Adaptive Stabilization of Uncertain Discrete-time Systems via Switching Control: The Method of Localization

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Abstract

In this chapter a new systematic switching control approach to adaptive stabilization of uncertain discrete-time systems with parametric uncertainty is presented. Our approach is based on a localization method which is conceptually different from supervisory adaptive control schemes and other existing switching adaptive schemes. Our approach allows for slow parameter drifting, infrequent large parameter jumps and unknown bound on exogenous disturbance. The unique feature of localization based switching adaptive control distinguishing it from conventional adaptive switching schemes is its rapid model falsification capabilities. In the LTI case this is manifested in the ability of the switching controller to quickly converge to a suitable stabilizing controller. We believe that the approach presented in this chapter is the first design of a switching controller which is applicable to a wide class of linear time invariant and time varying systems and which exhibits good transient performance. The performance of the proposed switching controllers is illustrated by many simulation examples.

1 Introduction

Control design for both linear and nonlinear dynamic systems with unknown parameters has been extensively studied over the last three decades. Despite significant advances in adaptive and robust control in recent years, control of systems with large-size uncertainty remains a difficult task. Not only are the control problems complicated, so is the analysis of stability and performance.

It is well-known [12, 24] that classical adaptive algorithms prior to 1980 were all based on the following set of standard assumptions or variations of them:

(i) An upper bound on the plant order is known;

(ii) The plant is minimum phase;

(iii) The sign of high frequency gain is known;

(iv) The uncertain parameters are constant, and the closed-loop system is free from measurement noise and input/output disturbances.

Classical adaptive algorithms are known to suffer from various robustness problems [34]. A number of attempts have been made since 1980 to relax the assumptions above. A major breakthrough occurred in the mid 1980s [17, 21, 35] for adaptive control of LTV plants with sufficiently small in the mean parameter variations. Later attempts were made for a broader class of systems. Fast varying continuous-time plants were treated in [36], assuming knowledge of the structure of the parameter variations. By using the concept of polynomial differential (integral) operators the problem of model reference adaptive control was dealt with in [32] for a certain class of continuous-time plants with fast time-varying parameters. An interesting approach based on some internal self-excitation mechanism was considered in [7] for a general class of LTV discrete-time systems. The global boundedness of the state was proved. However, it must be noted that the presence of such self-excitation signals in a closed-loop system is often undesirable.

In another research line, a number of switching control algorithms have been proposed recently by several authors [2, 6, 8, 20, 23, 24, 31], thus significantly weakening the assumptions in (i)-(iv). Both continuous and discrete linear time-invariant systems were considered. Research in this direction was originated by the pioneering works of Nussbaum [31] and Martensson [20]. Nussbaum considered the problem of finding a smooth stabilizing controller

$$\begin{cases} \dot{z}(t) = f(g(t), z(t)) \\ u(t) = g(y(t), z(t)) \end{cases} \tag{1}$$

for the one dimensional system

$$\begin{cases} \dot{x}(t) = ax(t) + qu(t) \\ y(t) = x(t) \end{cases} \tag{2}$$

with both $q \neq 0$ and $a > 0$ unknown. In [31] Nussbaum describes a whole family of controllers of the form (1) which achieve the desired stabilization of the system (2). For example, it was shown that every solution $(x(t), z(t))$ of

$$\begin{cases} \dot{x} = ax + qx(z^2+1)\cos(\pi z/2)\exp\, z^2 \\ \dot{z} = x(z^2+1) \end{cases} \tag{3}$$

has the property that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t)$ exists and is finite. We note that the structure of the adaptive controller is explicitly seen from (3). Another important result proved in [31] is that there exists no stabilizing controller for the plant (2) expressed in terms of polynomial or rational functions. A more general result was presented by Martensson [20]. In particular, it was shown in [20] that the only *a priori* information which is needed for adaptive stabilization of a minimal linear time-invariant plant is the order of a stabilizing controller. This assumption can even be removed if a slightly more complicated controller is used. Consider the following dynamic feedback problem: Given the plant

$$\begin{cases} \dot{x} = Ax + Bu,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m, \\ y = Cx,\ y \in \mathbf{R}^r \end{cases} \tag{4}$$

and the controller

$$\begin{cases} \dot{z} = Fz + Gy,\ z \in \mathbf{R}^l \\ u = Hz + Ky \end{cases} \tag{5}$$

where m, r are known and fixed, and n is allowed to be arbitrary. It is easy to see that this is equivalent to the static feedback problem

