we will seek connections between the two relations.

Lemma 1

Suppose Problem 2 has a unique optimal controller, then

$$\begin{aligned} R_k = {} & B_0' P_{k+1} B_0 + \Sigma \bar{B}_0' P_{k+1} \bar{B}_0 + E\left[B_d'(k+d) S_{k+1}\right] B_0 \\ & + E\left[B_d'(k+d) S_{k+1}^{d-1}\right] + B_0' P_{k+1}^{d-1} + R > 0. \end{aligned} \tag{28}$$

The optimal controller u_k is as

$$u_k = -R_k^{-1} T_k x_k - R_k^{-1} \sum_{j=0}^{d-1} T_k^j u_{j+k-d}, \tag{29}$$

with

$$T_k = B_0' P_{k+1} A + \Sigma \bar{B}_0' P_{k+1} \bar{A} + E\left[B_d'(k+d) S_{k+1}\right] A, \tag{30}$$

$$T_k^0 = B_0' P_{k+1} B_d + \Sigma \bar{B}_0' P_{k+1} \bar{B}_d + E\left[B_d'(k+d) S_{k+1}\right] B_d, \tag{31}$$

$$T_k^j = B_0' P_{k+1}^{j-1} + E\left[B_d'(k+d) S_{k+1}^{j-1}\right], \; j = 1, \ldots, d-1. \tag{32}$$

Therein, the matrices P_{k+1}, P_{k+1}^j, S_{k+1} and S_{k+1}^j, $j = 0, \ldots, d-1$, are the coefficients of the following relationships:

$$\lambda_{k-1} = P_k x_k + \sum_{j=0}^{d-1} P_k^j u_{j+k-d}, \tag{33}$$

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$$\lambda_{k+d-1} = S_k x_k + \sum_{j=0}^{d-1} S_k^j u_{j+k-d}, \tag{34}$$

where P_k and P_k^j satisfy the following coupled difference equations:

$$P_k = Q + A' P_{k+1} A + \Sigma \bar{A}' P_{k+1} \bar{A} - \left(A' P_{k+1} B_0 + \Sigma \bar{A}' P_{k+1} \bar{B}_0 + A' P_{k+1}^{d-1} \right) R_k^{-1} T_k, \tag{35}$$

$$P_k^0 = A' P_{k+1} B_d + \Sigma \bar{A}' P_{k+1} \bar{B}_d - \left(A' P_{k+1} B_0 + \Sigma \bar{A}' P_{k+1} \bar{B}_0 + A' P_{k+1}^{d-1} \right) R_k^{-1} T_k^0, \tag{36}$$

$$P_k^j = A' P_{k+1}^{j-1} - \left(A' P_{k+1} B_0 + \Sigma \bar{A}' P_{k+1} \bar{B}_0 + A' P_{k+1}^{d-1} \right) R_k^{-1} T_k^j, \ j = 1, \ldots, d-1, \tag{37}$$

while S_k and S_k^j, which are initialized by $S_{N+1} = 0$ and $S_{N+1}^j = 0$, contain the noises $\{\omega_k, \ldots, \omega_{k+d-1}\}$ and will be explicitly expressed in Lemma 3 of Appendix B.

Proof
See Appendix A. □

Lemma 2
The following identities hold for $k = 0, \ldots, N$:

$$E\left[B_d'(k+d) S_{k+1}\right] = \left(P_{k+1}^{d-1}\right)', \tag{38}$$

$$E\left[B_d'(k+d) S_{k+1}^j\right] = -\sum_{i=0}^{j} \left(T_{k+i+1}^{d-i-1}\right)' R_{k+i+1}^{-1} T_{k+i+1}^{j-i} + \sum_{i=0}^{d-2} \theta_{i,j} \left(P_{k+i+2}^{d-i-2}\right)' B_d + \theta_{d-1,j} \left(B_d' P_{k+d+1} B_d + \Sigma \bar{B}_d' P_{k+d+1} \bar{B}_d\right), \ j = 0, \ldots, d-1. \tag{39}$$

Proof
See Appendix B. □

Finally, it will be clarified that (28), (30)–(32), and (35)–(37) can be rewritten as (8)–(14) with the help of Lemma 2. In fact, the application of (38) and (39) in (28) and (30)–(32) yields (10), (9), (11), and (12) directly. From (9), it follows that

$$T_k' = A' P_{k+1} B_0 + \Sigma \bar{A}' P_{k+1} \bar{B}_0 + A' P_{k+1}^{d-1}.$$

Employ the aforementioned equation in (35)–(37). Then, (8), (13), and (14) can be derived immediately. This completes the proof of the necessity of Theorem 1.

4.1.2. Sufficiency. Given (8)–(14) and $R_k > 0, k = 0, \ldots, N$, we will show that the unique optimal controller of Problem 1 and the optimal cost are, respectively, as (16) and (17).

Define a value function by

$$\begin{aligned} V(k, \bar{x}_k) \doteq E \Bigg[& x_k' P_k x_k + 2 x_k' \sum_{j=0}^{d-1} P_k^j u_{j-d+k} + \sum_{j=0}^{d-1} u_{j-d+k}' \left(B_d' P_{k+j+1} B_d \right. \\ & \left. + \Sigma \bar{B}_d' P_{k+j+1} \bar{B}_d\right) u_{j-d+k} + 2 \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} u_{j-d+k}' B_d' P_{k+j+1}^{i-j-1} u_{i-d+k} \\ & - \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} \sum_{f=0}^{d-1} u_{j-d+k}' \left(T_{k+f}^{j-f}\right)' R_{k+f}^{-1} T_{k+f}^{i-f} u_{i-d+k} \Bigg], \end{aligned} \tag{40}$$

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where $\bar{x}_k$ represents the vector $\begin{pmatrix} x_k' & u_{k-1}' & \cdots & u_{k-d}' \end{pmatrix}'$. By applying (4), it yields

$$\begin{aligned}
&V(k,\bar{x}_k) - V(k+1,\bar{x}_{k+1})\\
=&E\Bigg[x_k'\left(P_k - A'P_{k+1}A - \Sigma\bar{A}'P_{k+1}\bar{A}\right)x_k + 2x_k'\sum_{j=0}^{d-1}P_k^j u_{j+k-d} - 2x_k'A'\sum_{i=0}^{d-1}P_{k+1}^{i-1}u_{i+k-d}\\
&-2x_k'\left(A'P_{k+1}B_d + \Sigma\bar{A}'P_{k+1}\bar{B}_d\right)u_{k-d} - u_{k-d}'\sum_{i=0}^{d-1}\left(T_k^0\right)'R_k^{-1}T_k^i u_{i+k-d}\\
&-\sum_{j=1}^{d-1}u_{j+k-d}'\left(T_k^j\right)'R_k^{-1}T_k^0 u_{k-d} - \sum_{j=1}^{d-1}\sum_{i=1}^{d-1}u_{j+k-d}'\left(T_k^j\right)'R_k^{-1}T_k^i u_{i+k-d}\\
&-u_k'\Bigg(B_0'P_{k+1}B_0 + \Sigma\bar{B}_0'P_{k+1}\bar{B}_0 + B_d'P_{k+d+1}B_d + \Sigma\bar{B}_d'P_{k+d+1}\bar{B}_d + B_0'P_{k+1}^{d-1}\\
&+\left(P_{k+1}^{d-1}\right)'B_0 - \sum_{f=1}^{d}\left(T_{k+f}^{d-f}\right)'R_{k+f}^{-1}T_{k+f}^{d-f}\Bigg)u_k - 2u_k'\left(B_0'P_{k+1}A + \Sigma\bar{B}_0'P_{k+1}\bar{A} + \left(P_{k+1}^{d-1}\right)'A\right)x_k\\
&-2u_k'\left(B_0'P_{k+1}B_d + \Sigma\bar{B}_0'P_{k+1}\bar{B}_d + \left(P_{k+1}^{d-1}\right)'B_d\right)u_{k-d}\\
&-2u_k'\sum_{j=1}^{d-1}\left(B_0'P_{k+1}^{j-1} + \left(P_{k+j+1}^{d-j-1}\right)'B_d - \sum_{f=1}^{j}\left(T_{k+f}^{d-f}\right)'R_{k+f}^{-1}T_{k+f}^{j-f}\right)u_{j+k-d}\Bigg].
\end{aligned}$$