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u} \\ \hat{y} = \hat{C}\hat{x},\ \hat{u} = \hat{K}\hat{y} \end{cases} \tag{6}$$

where $\hat{x} = (x^T z^T)^T$, $\hat{u} = (u^T z^T)^T$, $\hat{y} = (y^T z^T)^T$ and $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{K}$ are matrices of appropriate dimensions. Let the regulator be

$$\begin{cases} \hat{u} = g(h(k))N(h(k))\hat{y} \\ \dot{k} = ||\hat{y}||^2 + ||\hat{u}||^2 \end{cases} \tag{7}$$

where $N(h)$ is an 'almost periodic' and dense function and h and g are continuous, scalar functions satisfying a set of four assumptions (see [20] for more details). Martensson's result reads: "Assume that l is known so that there exists a fixed stabilizing controller of the form (5), and that the augmentation to the form (6) has been done. Then the controller (7) will stabilize the system in the sense that

$$(x(t),\ z(t),\ k(t)) \to (0,\ 0,\ k_\infty) \quad \text{as} \quad t \to \infty \tag{8}$$

where $k_\infty < \infty$ ".
One such set of functions given by Martensson is

$$h(k) = (\log\, k)^{1/2},\ k \geq 1,\ g(h) = (\sin\, h^{1/2} + 1)h^{1/2} \tag{9}$$

Martensson's method is based on a "dense" search over the control parameter space, allows for no measurement noise, and guarantees only asymptotic stability rather than exponential stability. These weaknesses were overcome in [8] where a finite switching control method was proposed for LTI systems with uncertain parameters satisfying some mild compactness assumptions. Different modifications of Martensson's controller aimed at achieving Lyapunov stability, avoiding dense search procedures, as well as extending this approach to discrete time systems have been reported recently (see, e. g., [2, 8, 19, 23]). However, the lack of exponential stability might result in poor transient performance as pointed out by many researchers; (see, for example, [8, 19] for simulation examples). Below we present a simple example of a controller based on a dense search over the parameter space. This controller is a simplified version of that presented in [19].
Example 1.1 The second order plant

$$x(t+1) = a_1 x(t) + a_2 x(t-1) + bu(t) + \xi(t),\ x, u \in \mathbf{R}, \quad a_{1,2} \in \mathbf{R},\ b \neq 0 \tag{10}$$

with $a_{1,2},\ b \neq 0$ being arbitrary unknown constants and $\sup_{t \geq t_0} |\xi(t)| < \infty$ has to be controlled by the switching controller

$$u(t) = k(t)x(t) \tag{11}$$

where $k(0) = h(1)$ and $k(t) = h(i)$, $t \in (t_i,\ t_{i+1}]$ and $h(i)$ is a function dense in $\mathbf{R}$ defined so that it successively looks at each interval $[-p,\ p]$, $p \in \mathbf{N}$ and tries points $1/2^p$ apart, namely,

$$\begin{array}{lll} h(1) = 1 & h(4) = -0.5 & h(7) = 1.75 \\ h(2) = 0.5 & h(5) = -1 & \quad etc. \\ h(3) = 0 & h(6) = \ \ 2 & \end{array}$$

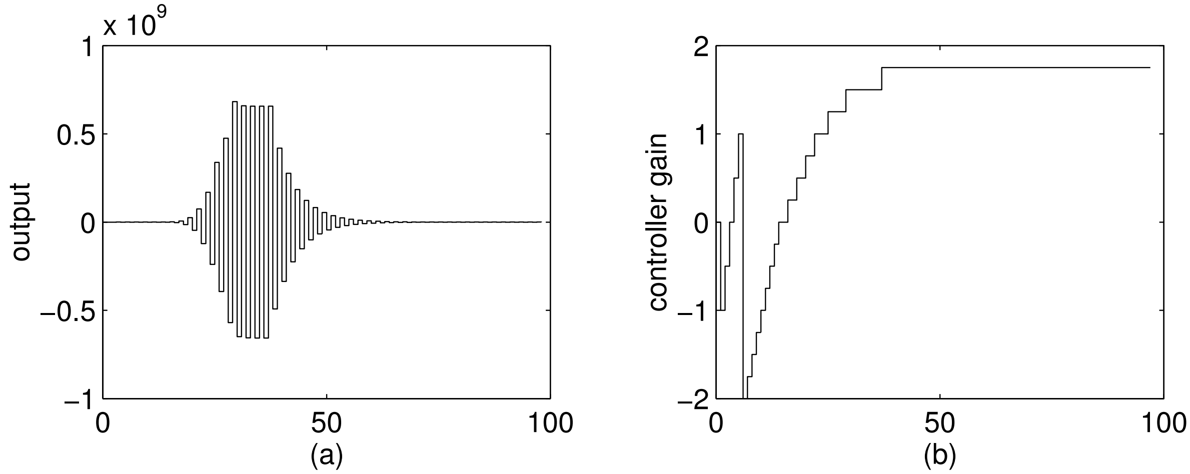

Figure 1: Example of a dense search

The system performance is monitored using a function

$$\Gamma(t) = M(t_{i-1})\beta(t_{i-1})^{(t-t_{i-1})}|x(t_{i-1})| + \nu(t_{i-1}) \tag{12}$$