In view of (10), (9), (11), and (12), the aforementioned equation is further written as

$$\begin{aligned}
&V(k,\bar{x}_k) - V(k+1,\bar{x}_{k+1})\\
=&E\Bigg[x_k'\left(P_k - A'P_{k+1}A - \Sigma\bar{A}'P_{k+1}\bar{A}\right)x_k + 2x_k'\sum_{j=0}^{d-1}P_k^j u_{j+k-d} - 2x_k'A'\sum_{i=0}^{d-1}P_{k+1}^{i-1}u_{i+k-d}\\
&-2x_k'\left(A'P_{k+1}B_d + \Sigma\bar{A}'P_{k+1}\bar{B}_d\right)u_{k-d} - u_{k-d}'\sum_{i=0}^{d-1}\left(T_k^0\right)'R_k^{-1}T_k^i u_{i+k-d}\\
&-\sum_{j=1}^{d-1}u_{j+k-d}'\left(T_k^j\right)'R_k^{-1}T_k^0 u_{k-d} - \sum_{j=1}^{d-1}\sum_{i=1}^{d-1}u_{j+k-d}'\left(T_k^j\right)'R_k^{-1}T_k^i u_{i+k-d}\\
&-u_k'(R_k - R)u_k - 2u_k'T_k x_k - 2u_k'\sum_{j=0}^{d-1}T_k^j u_{j+k-d}\Bigg].
\end{aligned}$$

Because $R_k > 0$, we can complete the square in the earlier equation as

$$\begin{aligned}
&V(k,\bar{x}_k) - V(k+1,\bar{x}_{k+1}) = E\Bigg[x_k'\left(P_k - A'P_{k+1}A - \Sigma\bar{A}'P_{k+1}\bar{A} + T_k'R_k^{-1}T_k\right)x_k\\
&+2x_k'\sum_{j=1}^{d-1}\left(P_k^j - A'P_{k+1}^{j-1} + T_k'R_k^{-1}T_k^j\right)u_{j+k-d} + 2x_k'\left(P_k^0 + T_k'R_k^{-1}T_k^0 - A'P_{k+1}B_d - \Sigma\bar{A}'P_{k+1}\bar{B}_d\right)u_{k-d}\\
&+u_k'Ru_k - \left(u_k + R_k^{-1}T_kx_k + R_k^{-1}\sum_{j=0}^{d-1}T_k^ju_{j+k-d}\right)'R_k\left(u_k + R_k^{-1}T_kx_k + R_k^{-1}\sum_{i=0}^{d-1}T_k^iu_{i+k-d}\right)\Bigg].
\end{aligned}$$

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Furthermore, from (8), (13), and (14), it follows that

$$\begin{aligned}
&V(k,\bar{x}_k) - V(k+1,\bar{x}_{k+1}) \\
&= E\left[x_k' Q x_k + u_k' R u_k - \left(u_k + R_k^{-1} T_k x_k + R_k^{-1}\sum_{j=0}^{d-1} T_k^j u_{j+k-d}\right)' R_k \left(u_k + R_k^{-1} T_k x_k \right.\right. \\
&\quad \left.\left. + R_k^{-1}\sum_{i=0}^{d-1} T_k^i u_{i+k-d}\right)\right].
\end{aligned} \tag{41}$$

(40) and $P_{N+1}^j = 0,\ j = 0,\dots,d-1$ imply that $V(N+1,\bar{x}_{N+1}) = E\left[x_{N+1}' P_{N+1} x_{N+1}\right]$. Adding from $k=0$ to $k=N$ on both sides of (41) produces

$$\begin{aligned}
J &= \sum_{k=0}^{N} E\left[x_k' Q x_k + u_k' R u_k\right] + V(N+1,\bar{x}_{N+1}) \\
&= V(0,\bar{x}_0) + E\left[\sum_{k=0}^{N}\left(u_k + R_k^{-1} T_k x_k + R_k^{-1}\sum_{j=0}^{d-1} T_k^j u_{j+k-d}\right)' R_k \left(u_k + R_k^{-1} T_k x_k \right.\right. \\
&\quad \left.\left. + R_k^{-1}\sum_{i=0}^{d-1} T_k^i u_{i+k-d}\right)\right].
\end{aligned}$$

As $R_k > 0$, the unique optimal controller must be as (16), and the optimal cost is $V(0,\bar{x}_0)$, that is, (17). Thus, the proof of sufficiency is completed. □

4.2. Derivation of Theorem 2

The change of the system from (3) to (1) does not cause any essential differences. The main procedures in the derivation of Theorem 2 are the same as those in Theorem 1. Therefore, only major differences in the proof will be listed below.

- The equilibrium condition in the maximum principle becomes

$$0 = E\left[\sum_{i=0}^{d} B_i'(k+i)\lambda_{k+i}|\mathcal{F}_{k-1}\right] + Ru_k,$$

  where $B_i(k+i) = B_i + \sum_{j=1}^{r}\omega_k(j)\bar{B}_{i,j}$.
- Accordingly, it is necessary to establish the additional relations between the costate and the state

$$\lambda_{k+i} = S_k(i)x_k + \sum_{j=0}^{d-1} S_k^j(i)u_{j+k-d},\ i = 0,\dots,d-2,$$

  and to set up identities like (38) and (39) for $S_k(i),\ S_k^j(i),\ i = 0,\dots,d-2$.