For each $i > 1$ such that $t_{i-1} \neq \infty$, the switching instant is defined as

$$t_i = \begin{cases} \min\{t : t > t_{i-1},\ |x(t)| > M(t_{i-1})\beta(t_{i-1})^{(t-t_i)}|x(t_{i-1})| + \nu(t_{i-1})\} & \text{if this exists} \\ \infty \quad \text{otherwise} & \end{cases} \tag{13}$$

where $0 < M(t)$, $0 < \beta(t) < 1$ and $0 < \nu(t)$ are strictly positive increasing functions satisfying the following conditions $\lim_{t\to\infty} M(t) = +\infty$, $\lim_{t\to\infty} \beta(t) = 1$, $\lim_{t\to\infty} \nu(t) = +\infty$. The behaviour of the closed loop system with $a_1 = -2.2$, $a_2 = 0.3$ and $b = 1$ is illustrated in Fig. 1(a)-(b). $\diamondsuit$

A different switching control approach, called hysteresis switching, was reported in a number of papers [22, 27, 37] in the context of adaptive control. In these papers, the hysteresis switching is used to swap between a number of "standard" adaptive controllers operating in regimes of the parameter space. The switching, in these cases, is used to avoid the "stabilizability" problem in adaptive controllers.

Conventional switching control techniques are all based on some mechanism of an exhaustive search over the entire set of potential controllers (either a continuum set [20] or a finite set [8]). A major drawback is that the search may converge very slowly, resulting in excessive transients which renders the system "unstable" in a practical sense. This phenomenon can take place even if the closed-loop system is exponentially stable. To alleviate this problem, several new switching control schemes have been proposed recently. The so-called supervisory control of LTI systems for adaptive set-point tracking is proposed by Morse [25, 26] to improve the transient response. An further extension of Morse's approach is given in [13]. A very similar, in spirit, supervisory control scheme for model reference adaptive control is analysed in [29]. The main idea of supervisory control is to orchestrate the process of switching into feedback controllers from a pre-computed finite (continuum) set of fixed controllers based on certain on-line estimation. This represents a significant departure from from traditional estimator based tuning algorithms which usually employ recursive or dynamic parameter tuning schemes. This approach has apparently significantly improved the quality of regulation, thus demonstrating that switching control if properly performed is no longer just a nice theoretical toy but a powerful tool for high performance control systems design. However, several issues still remain unresolved. For example,

(i) a finite convergence of switching is not guaranteed. This aspect is especially important in situations when convergence of switching is achievable. It seems intuitively that in adaptive control of a linear time invariant system it is desirable that the adaptive controller "converges" to a linear time invariant controller;

(ii) the analysis of the closed loop stability is quite complicated and often dependent on the system architecture. Without a simpler proof and better understanding of the "hidden" mechanisms of supervisory switching control its design will remain primarily a matter of trial and error.

In this chapter, we present a new approach to switching adaptive control for uncertain discrete time systems. This approach is based on a localization method, and is conceptually different from the supervisory control schemes and other switching schemes. The localization method was initially proposed by the authors for LTI systems [39]. This method has the unique feature of fast convergence for switching. That is, it can localize a suitable stabilizing controller very quickly, hence the name of localization. Later this method was extended to LTV plants in [40] . By utilizing the high speed of localization and the rate of admissible parameter variations exponential stability of the closed-loop system was proved. The main contribution of this chapter is a unified description of the method of localization. We show that this method is also easy to implement, has no bursting phenomenon, and can be modified to work with or without a known bound on the exogenous disturbance.

To highlight the principal differences between the proposed framework and existing switching control schemes, in particular, supervisory switching control, we outline potential advantages of localization based switching control:

(i) The switching controller is finitely convergent provided that the system is time-invariant.

Depending on how the switching controller is practically implemented the absence of this property could potentially have far reaching implications;

(ii) Unlike conventional switching control based on an exhaustive search over the parameter space, the switching converges rapidly thus guaranteeing a high quality of regulation;

(iii) The closed loop stability analysis is comparatively simple even in the case of linear time varying plants. This is in sharp contrast with supervisory switching control where the stability analysis is quite complicated and depends on the system architecture;

(iv) Localization based switching control is directly applicable to both linear time invariant and time varying systems;

(v) The localization technique provides a clear understanding of the control mechanism which is important in applications.

The rest of this chapter is organized as follows. Section 2 introduces the class of LTI systems to be controlled and states the switching adaptive stabilization problem. Two different localization principles are studied in Sections 3,4. We also study a problem of optimal localization, which allows us to obtain guaranteed lower bounds on the number of controllers discarded at each switching instant and adaptive stabilization in the presence of unknown exogenous disturbance. Simulation examples are given in Section 5 to demonstrate the fast switching capability of the localization method. Conclusions are reached in Section 6.

2 Problem Statement

We consider a general class of LTI discrete-time plants in the following form:

$$D(z^{-1})y(t) = N(z^{-1})u(t) + \xi(t-1) + \eta(t-1) \tag{14}$$

where $u(t)$ is the input, $y(t)$ is the output, $\xi(t)$ is the exogenous disturbance, $\eta(t)$ represents the unmodelled dynamics (to be specified later), z^{-1} is the unit delay operator,

$$N(z^{-1}) = n_1 z^{-1} + n_2 z^{-2} + \cdots + n_n z^{-n} \tag{15}$$

$$D(z^{-1}) = 1 + d_1 z^{-1} + \cdots + d_n z^{-n} \tag{16}$$

Remark 2.1 *By using simple algebraic manipulations, measurement noise and input disturbance are easily incorporated into the model (14). In this case, $y(t)$, $u(t)$, and $\xi(t)$*

represent the measured output, computed input and (generalized) exogenous disturbance, respectively. For example, if a linear time-invariant discrete-time plant is described by

$$y(z) = \frac{N(z^{-1})}{D(z^{-1})}(u(z) + d(z)) + q(z)$$

where $d(z)$ and $q(z)$ are the input disturbance and plant noise, respectively, the plant can be rewritten as

$$D(z^{-1})y(z) = N(z^{-1})u(z) + (N(z^{-1})d(z) + D(z^{-1})q(z^{-1}))$$

Consequently, the exogenous input $\xi(z)$ is $N(z^{-1})d(z) + D(z^{-1})q(z^{-1})$. $\diamondsuit$

We will denote by θ the vector of unknown parameters, i.e.,

$$\theta = (n_n, \cdots, n_2, d_n, \cdots, d_1, n_1)^T \tag{17}$$

Throughout the paper, we will use the following nonminimal state space description of the plant (14):

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + E\ (\xi(t) + \eta(t)) \tag{18}$$

where

$$x(t) = \left[\begin{array}{cc|cc} u(t-n+1)\cdots & u(t-1) & y(t-n+1) & \cdots y(t)) \end{array}\right]^T \tag{19}$$

and the matrices $A(\theta)$, $B(\theta)$ and E are constructed in a standard way

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots 0 & 0 & 0 & \cdots & 0 & 1 \\ n_n & n_{n-1} & \cdots & n_2 & -d_n & \cdots & -d_2 & -d_1 \end{bmatrix} \tag{20}$$

$$B(\theta) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline 0 \\ \vdots \\ 0 \\ n_1 \end{bmatrix}; \quad E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

We also define the regressor vector

$$\phi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{22}$$

Then, (14) can be rewritten as

$$y(t) = \theta^T \phi(t-1) + \xi(t-1) + \eta(t-1) \tag{23}$$

The following assumptions are used throughout this section:

(A1) The order n of the nominal plant (excluding the unmodelled dynamics) is known;

(A2) A compact set $\Omega \in \mathbf{R}^{2n}$, is known such that $\theta \in \Omega$;

(A3) The plant (14) without unmodelled dynamics (i.e., $\eta(t) \equiv 0$) is stabilizable over Ω. That is, for any $\theta \in \Omega$, there exists a linear time-invariant controller $C(z^{-1})$ such that the closed-loop system is exponentially stable;

(A4) The exogenous disturbance ξ is uniformly bounded, i.e., for all $t_0 \in \mathbf{N}$

$$\sup_{t \geq t_0} |\xi(t)| \leq \bar{\xi} \tag{24}$$

for some known constant $\bar{\xi}$;