4.3. Technical difficulties for the multiple delay case

First, we will review the LQR problem for the single input-delay case investigated in [20]. The problem is to minimize the following cost function

$$J = E\left(\sum_{k=0}^{N} x_k' Q x_k + \sum_{k=d}^{N} u_{k-d}' R u_{k-d} + x_{N+1}' P_{N+1} x_{N+1}\right)$$

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subject to the system equation

$$x_{k+1} = \left(A + \omega_k \bar{A}\right) x_k + \left(B + \omega_k \bar{B}\right) u_{k-d},\ k = 0, \ldots, N$$

over $\mathcal{F}_{k-1}$−measurable u_k. Based on the maximum principle, the main idea is to set up the following relation

$$\lambda_{k-1} = P_k^1 x_k + \sum_{i=2}^{d+1} P_k^i E\left[x_k | \mathcal{F}_{k-d+i-3}\right],\ k = d, \ldots, N+1. \tag{42}$$

The optimal controller is shown to be the following predictor form [20]

$$u_k = -\Upsilon_{k+d}^{-1} M_{k+d} \hat{x}_{k+d|k},\ k = 0, 1, \ldots, N-d, \tag{43}$$

where

$$\hat{x}_{k+d|k} \doteq E\left[x_{k+d} | \mathcal{F}_{k-1}\right] = A^d x_k + \sum_{i=1}^{d} A^{i-1} B u_{k-i}. \tag{44}$$

The gain matrix is given by the following Riccati-ZXL difference equation

$$Z_k = A' Z_{k+1} A + \sigma^2 \bar{A}' X_{k+1} \bar{A} + Q - L_k, \tag{45}$$

$$X_k = Z_k + \sum_{i=0}^{d-1} (A')^i L_{k+i} A^i, \tag{46}$$

$$L_k = M_k' \Upsilon_k^{-1} M_k, \tag{47}$$

$$\Upsilon_k = B' Z_{k+1} B + \sigma^2 \bar{B}' X_{k+1} \bar{B} + R, \tag{48}$$

$$M_k = B' Z_{k+1} A + \sigma^2 \bar{B}' X_{k+1} \bar{A}, \tag{49}$$

with the terminal values $Z_{N+1} = P_{N+1}$ and $X_{N+1} = P_{N+1}$.

From the technical viewpoint, the difficulty arising in the generalization from the single input-delay case to the multiple input-delay case lies in the establishment of the relation between the state and the costate. More specifically, the relation (42) is sufficient to derive the optimal controller in [20]. However, this is not true for the multiple delay case. Except the relation between λ_{k-1} and x_k (33), another one between λ_{k+d-1} and x_k (34) is established to compute the optimal controller. The construction of (33) is similar to that of (42) and is simple, but that of (34) is much more complicated.

Another fundamental obstacle is that the optimal controller for the multiple delay case does not process a simple predictor form like (43). The feature of (43) is that only two variables, Υ_k and M_k, are needed to determine its feedback gains. But for the multiple delay case, $d+2$ variables, $R_k, T_k, T_k^0, \ldots, T_k^{d+1}$, are necessary to give the optimal gain matrices. (An example will be given in the next section to further clarify this point). This is the one of the reasons why the coupled difference equations (8)–(14) can not be simplified like (45)–(49).

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5. NUMERICAL EXAMPLES

Before we give numerical examples, let us see how to solve the coupled difference equations (8)–(14). In fact, (8)–(14) are backward recursions, and they have solutions for $k = N, \dots, 0$ if and only if R_k is nonsingular. The solution is a set of matrix sequences $\{R_k, T_k, T_k^0, \dots, T_k^{d-1}, P_k, P_k^0, \dots, P_k^{d-1}\}_{k=N,\dots,0}$. It is worth noting that, for each k, the correct order of computation should be (10)–(12), (8), (13), and (14). Moreover, the derivation of $\left\{R_k, T_k, T_k^j, P_k, P_k^j, j = 0, \dots, d-1\right\}$ needs the previous values $\left\{R_{k+i}, T_{k+i}, T_{k+i}^j, P_{k+1}, P_{k+1+i}, P_{k+1+i}^j, i = 1, \dots, d, j = 0, \dots, d-1\right\}$.

5.1. *Example 1*

Consider the system (3) and the cost function (2) with

$$A = 1.5,\ \bar{A} = 0.8,\ B_0 = -2,\ \bar{B}_0 = 1.2,\ B_d = 1,\ \bar{B}_d = -0.2,\ d = 1,\ \Sigma = 1,$$
$$N = 0,\ R = 1,\ Q = 1,\ P_{N+1} = 1.$$

The solution to (8)–(14) can be easily derived as

$$R_0 = 6.44,\ T_0 = -2.04,\ T_0^0 = -2.24,\ P_0 = 3.2438,\ P_0^0 = 0.6304.$$

Theorem 1 implies that the unique optimal controller and the optimal cost are, respectively,

$$u_0^\star = -R_0^{-1}T_0x_0 - R_0^{-1}T_0^0u_{-1} = 0.3168x_0 + 0.3478u_{-1},$$

and

$$\begin{aligned} J^\star &= P_0x_0^2 + 2P_0^0x_0u_{-1} + \left(B_1'P_1B_1 + \Sigma\bar{B}_1'P_1\bar{B}_1 - \left(T_0^0\right)'R_0^{-1}T_0^0\right)u_{-1}^2 \\ &= 3.2438x_0^2 + 1.2609x_0u_{-1} + 0.2609u_{-1}^2. \end{aligned}$$

Next, we will verify in a direct way that the aforementioned $u_0^\star$ and $J^\star$ are indeed optimal. Based on the fact that the admissible control u_0 is a function of x_0 and u_{-1} and thus is deterministic, the cost function (2) is computed as

$$\begin{aligned} J &= E\left[x_0^2 + u_0^2 + x_1^2\right] \\ &= x_0^2 + u_0^2 + (B_0u_0 + Ax_0 + B_1u_{-1})^2 + \left(\bar{B}_0u_0 + \bar{A}x_0 + \bar{B}_1u_{-1}\right)^2 \\ &= \left(1 + B_0^2 + \bar{B}_0^2\right)\left[u_0 + \frac{\left(AB_0 + \bar{A}\bar{B}_0\right)x_0 + \left(B_0B_1 + \bar{B}_0\bar{B}_1\right)u_{-1}}{1 + B_0^2 + \bar{B}_0^2}\right]^2 \\ &\quad + \left[1 + A^2 + \bar{A}^2 - \frac{\left(AB_0 + \bar{A}\bar{B}_0\right)^2}{1 + B_0^2 + \bar{B}_0^2}\right]x_0^2 + \left[B_1^2 + \bar{B}_1^2 - \frac{\left(B_0B_1 + \bar{B}_0\bar{B}_1\right)^2}{1 + B_0^2 + \bar{B}_0^2}\right]u_{-1}^2 \\ &\quad + 2\left[AB_1 + \bar{A}\bar{B}_1 - \frac{\left(AB_0 + \bar{A}\bar{B}_0\right)\left(B_0B_1 + \bar{B}_0\bar{B}_1\right)}{1 + B_0^2 + \bar{B}_0^2}\right]x_0u_{-1} \\ &= 6.44[u_0 - 0.3168x_0 - 0.3478u_{-1}]^2 + 3.2438x_0^2 + 1.2609x_0u_{-1} + 0.2609u_{-1}^2, \end{aligned}$$

which means $J \geqslant J^\star$ and J reaches $J^\star$ at $u_0^\star$. Hence, $u_0^\star$ and $J^\star$ are optimal. This demonstrates the correctness of our results.

5.2. *Example 2*

Consider the system (3) and the cost function (2) with

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$$A = 2,\ \bar{A} = 1,\ B_0 = -2,\ \bar{B}_0 = 1,\ B_d = 1,\ \bar{B}_d = 3,\ d = 2,\ \Sigma = 1,$$
$$N = 1,\ R = 1,\ Q = 1,\ P_{N+1} = 1.$$

Direct computation produces the solution to (8)–(14) as

$$R_1 = 6,\ T_1 = -3,\ T_1^0 = 1,\ T_1^1 = 0,\ P_1 = 4.5,\ P_1^0 = 5.5,\ P_1^1 = 0,$$
$$R_0 = 23.5,\ T_0 = -13.5,\ T_0^0 = 4.5,\ T_0^1 = -11,\ P_0 = 15.7447,\ P_0^0 = 25.0851,\ P_0^1 = 4.6809.$$

From Theorem 1, it follows that the optimal controller is

$$u_0^\star = 0.5745x_0 + 0.4681u_{-1} - 0.1915u_{-2},$$
$$u_1^\star = 0.5000x_1^\star + 0u_0 - 0.1667u_{-1}.$$

If the aforementioned optimal controller was a predictor form like (43), then its coefficients would have to satisfy the following relation

$$\frac{0.5}{0.5745} = \frac{0}{0.4681} = \frac{-0.1667}{-0.1915},$$

which obviously does not hold. Therefore, the optimal controller in the multiple input-delay case fails to admit a predictor form as in the single input-delay case.

6. CONCLUSION

This paper resolves the LQR problem for discrete-time systems with multiplicative noise and multiple input delays. A necessary and sufficient condition for the problem to admit a unique optimal controller is given. Under this condition, the optimal feedback controller and the optimal cost are given via coupled difference equations. The approach is based on the maximum principle and the main idea is to establish the relations between the costate and the state. We expect that the results in this paper pave new ways for optimal control of stochastic systems with multiple delays in both state and control variables. In addition, this paper focuses on the finite-horizon LQR problem. The infinite-horizon LQR problem and the stabilization problem are worth considering in the future.

APPENDIX A: PROOF OF LEMMA 1

Proof

This lemma is to be shown inductively on k. Define

$$J(k) \doteq \sum_{i=k}^{N} E\left[x_i' Q x_i + u_i' R u_i\right] + E\left[x_{N+1}' P_{N+1} x_{N+1}\right],\ k = N, \ldots, 0. \tag{A.1}$$

First, consider the case of $k = N$. By applying (4), $J(N)$ defined via (A.1) is computed as

$$\begin{aligned} J(N) = E\, \{ & u_N' \left[R + E\left(B_0'(N) P_{N+1} B_0(N)\right)\right] u_N \\ & + 2u_N' \left[E\left(B_0'(N) P_{N+1} B_d(N)\right) u_{N-d} + E\left(B_0'(N) P_{N+1} A(N)\right) x_N\right] \\ & + u_{N-d}' E\left(B_d(N)' P_{N+1} B_d(N)\right) u_{N-d} + 2u_{N-d}' E\left(B_d(N)' P_{N+1} A(N)\right) x_N \\ & + x_N' E\left(A'(N) P_{N+1} A(N)\right) x_N + x_N' Q x_N \}, \end{aligned}$$

where the fact that x_N, u_N and u_{N-d} are independent of the noise ω_N has been used. Because Problem 1 has a unique solution so does $\min_{u_N} J(N)$. Therefore, the weighting matrix of u_N in $J(N)$ must be positive definite, that is,

$$R_N = R + E\left[B_0'(N) P_{N+1} B_0(N)\right] = R + B_0' P_{N+1} B_0 + \Sigma \bar{B}_0' P_{N+1} \bar{B}_0 > 0.$$

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To solve the optimal u_N, substituting (5) and (4) into (7) yields

$$\begin{aligned}0 &= E\left[B_0'(N)P_{N+1}x_{N+1}|\mathcal{F}_{N-1}\right] + Ru_N \\ &= E\left[B_0'(N)P_{N+1}A(N)\right]x_N + E\left[B_0'(N)P_{N+1}B_d(N)\right]u_{N-d} + R_N u_N \\ &= \left(B_0'P_{N+1}A + \Sigma\bar{B}_0'P_{N+1}\bar{A}\right)x_N + \left(B_0'P_{N+1}B_d + \Sigma\bar{B}_0'P_{N+1}\bar{B}_d\right)u_{N-d} + R_N u_N.\end{aligned}$$

From $R_N > 0$, it is easily seen that the optimal u_N is as (29).

Next, we show that relations (33) and (34) hold for $k = N$. For $d > 1$, (34) is trivial because $\lambda_{N+d-1} = 0$. For $d = 1$, (34) can be derived by employing (4), (5) and the optimal u_N, that is,

$$\begin{aligned}\lambda_N &= P_{N+1}A(N)x_N + P_{N+1}B_0(N)u_N + P_{N+1}B_d(N)u_{N-d} \\ &= \left[P_{N+1}A(N) - P_{N+1}B_0(N)R_N^{-1}T_N\right]x_N + \left[P_{N+1}B_d(N) - P_{N+1}B_0(N)R_N^{-1}T_N^0\right]u_{N-d}.\end{aligned} \tag{A.2}$$

By plugging (A.2) into (6), one obtains

$$\begin{aligned}\lambda_{N-1} = {} & E\left\{\left[A'(N)P_{N+1}A(N) - A'(N)P_{N+1}B_0(N)R_N^{-1}T_N\right]x_N\right. \\ & \left.+\left[A'(N)P_{N+1}B_d(N) - A'(N)P_{N+1}B_0(N)R_N^{-1}T_N^0\right]u_{N-d}|\mathcal{F}_{N-1}\right\} + Qx_N \\ = {} & \left[A'P_{N+1}A + \Sigma\bar{A}'P_{N+1}\bar{A} - \left(A'P_{N+1}B_0 + \Sigma\bar{A}'P_{N+1}\bar{B}_0\right)R_N^{-1}T_N + Q\right]x_N \\ & + \left[A'P_{N+1}B_d + \Sigma\bar{A}'P_{N+1}\bar{B}_d - \left(A'P_{N+1}B_0 + \Sigma\bar{A}'P_{N+1}\bar{B}_0\right)R_N^{-1}T_N^0\right]u_{N-d},\end{aligned}$$

which is just (33) with $k = N$.

Inductively, suppose when $k \geqslant n+1$, R_k defined by (28) is positive definite; the optimal u_k is as (29); λ_{k-1} and λ_{k+d-1} can be expressed as (33) and (34), respectively. We shall verify this claim for $k = n$. First, $R_n > 0$ will be shown. For convenience, denote the value of $J(n+1)$ with $u_k,\ k \geqslant n+1$, being optimal by $J^\star(n+1)$ and

$$\hat{J}(n) \doteq E\left[x_n'Qx_n + u_n'Ru_n\right] + J^\star(n+1). \tag{A.3}$$

According to the dynamic programming principle, that Problem 2 admits a unique optimal controller implies that $\min_{u_n} \hat{J}(n)$ has a unique solution. Hence, the weighting matrix of u_n in $\hat{J}(n)$ must be positive definite. The following calculation reveals that R_n is just the weighting matrix. Let us compute $J^\star(n+1)$ first. In view of (4), (6), and (7), it leads to

$$\begin{aligned}& E\left[x_k'\lambda_{k-1} - x_{k+1}'\lambda_k\right] \\ &= E\left\{x_k'E\left[A'(k)\lambda_k|\mathcal{F}_{k-1}\right]\right\} + E\left[x_k'Qx_k\right] - E\left[x_k'A'(k)\lambda_k\right] - E\left[u_k'B_0'(k)\lambda_k\right] - E\left[u_{k-d}'B_d'(k)\lambda_k\right] \\ &= E\left[x_k'Qx_k\right] - E\left[u_k'B_0'(k)\lambda_k\right] - E\left[u_{k-d}'B_d'(k)\lambda_k\right] \\ &= E\left[x_k'Qx_k + u_k'Ru_k\right] + E\left[u_k'B_d'(k+d)\lambda_{k+d}\right] - E\left[u_{k-d}'B_d'(k)\lambda_k\right],\ k \geqslant n+1.\end{aligned}$$