(A5) The unmodelled dynamics is arbitrary subject to

$$|\eta(t)| \leq \bar{\eta}(t) = \epsilon \sup_{0 \leq k \leq t} \sigma^{t-k} ||x(k)|| \tag{25}$$

for some constants $\epsilon > 0$ and $0 \leq \sigma < 1$ which represent the "size" and "decay rate" of the unmodelled dynamics, respectively;

Remark 2.2 *Assumption (A1) can be relaxed to that only an upper bound $n_{\max}$ is known. Assumption (A4) will be used in Sections 3-4 and will be relaxed to allow $\bar{\xi}$ to be unknown in Sections 2.2.2 and 2.3.1 where an estimation scheme is given for $\bar{\xi}$.* $\diamondsuit$

Remark 2.3 *We note that the assumptions outlined above are quite standard and have been used in adaptive control to derive stability results for systems with unmodelled dynamics (see, e.g., [7, 16, 21, 30] for more details).* $\diamondsuit$

The switching controller to be designed will be of the following form:

$$u(t) = K_{i(t)} x(t) \tag{26}$$

where $K_{i(t)}$ is the control gain applied at time t, and $i(t)$ is the switching index at time t, taking value in a finite index set I. The objective of the control design is to determine the set of control gains

$$K_I = \{K_i, i \in I\} \tag{27}$$

and an on-line switching algorithm for $i(t)$ so that the closed-loop system will be "stable" in some sense.

We note that switching controllers can be classified according to the logic governing the process of switching. Here are some typical examples.

1. Conventional Switching Control

The switching index is defined as

$$i(t) = \begin{cases} i(t-1) & \text{if} \quad \mathbf{G}_t \leq 0 \\ i(t-1)+1 & \text{otherwise} \end{cases} \tag{28}$$

where $\mathbf{G}_t$ is some appropriately chosen performance index. This type of switching control is finitely convergent and based on an exhaustive search over the parameter space (see, for example, [8, 9]).

2. Supervisory Switching Control

The switching index is defined as

$$i(t) = \begin{cases} i(t-1) & \text{if} \quad t - s(t) < t_d \\ \arg\ \min_{i \in I} & |e_i(t)| \end{cases} \tag{29}$$

where $s(t)$ is the time of the most recent switching, t_d is a positive dwell time, and $e_i(t)$, $\forall i \in I$ is a weighted prediction error computed for the ith nominal system. This type of switching control has been extensively studied recently by a number of researchers (see, e.g., [25, 26]). The proof of the closed-loop stability in this case is not dependent on finite convergence of the switching process, furthermore, supervisory switching control is not finitely convergent in general.

3 Direct Localization Principle

The switching algorithms to be used in this section are based on a *localization* technique. This technique, originally used in [39] for LTI plants, allows us to falsify incorrect controllers very rapidly while guaranteeing exponential stability of the closed-loop system. In this section, we describe a direct localization principle (see, e.g. [40]) for LTI plants which is slightly different from [39] but is readily extended to LTV plants. The main idea behind this

principle consists of simultaneous falsification of potentially stabilizing controllers based explicitly on the model of the controlled plant. That implies the use of some effective mechanism of discarding controllers inconsistent with the measurements.
The specific notion of stability to be used in this section is described below:

Definition 3.1 *The system (14) satisfying (A1)-(A5) is said to be globally $\bar{\xi}$-exponentially stabilized by the controller (26) if there exist constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : R_+ \to R_+$ with $M_2(0) = 0$ such that*

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi}) \tag{30}$$

holds for all $t_0 \geq 0$, $x(t_0)$, $\bar{\xi} \geq 0$, and $\xi(\cdot)$ and $\eta(\cdot)$ satisfying (A4)-(A5), respectively.

The definition above yields exponential stability of the closed loop system provided that $\bar{\xi} = 0$ and exponential attraction of the states to an origin centred ball whose radius is related to the magnitude of the exogenous disturbance.
First, we decompose the parameter set Ω to obtain a *finite cover* $\{\Omega_i\}_{i=1}^{L}$ which satisfies the following conditions:

C1. $\Omega_i \subset \Omega$, $\Omega_i \neq \{\ \}$, $i = 1, \cdots, L$;

C2. $\bigcup_{i=1}^{L} \Omega_i = \Omega$;

C3. For each $i = 1, \cdots, L$, let θ_i and $r_i > 0$ denote the "centre" and "radius" of Ω_i, i.e., $\theta_i \in \Omega_i$ and $||\theta - \theta_i|| \leq r_i$ for all $\theta \in \Omega_i$. Then, there exist K_i, $i = 1, \cdots, L$, such that

$$|\lambda_{\max}(A(\theta) + B(\theta)K_i)| < 1, \ \forall \ \|\theta - \theta_i\| \leq r_i, \ i = 1, \cdots, L. \tag{31}$$