Adding from $k = n+1$ to $k = N$ on the two sides of the aforementioned equation, we have

$$E\left[x_{n+1}'\lambda_n - x_{N+1}'P_{N+1}x_{N+1}\right] = E\left[x_{n+1}'\lambda_n - x_{N+1}'\lambda_N\right] = \sum_{k=n+1}^{N} E\left[x_k'\lambda_{k-1} - x_{k+1}'\lambda_k\right]$$

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$$
\begin{aligned}
&= \sum_{k=n+1}^{N} E\left[x_k' Q x_k + u_k' R u_k\right] + \sum_{k=n+d+1}^{N} E\left[u_{k-d}' B_d'(k)\lambda_k\right] - \sum_{k=n+1}^{N} E\left[u_{k-d}' B_d'(k)\lambda_k\right] \\
&= \sum_{k=n+1}^{N} E\left[x_k' Q x_k + u_k' R u_k\right] - \sum_{k=n+1}^{n+d} E\left[u_{k-d}' B_d'(k)\lambda_k\right],
\end{aligned}
$$

where $\lambda_k = 0$ for $k > N$ has been applied. From the aforementioned equation, it can be easily obtained that

$$
J^\star(n+1) = E\left[x_{n+1}'\lambda_n\right] + \sum_{k=n+1}^{n+d} E\left[u_{k-d}' B_d'(k)\lambda_k\right]. \tag{A.4}
$$

To compute the weighting matrix of u_n in $\hat{J}(n)$, let $x_n = 0$ and $u_{n-i} = 0$ for $i = 1, \ldots, d$. (A.3) and (A.4) imply that

$$
\hat{J}(n) = E\left[u_n' R u_n + x_{n+1}'\lambda_n + u_n' B_d'(n+d)\lambda_{n+d}\right]. \tag{A.5}
$$

According to the inductive assumption, (33) and (34) hold for $k = n+1$, that is,

$$
\lambda_n = P_{n+1}x_{n+1} + \sum_{j=0}^{d-1} P_{n+1}^j u_{j+n+1-d}, \tag{A.6}
$$

$$
\lambda_{n+d} = S_{n+1}x_{n+1} + \sum_{j=0}^{d-1} S_{n+1}^j u_{j+n+1-d}. \tag{A.7}
$$

Combined with $u_{n-i} = 0,\ i = 1, \ldots, d$, and $x_{n+1} = B_0(n)u_n$, substitution of (A.6) and (A.7) into (A.5) yields

$$
\begin{aligned}
\hat{J}(n) = {} & E\left\{u_n'\left[B_0' P_{n+1} B_0 + \Sigma \bar{B}_0' P_{n+1}\bar{B}_0 + B_0' P_{n+1}^{d-1} + E\left(B_d'(n+d)S_{n+1}\right)B_0\right.\right. \\
& \left.\left. + E\left(B_d'(n+d)S_{n+1}^{d-1}\right)\right]u_n\right\} \\
= {} & E\left[u_n' R_n u_n\right],
\end{aligned}
$$

where the fact that P_{n+1} and P_{n+1}^j are deterministic while S_{n+1} and S_{n+1}^j contain the noises $\{\omega_{n+1}, \ldots, \omega_{n+d}\}$ are used. Thus, $R_n > 0$ has been shown.

Secondly, we will compute the optimal u_n. Substitution of (A.6), (A.7), and (4) into (7) yields

$$
\begin{aligned}
0 = {} & E\left[\left(B_0'(n)P_{n+1} + B_d'(n+d)S_{n+1}\right)x_{n+1}\right. \\
& \left. + \sum_{j=0}^{d-1}\left(B_0'(n)P_{n+1}^j + B_d'(n+d)S_{n+1}^j\right)u_{j+n+1-d} \middle| \mathcal{F}_{n-1}\right] + Ru_n \\
= {} & E\left\{\left[B_0'(n)P_{n+1}A(n) + B_d'(n+d)S_{n+1}A(n)\right]x_n + \left[B_0'(n)P_{n+1}B_d(n)\right.\right. \\
& \left. + B_d'(n+d)S_{n+1}B_d(n)\right]u_{n-d} + \sum_{j=1}^{d-1}\left[B_0'(n)P_{n+1}^{j-1} + B_d'(n+d)S_{n+1}^{j-1}\right]u_{j+n-d} \\
& \left. + \left[B_0'(n)P_{n+1}B_0(n) + B_d'(n+d)S_{n+1}B_0(n) + B_0'(n)P_{n+1}^{d-1} + B_d'(n+d)S_{n+1}^{d-1}\right]u_n \middle| \mathcal{F}_{n-1}\right\} \\
& + Ru_n.
\end{aligned}
$$

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Notice that $x_n, u_n, \ldots, u_{n-d}$ are all $\mathcal{F}_{n-1}$-measurable and the coefficient matrices in the aforementioned equation are independent of $\mathcal{F}_{n-1}$. Also, recall that P and S in the aforementioned equation have been assumed to be deterministic and $\mathcal{F}_{[n+1,n+d]}$-measurable, respectively. So one has

$$\begin{aligned} 0 =& \left\{B_0' P_{n+1} A + \Sigma \bar{B}_0' P_{n+1} \bar{A} + E\left[B_d'(n+d) S_{n+1}\right] A\right\} x_n \\ &+ \left\{B_0' P_{n+1} B_d + \Sigma \bar{B}_0' P_{n+1} \bar{B}_d + E\left[B_d'(n+d) S_{n+1}\right] B_d\right\} u_{n-d} \\ &+ \sum_{j=1}^{d-1} \left\{B_0' P_{n+1}^{j-1} + E\left[B_d'(n+d) S_{n+1}^{j-1}\right]\right\} u_{j+n-d} + R_n u_n. \end{aligned}$$

In view of $R_n > 0$, the optimal u_n can be derived as (29).

Finally, we show (33) and (34) for $k = n$. From (4), (6), and (A.6), λ_{n-1} is derived as

$$\begin{aligned} \lambda_{n-1} =& E\Bigg[A'(n) P_{n+1} A(n) x_n + A'(n) P_{n+1} B_0(n) u_n + A'(n) P_{n+1} B_d(n) u_{n-d} \\ &+ \sum_{j=0}^{d-1} A'(n) P_{n+1}^{j} u_{j+n+1-d} | \mathcal{F}_{n-1}\Bigg] + Q x_n \\ =& \left(A' P_{n+1} A + \Sigma \bar{A}' P_{n+1} \bar{A} + Q\right) x_n + \left(A' P_{n+1} B_0 + \Sigma \bar{A}' P_{n+1} \bar{B}_0 + A' P_{n+1}^{d-1}\right) u_n \\ &+ \left(A' P_{n+1} B_d + \Sigma \bar{A}' P_{n+1} \bar{B}_d\right) u_{n-d} + \sum_{j=0}^{d-2} A' P_{n+1}^{j} u_{j+n+1-d}. \end{aligned}$$

Applying the optimal u_n in the aforementioned equation generates (33) directly. In addition, from the inductive assumption, (33) holds for $k = n + d$, that is,

$$\lambda_{n+d-1} = P_{n+d} x_{n+d} + \sum_{j=0}^{d-1} P_{n+d}^{j} u_{j+n}.$$

By employing (4) and (29) with $k = n + d - 1, \ldots, n$ in the aforementioned identity, (34) for $k = n$ can be obtained, and its coefficient matrices contain the noises $\{\omega_n, \ldots, \omega_{n+d-1}\}$. So far, the case of $k = n$ has been clarified. The proof is completed in an inductive way. □

APPENDIX B: PROOF OF LEMMA 2

Before showing Lemma 2, we need to give explicit expressions of S_k and S_k^j, $j = 0, \ldots, d-1$.

Lemma 3

Define a set of matrices $\Phi_k^s, \Phi_k^{s,j}, \Pi_k^s$ and $\Pi_k^{s,j}$, $k = 0, \ldots, N$, by the recursion

$$\Phi_k^{s+1} = \Phi_{k+s}^1 \Phi_k^s + \sum_{f=0}^{s-1} \Phi_{k+s}^{1,f+d-s} \Pi_k^f, \tag{B.1}$$

$$\Phi_k^{s+1,j} = \Phi_{k+s}^1 \Phi_k^{s,j} + \sum_{f=0}^{s-1} \Phi_{k+s}^{1,f+d-s} \Pi_k^{f,j} + \Phi_{k+s}^{1,j-s}, \tag{B.2}$$

$$\Pi_k^s = \Pi_{k+s}^0 \Phi_k^s + \sum_{f=0}^{s-1} \Pi_{k+s}^{0,f+d-s} \Pi_k^f, \tag{B.3}$$

$$\Pi_k^{s,j} = \Pi_{k+s}^0 \Phi_k^{s,j} + \sum_{f=0}^{s-1} \Pi_{k+s}^{0,f+d-s} \Pi_k^{f,j} + \Pi_{k+s}^{0,j-s},\ s = 0, \ldots, d-1,\ j \leqslant d-1, \tag{B.4}$$

with initial values

$$\Phi_k^0 = I,\ \Phi_k^{0,j} = 0, \tag{B.5}$$

$$\Phi_k^1 = A(k) - B_0(k) R_k^{-1} T_k, \tag{B.6}$$

$$\Phi_k^{1,j} = \theta_{j,0} B_d(k) - B_0(k) R_k^{-1} T_k^j, \tag{B.7}$$

$$\Pi_k^0 = -R_k^{-1} T_k,\ \Pi_k^{0,j} = -R_k^{-1} T_k^j,\ j \leqslant d-1, \tag{B.8}$$

where R_k, T_k and T_k^j, $j \leqslant d-1$, are defined by (28) and (30)–(32). Let the controller be the optimal one, that is, (29). Then ,x_{k+s+1} and u_{k+s} can be represented as

$$x_{k+s+1} = \Phi_k^{s+1} x_k + \sum_{j=0}^{d-1} \Phi_k^{s+1,j} u_{j+k-d}, \tag{B.9}$$

$$u_{k+s} = \Pi_k^s x_k + \sum_{j=0}^{d-1} \Pi_k^{s,j} u_{j+k-d},\ s = 0, \ldots, d-1. \tag{B.10}$$

Moreover, S_k and S_k^j in (34) can be calculated by

$$S_k = P_{k+d} \Phi_k^d + \sum_{f=0}^{d-1} P_{k+d}^f \Pi_k^f, \tag{B.11}$$

$$S_k^j = P_{k+d} \Phi_k^{d,j} + \sum_{f=0}^{d-1} P_{k+d}^f \Pi_k^{f,j}. \tag{B.12}$$

Proof
First of all, (B.9) and (B.10) will be shown inductively with respect to $s = 0, \ldots, d-1$. When $s = 0$, (B.10) is actually the expression of the optimal controller (29). Substituting (29) into (4) results in

$$x_{k+1} = \left[A(k) - B_0(k) R_k^{-1} T_k\right] x_k + \left[B_d(k) - B_0(k) R_k^{-1} T_k^0\right] u_{k-d} - B_0(k) R_k^{-1} \sum_{j=1}^{d-1} T_k^j u_{j+k-d},$$

which is (B.9) with $s = 0$. Now, suppose (B.9) and (B.10) hold for $s = 0, \ldots, n-1$ and arbitrary k. Thus, we have

$$x_{k+n+1} = \Phi_{k+n}^1 x_{k+n} + \sum_{j=0}^{d-1} \Phi_{k+n}^{1,j} u_{j+k+n-d}, \tag{B.13}$$

$$u_{k+n} = \Pi_{k+n}^0 x_{k+n} + \sum_{j=0}^{d-1} \Pi_{k+n}^{0,j} u_{j+k+n-d}, \tag{B.14}$$

$$x_{k+n} = \Phi_k^n x_k + \sum_{j=0}^{d-1} \Phi_k^{n,j} u_{j+k-d}, \tag{B.15}$$

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$$u_{k+f} = \Pi_k^f x_k + \sum_{j=0}^{d-1} \Pi_k^{f,j} u_{j+k-d}, \ f = 0, \ldots, n-1. \tag{B.16}$$

Rewrite (B.13) and (B.14) as

$$x_{k+n+1} = \Phi_{k+n}^1 x_{k+n} + \sum_{j=0}^{d-1} \Phi_{k+n}^{1,j-n} u_{j+k-d} + \sum_{f=0}^{n-1} \Phi_{k+n}^{1,f-n+d} u_{k+f},$$
$$u_{k+n} = \Pi_{k+n}^0 x_{k+n} + \sum_{j=0}^{d-1} \Pi_{k+n}^{0,j-n} u_{j+k-d} + \sum_{f=0}^{n-1} \Pi_{k+n}^{0,f-n+d} u_{k+f},$$

where $\Phi_{k+n}^{1,j-n} = 0$ and $\Pi_{k+n}^{0,j-n} = 0$ if $j < n$. By substituting (B.15) and (B.16) into the aforementioned two identities, (B.9) and (B.10) with $s = n$ can be deduced directly.

Next, (B.11) and (B.12) are to be proven. According to (33), λ_{k+d-1} can be expressed as

$$\lambda_{k+d-1} = P_{k+d} x_{k+d} + \sum_{f=0}^{d-1} P_{k+d}^f u_{k+f}. \tag{B.17}$$

Just now, it has been proven that

$$x_{k+d} = \Phi_k^d x_k + \sum_{j=0}^{d-1} \Phi_k^{d,j} u_{j+k-d}, \ u_{k+f} = \Pi_k^f x_k + \sum_{j=0}^{d-1} \Pi_k^{f,j} u_{j+k-d}.$$

Substituting the aforementioned equations into (B.17) yields an express of λ_{k+d-1} like (34) with the coefficient matrices given by (B.11) and (B.12). Hence, S_k and S_k^j in (34) can be written as (B.11) and (B.12). This ends the proof. □

Next, Lemma 2 is to be verified.

Proof

This lemma is to be shown inductively on $k = N, \ldots, 0$. The case of $k = N$ is trivial because both sides of (38) and (39) are zero in this case.