Conditions C1 - C2 basically say that the uncertainty set Ω is presented as a finite union of nonempty subsets while condition C3 defines each subset Ω_i as being stabilizable by a single LTI controller K_i. It is well-known that such a finite-cover can be found under assumptions (A1)-(A3) (see, e.g., [8, 24, 25] for technical details and examples). More specifically, there exist (sufficiently large) L, (sufficiently small) r_i, and suitable K_i, $i = 1, \cdots, L$, such that (C1)-(C3) hold. Leaving apart the computational aspects of decomposing the uncertainty set satisfying conditions C1 - C3 we just note that decomposition can be conducted off - line, moreover, some additional technical assumptions (see, for example, C3' below) make the process of decomposing pretty trivial. The computational complexity of decomposing the uncertainty set, in general, depends on many factors including the "size" of the set, its dimension and "stabilizability" properties, and has to be evaluated on a case by case basis.

The key observation used in the localization technique is the following fact: Given any parameter vector $\theta \in \Omega_j$ and a control gain $K_{i(t)}$ for some $i(t), j = 1, \cdots, L$. If $i(t) = j$, then it follows from

$$y(t) = \theta^T \phi(t-1) + \xi(t-1) + \eta(t-1) \tag{32}$$

that

$$|\theta_j^T \phi(t-1) - y(t)| \leq r_j ||\phi(t-1)|| + \bar{\xi} + \bar{\eta}(t-1) \tag{33}$$

This observation leads to a simple localization scheme by elimination: If the above inequality is violated at any time instant, we know that the switching index $i(t)$ is wrong (i.e., $i(t) \neq j$), so it can be eliminated. In identification theory this concept is sometimes referred to as falsification; see, e.g. a survey [15] and references therein. The unique feature of the localization technique comes from the fact that violation of (33) allows us not only to eliminate $i(t)$ from the set of possible controller indices, but many others. This is the key point! As a result, a correct controller can be found very quickly.

We now describe the localization algorithm. Let $I(t)$ denote the set of "admissible" control gain indices at time t and initialise it to be

$$I(t_0) = \{1, 2, \cdots, L\} \tag{34}$$

Choose any initial switching index $i(t_0) \in I(t_0)$. For $t > t_0$, define

$$\hat{I}(t) = \{j : (33) \text{ holds },\ j = 1, \cdots, L\} \tag{35}$$

Then, the localization algorithm is simply given by

$$I(t) = I(t-1) \bigcap \hat{I}(t), \quad \forall t > t_0 \tag{36}$$

The switching index is updated by taking[1]

$$i(t) = \begin{cases} i(t-1) & \text{if } t > t_0 \text{ and } i(t-1) \in I(t) \\ \text{any member of } I(t) & \text{otherwise} \end{cases} \tag{37}$$

A simple geometrical interpretation of localization algorithm (36) is given in Fig. 2. One possible way to view the localization technique is to interpret it as family set identification of a special type, that is, family set identification conducted on a finite set of elements. Interpreted in this way the localization technique represents a significant departure from traditional family set identification ideas. Either strip depicted in Fig. 2 contains only those

[1] In fact, we will see in Section 2.2.1 that there may be "clever" ways of selecting $i(t)$ when $i(t-1)$ is falsified.

elements which are consistent with the measurement of the input/output pair $\{y(t), u(t-1)\}$. The high falsifying capability of the proposed algorithm observed in simulations can informally be explained in the following way. Let the index $i(t)$ be falsified, then the discrete set of elements $\{\theta_i : i \in I(t)\}$ consistent with all the past measurements is separated from the point $\theta_{i(t)}$ by one of the hyperplanes

$$\theta_j^T \phi(t-1) = y(t) + r_j ||\phi(t-1)|| + \bar{\xi} + \bar{\eta}(t-1) \tag{38}$$

or

$$\theta_j^T \phi(t-1) = y(t) - r_j ||\phi(t-1)|| - \bar{\xi} - \bar{\eta}(t-1) \tag{39}$$

dividing the parameter space into two half-spaces. It is also clear that every element belonging to the half-space containing the point $\theta_{i(t)}$ is falsified by the algorithm of localization (36) at the switching instant t. We note that the rigorous analysis of the problem of optimal localization conducted in Section 2.2.1 allows us to derive a guaranteed lower bound on the number of controllers falsified at an arbitrary switching instant. A different non-identification based interpretation of localization can be given in terms of the prediction errors $e_j = |\theta_j^T \phi(t-1) - y(t)|$, $j = 1, 2, \cdots L$ computed for the entire set of "nominal" models. Thus, any model giving a large prediction error is falsified. The following technical lemma describes the main properties of the algorithm of localization (36).