Suppose (38) and (39) hold for $k \geqslant n$. Then, the application of (38) and (39) in (28) and (30)–(32) yields

$$\begin{aligned} R_k = {} & B_0' P_{k+1} B_0 + \Sigma \bar{B}_0' P_{k+1} \bar{B}_0 + \left(P_{k+1}^{d-1}\right)' B_0 + B_0' P_{k+1}^{d-1} - \sum_{i=1}^{d} \left(T_{i+k}^{d-i}\right)' R_{i+k}^{-1} T_{i+k}^{d-i} \\ & + B_d' P_{k+d+1} B_d + \Sigma \bar{B}_d' P_{k+d+1} \bar{B}_d + R, \end{aligned} \tag{B.18}$$

$$T_k = B_0' P_{k+1} A + \Sigma \bar{B}_0' P_{k+1} \bar{A} + \left(P_{k+1}^{d-1}\right)' A, \tag{B.19}$$

$$T_k^0 = B_0' P_{k+1} B_d + \Sigma \bar{B}_0' P_{k+1} \bar{B}_d + \left(P_{k+1}^{d-1}\right)' B_d, \tag{B.20}$$

$$T_k^j = B_0' P_{k+1}^{j-1} - \sum_{i=1}^{j} \left(T_{i+k}^{d-i}\right)' R_{i+k}^{-1} T_{i+k}^{j-i} + \left(P_{j+k+1}^{d-j-1}\right)' B_d, \ j = 1, \ldots, d-1, \ k \geqslant n. \tag{B.21}$$

So, T_k' is derived as

$$T_k' = A' P_{k+1} B_0 + \Sigma \bar{A}' P_{k+1} \bar{B}_0 + A' P_{k+1}^{d-1}.$$

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Thus, (35)–(37) become

$$P_k = Q + A'P_{k+1}A + \Sigma\bar{A}'P_{k+1}\bar{A} - T_k'R_k^{-1}T_k, \tag{B.22}$$

$$P_k^0 = A'P_{k+1}B_d + \Sigma\bar{A}'P_{k+1}\bar{B}_d - T_k'R_k^{-1}T_k^0, \tag{B.23}$$

$$P_k^j = A'P_{k+1}^{j-1} - T_k'R_k^{-1}T_k^j,\ j = 1,\dots,d-1,\ k \geqslant n. \tag{B.24}$$

Next, (38) and (39) are to be shown for $k = n-1$. The following equations will be proved inductively on $m = 1,\dots,d$:

$$E\left[B_d'(n_1)S_n\right] = \left(P_{n_m}^{m-1}\right)' E\left[\Phi_n^{d-m}\right] + \sum_{f=0}^{d-1-m}\left[B_d'P_{n+d}^f - \sum_{i=d-m}^{d-1} T_{n+i}^{d-i-1\,\prime} R_{n+i}^{-1} T_{n+i}^{f+d-i}\right] E\left[\Pi_n^f\right], \tag{B.25}$$

$$\begin{aligned} E\left[B_d'(n_1)S_n^j\right] = {} & \left(P_{n_m}^{m-1}\right)' E\left[\Phi_n^{d-m,j}\right] \\ & + \sum_{f=0}^{d-1-m}\left[B_d'P_{n+d}^f - \sum_{i=d-m}^{d-1}\left(T_{n+i}^{d-i-1}\right)' R_{n+i}^{-1} T_{n+i}^{f+d-i}\right] E\left[\Pi_n^{f,j}\right] \\ & - \sum_{i=d-m}^{d-1}\left(T_{n+i}^{d-i-1}\right)' R_{n+i}^{-1} T_{n+i}^{j-i} + \sum_{i=d-m}^{d-2}\theta_{i,j}\left(P_{n+i+1}^{d-i-2}\right)' B_d \\ & + \theta_{d-1,j}\left(B_d'P_{n+d}B_d + \Sigma\bar{B}_d'P_{n+d}\bar{B}_d\right), \end{aligned} \tag{B.26}$$

where $n_m \doteq n + d - m$. First, consider $m = 1$. Applying (B.1)–(B.4), we obtain

$$\Phi_n^d = \Phi_{n_1}^1\Phi_n^{d-1} + \sum_{f=0}^{d-2}\Phi_{n_1}^{1,f+1}\Pi_n^f,$$

$$\Phi_n^{d,j} = \Phi_{n_1}^1\Phi_n^{d-1,j} + \sum_{f=0}^{d-2}\Phi_{n_1}^{1,f+1}\Pi_n^{f,j} + \Phi_{n_1}^{1,j-d+1},$$

$$\Pi_n^{d-1} = \Pi_{n_1}^0\Phi_n^{d-1} + \sum_{f=0}^{d-2}\Pi_{n_1}^{0,f+1}\Pi_n^f,$$

$$\Pi_n^{d-1,j} = \Pi_{n_1}^0\Phi_n^{d-1,j} + \sum_{f=0}^{d-2}\Pi_{n_1}^{0,f+1}\Pi_n^{f,j} + \Pi_{n_1}^{0,j-d+1}.$$

Employing the aforementioned equations in (B.11) and (B.12) generates

$$E\left[B_d'(n_1)S_n\right] = XE\left[\Phi_n^{d-1}\right] + \sum_{f=0}^{d-2}\left[Y_f + B_d'P_{n+d}^f\right]E\left[\Pi_n^f\right], \tag{B.27}$$

$$E\left[B_d'(n_1)S_n^j\right] = XE\left[\Phi_n^{d-1,j}\right] + \sum_{f=0}^{d-2}\left[Y_f + B_d'P_{n+d}^f\right]E\left[\Pi_n^{f,j}\right] + Y_{j-d}, \tag{B.28}$$

where

$$\begin{aligned} X &= E\left[B_d'(n_1)P_{n+d}\Phi_{n_1}^1\right] + B_d'P_{n+d}^{d-1}\Pi_{n_1}^0, \\ Y_f &= E\left[B_d'(n_1)P_{n+d}\Phi_{n_1}^{1,f+1}\right] + B_d'P_{n+d}^{d-1}\Pi_{n_1}^{0,f+1},\ f = -1,\dots,d-2. \end{aligned}$$

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By means of (B.6)–(B.8), X and Y_f can be calculated as

$$X = E\left[B_d'(n_1)P_{n+d}A(n_1)\right] - \left\{E\left[B_d'(n_1)P_{n+d}B_0(n_1)\right] + B_d'P_{n+d}^{d-1}\right\}R_{n_1}^{-1}T_{n_1},$$
$$Y_f = \theta_{f+1,0}E\left[B_d'(n_1)P_{n+d}B_d(n_1)\right] - \left\{E\left[B_d'(n_1)P_{n+d}B_0(n_1)\right] + B_d'P_{n+d}^{d-1}\right\}R_{n_1}^{-1}T_{n_1}^{f+1}.$$

From (B.20) and (B.23), it is easy to obtain

$$X = \left(P_{n_1}^0\right)',$$
$$Y_f = \theta_{f+1,0}\left(B_d'P_{n+d}B_d + \Sigma\bar{B}_d'P_{n+d}\bar{B}_d\right) - \left(T_{n_1}^0\right)'R_{n_1}^{-1}T_{n_1}^{f+1}.$$

By applying the aforementioned equations in (B.27) and (B.28), (B.25) and (B.26) for $m = 1$ can be derived.