Lemma 3.1 *Given the uncertain system (14) satisfying Assumptions (A1)-(A5), suppose the finite cover $\{\Omega_i\}_{i=1}^L$ of Ω satisfies Conditions (C1)-(C3). Then, the localization algorithm given in (34)-(37) applied to a LTI plant (14) possesses the following properties:*

(i) $I(t) \neq \{\ \}, \quad \forall t \geq t_0$;

(ii) *There exists a switching index $j \in I(t)$ for all $t \geq t_0$ such that the closed-loop system with $u(t) = K_j x(t)$ is globally exponentially stable.*

Proof. The proof is trivial: Suppose the parameter vector θ for the true plant is in Ω_j for some $j \in \{1, \cdots, L\}$. Then, the localization algorithm guarantees that $j \in \hat{I}(t)$ for all t. Hence, both (i) and (ii) hold. $\diamondsuit$

To guarantee exponential stability of the closed-loop system, we need a further property of the finite cover of Ω. To explain this, we first introduce the notion of quadratic stability [3].

Definition 3.2 *A given set of matrices $\{A(\theta) : \theta \in \Omega\}$ is called quadratically stable if there exist symmetric positive-definite matrices H, Q such that*

$$A^T(\theta) H A(\theta) - H \leq -Q, \quad \forall\ \theta \in \Omega \tag{40}$$

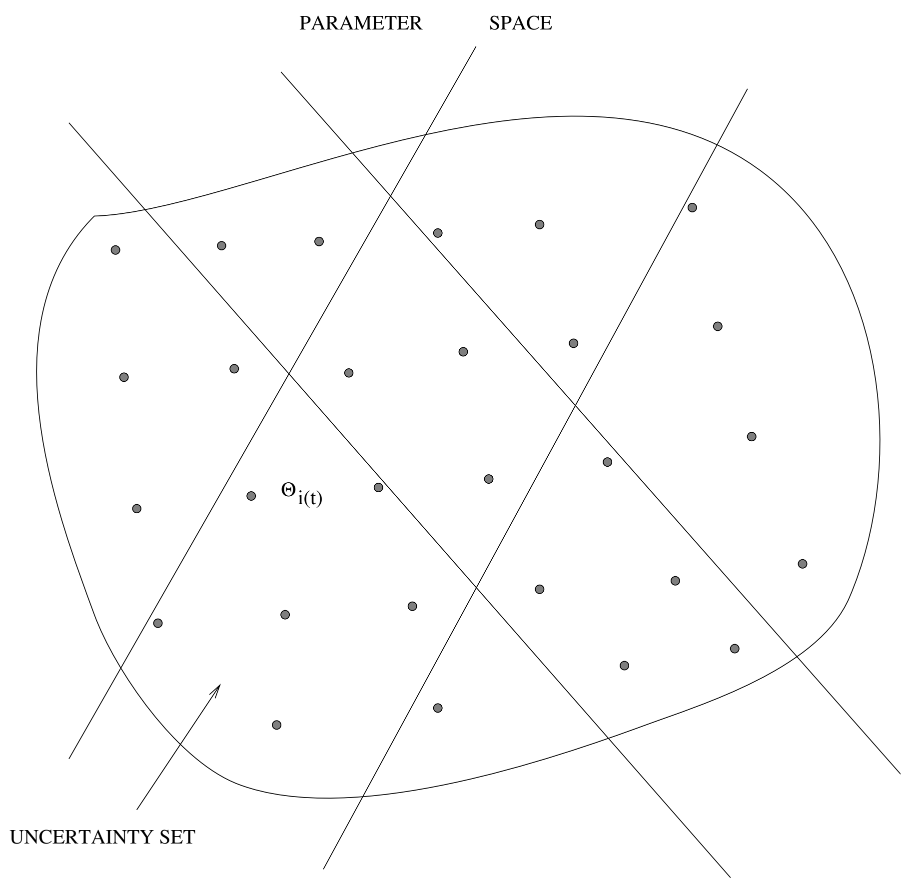

Figure 2: LOCALIZATION

It is obvious that the finite cover $\{\Omega\}_{i=1}^{L}$ of Ω can always be made such that each Ω_i is "small" enough for the corresponding family of the "closed-loop" matrices $\{A(\theta)+B(\theta)K_i : \theta \in \Omega_i\}$ to be quadratically stable with some K_i.