Inductively, assume that (B.25) and (B.26) are true for $m = t-1$. It will be shown that they hold for $m = t$. Equations (B.1)–(B.8) imply that Φ_k^s, $\Phi_k^{s,j}$, Π_k^s and $\Pi_k^{s,j}$ are $\mathcal{F}_{[k,k+s-1]}$-measurable. So, $\Phi_{n_t}^1$ is independent of Φ_n^{d-t} and $\Phi_n^{d-t,j}$, and $\Phi_{n_t}^{1,f+t}$ is independent of Π_n^f and $\Pi_n^{f,j}$ for $f = 0,\ldots,d-t-1$. Hence, it can be deduced from (B.1)–(B.4) that

$$E\left[\Phi_n^{d-t+1}\right] = E\left[\Phi_{n_t}^1\right]E\left[\Phi_n^{d-t}\right] + \sum_{f=0}^{d-t-1}E\left[\Phi_{n_t}^{1,f+t}\right]E\left[\Pi_n^f\right],$$
$$E\left[\Phi_n^{d-t+1,j}\right] = E\left[\Phi_{n_t}^1\right]E\left[\Phi_n^{d-t,j}\right] + \sum_{f=0}^{d-t-1}E\left[\Phi_{n_t}^{1,f+t}\right]E\left[\Pi_n^{f,j}\right] + E\left[\Phi_{n_t}^{1,j-d+t}\right],$$
$$E\left[\Pi_n^{d-t}\right] = \Pi_{n_t}^0E\left[\Phi_n^{d-t}\right] + \sum_{f=0}^{d-t-1}\Pi_{n_t}^{0,f+t}E\left[\Pi_n^f\right],$$
$$E\left[\Pi_n^{d-t,j}\right] = \Pi_{n_t}^0E\left[\Phi_n^{d-t,j}\right] + \sum_{f=0}^{d-t-1}\Pi_{n_t}^{0,f+t}E\left[\Pi_n^{f,j}\right] + \Pi_{n_t}^{0,j-d+t}.$$

Substitution of the aforementioned equations into (B.25) and (B.26) with $m = t-1$ results in

$$E\left[B_d'(n_1)S_n\right] = XE\left[\Phi_n^{d-t}\right] + \sum_{f=0}^{d-t-1}\left[Y_f - \sum_{i=d-t+1}^{d-1}\left(T_{n+i}^{d-i-1}\right)'R_{n+i}^{-1}T_{n+i}^{f+d-i} + B_d'P_{n+d}^f\right]E\left[\Pi_n^f\right], \tag{B.29}$$

$$\begin{aligned}E\left[B_d'(n_1)S_n^j\right] = {}& XE\left[\Phi_n^{d-t,j}\right] + \sum_{f=0}^{d-t-1}\left[Y_f - \sum_{i=d-t+1}^{d-1}\left(T_{n+i}^{d-i-1}\right)'R_{n+i}^{-1}T_{n+i}^{f+d-i} + B_d'P_{n+d}^f\right]E\left[\Pi_n^{f,j}\right]\\ &+ Y_{j-d} - \sum_{i=d-t+1}^{d-1}\left(T_{n+i}^{d-i-1}\right)'R_{n+i}^{-1}T_{n+i}^{j-i} + \sum_{i=d-t+1}^{d-2}\theta_{i,j}\left(P_{n+i+1}^{d-i-2}\right)'B_d\\ &+ \theta_{d-1,j}\left(B_d'P_{n+d}B_d + \Sigma\bar{B}_d'P_{n+d}\bar{B}_d\right),\end{aligned} \tag{B.30}$$

where

$$X = \left(P_{n_t+1}^{t-2}\right)'E\left[\Phi_{n_t}^1\right] + \left(-\sum_{i=d-t+1}^{d-1}\left(T_{n+i}^{d-i-1}\right)'R_{n+i}^{-1}T_{n+i}^{-t-i} + B_d'P_{n+d}^{d-t}\right)\Pi_{n_t}^0,$$
$$Y_f = \left(P_{n_t+1}^{t-2}\right)'E\left[\Phi_{n_t}^{1,f+t}\right] + \left(-\sum_{i=d-t+1}^{d-1}\left(T_{n+i}^{d-i-1}\right)'R_{n+i}^{-1}T_{n+i}^{-t-i} + B_d'P_{n+d}^{d-t}\right)\Pi_{n_t}^{0,f+t},$$
$$f = -1,\ldots,d-t-1.$$

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By employing (B.6)–(B.8), it is easy to obtain

$$X = \left(P_{n_t+1}^{t-2}\right)' A - \left[\left(P_{n_t+1}^{t-2}\right)' B_0 + B_d' P_{n+d}^{d-t} - \sum_{i=d-t+1}^{d-1} \left(T_{n+i}^{d-i-1}\right)' R_{n+i}^{-1} T_{n+i}^{-t-i} \right] R_{n_t}^{-1} T_{n_t},$$

$$Y_f = \theta_{f+t,0} \left(P_{n_t+1}^{t-2}\right)' B_d - \left[\left(P_{n_t+1}^{t-2}\right)' B_0 + B_d' P_{n+d}^{d-t} - \sum_{i=d-t+1}^{d-1} \left(T_{n+i}^{d-i-1}\right)' R_{n+i}^{-1} T_{n+i}^{-t-i} \right] R_{n_t}^{-1} T_{n_t}^{f+t}.$$

From (B.21) and (B.24), $\left(T_{n_t}^{t-1}\right)'$ and $\left(P_{n_t}^{t-1}\right)'$ are

$$\left(T_{n_t}^{t-1}\right)' = \left(P_{n_t+1}^{t-2}\right)' B_0 - \sum_{i=d-t+1}^{d-1} \left(T_{i+n}^{d-i-1}\right)' R_{i+n}^{-1} T_{i+n}^{-i-t} + B_d' P_{n+d}^{d-t},$$

$$\left(P_{n_t}^{t-1}\right)' = \left(P_{n_t+1}^{t-2}\right)' A - \left(T_{n_t}^{t-1}\right)' R_{n_t}^{-1} T_{n_t}.$$

Thus, X and Y_f can be further calculated as

$$X = \left(P_{n_t}^{t-1}\right)', \; Y_f = \theta_{f+t,0} \left(P_{n_t+1}^{t-2}\right)' B_d - \left(T_{n_t}^{t-1}\right)' R_{n_t}^{-1} T_{n_t}^{f+t}.$$

Employing the aforementioned equations in (B.29) and (B.30) leads to (B.25) and (B.26) for $m = t$. As a result, we have shown (B.25) and (B.26) for $m = 0, \dots, d$ in an inductive way. In particular, setting $m = d$ yields

$$E\left[B_d'(n_1) S_n\right] = \left(P_n^{d-1}\right)' E\left[\Phi_n^0\right],$$

$$E\left[B_d'(n_1) S_n^j\right] = \left(P_n^{d-1}\right)' E\left[\Phi_n^{0,j}\right] - \sum_{i=0}^{d-1} \left(T_{n+i}^{d-i-1}\right)' R_{n+i}^{-1} T_{n+i}^{j-i} + \sum_{i=0}^{d-2} \theta_{i,j} \left(P_{n+i+1}^{d-i-2}\right)' B_d + \theta_{d-1,j} \left(B_d' P_{n+d} B_d + \Sigma \bar{B}_d' P_{n+d} \bar{B}_d\right).$$

From (B.5), it follows that $\Phi_n^0 = I$ and $\Phi_n^{0,j} = 0$. Also, in consideration of $T_{k+i}^{j-i} = 0$ for $j < i$, the aforementioned equations can be further written as (38) and (39) with $k = n - 1$. Thus, the proof is completed. □

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