In view of the observation above, we replace the Condition (C3) with the following:

C3'. For each $i = 1, \cdots, L$, let θ_i and $r_i > 0$ denote the "centre" and "radius" of Ω_i, i.e., $\theta_i \in \Omega_i$ and $||\theta - \theta_i|| \leq r_i$ for all $\theta \in \Omega_i$. Then, there exist control gain matrices K_i, symmetric positive-definite matrices H_i and Q_i, $i = 1, \cdots, L$, and a positive number q such that

$$\begin{array}{c}(A(\theta) + B(\theta)K_i)^T H_i (A(\theta) + B(\theta)K_i) - H_i \leq -Q_i, \\ \forall\ \|\theta - \theta_i\| \leq (r_i + q), i = 1, \cdots, L\end{array} \tag{41}$$

Remark 3.1 *Condition C3' requires that every subset Ω_i obtained as a result of decomposition be quadratically stabilized by a single LTI controller. We also note that a finite cover which satisfies (C1)-(C2) and (C3') is guaranteed to exist. Moreover, Condition C3' translated as one requiring the existence of a common quadratic Lyapunov function for any subset Ω_i further facilitates the process of decomposition.* $\diamondsuit$

The following theorem contains the main result for the LTI case:

Theorem 3.1 *Given a LTI plant (14) satisfying Assumptions (A1)-(A5). Let $\{\Omega_i\}_{i=1}^{L}$ be a finite cover of Ω satisfying Conditions (C1)-(C2) and (C3'). Then, the localization algorithm given in (34)-(37) will guarantee the following properties when ϵ (i.e., the "size" of unmodelled dynamics) is sufficiently small:*

(i) *The closed-loop system is globally $\bar{\xi}$-exponentially stable, i.e., there exist constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$, $M_2(0) = 0$ such that*

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi}) \tag{42}$$

holds for all $t \geq t_0$ and $x(t_0)$;

(ii) *The switching sequence $\{i(t_0), i(t_0 + 1), \cdots\}$ is finitely convergent, i.e., $i(t) =$const, $\forall\ t \geq t^*$ for some t^*.*

Proof. See Appendix A. $\diamondsuit$

The proof of the theorem presented in Appendix A is based on the observation that between any two consecutive switchings the closed loop system behaves as an exponentially stable LTI system subject to small parametric perturbations and bounded exogenous disturbance. This is the key point offering a clear understanding of the control mechanisms.

It follows from the proof of Theorem 3.1 that the constant M_1 in the bound (42) is proportional to the total number of switchings made by the controller while the parameter ρ is dependent on the "stabilizability" property of the uncertainty set Ω. This further emphasizes the importance of fast switching capabilities of the controller for achieving good transient performance.

3.1 Optimal Localization

The localization scheme described above allows an arbitrary new switching index in $I(t)$ to be used when a switching occurs. That is, when the previous switching index $i(t-1)$ is eliminated from the current index set $I(t)$, any member of $I(t)$ can be used for $i(t)$. The problem of optimal localization addresses the issue of optimal selection of the new switching index at each switching instant so that the set of admissible switching indices $I(t)$ is guaranteed to be pruned down as rapidly as possible. The problem of optimal localization is solved in this section in terms of the indices of localization defined below. In the following for notational convenience we drop when possible the index t from the description of the set of indices $I(t)$. Also we make the technical assumption

(A7) $r_i = r_j = r, \ \forall i, j = 1, 2, \cdots, L,$ and some $r > 0$.

For any set $I \subset \{1, 2, \cdots, L\}$, $\Theta = \{\theta_i : i \in I\}$, a fixed $j \in I$ and any $z \neq 0$, $z \in \mathbf{R}^{2n}$, define the function

$$\lambda(z, j, \Theta) = |\{\theta_i : (\theta_i - \theta_j)^T z \geq 0, \ i \in I\}| \tag{43}$$

where $|\cdot|$ denotes the cardinal number of a set. Then

$$\mathrm{ind}(\theta_j, \Theta) = \min_{||z||=1} \lambda(z, j, \Theta) \tag{44}$$

will be referred to as the index of localization of the element θ_j with respect to the set Θ.

Lemma 3.2 *The index of localization* $\mathrm{ind}(\theta_j, \Theta)$ *represents a guaranteed lower bound on the number of indices discarded from the localization set* $I(t)$ *at the next switching instant provided that* $u(t) = K_j x(t)$.

Proof: Without loss of generality we assume that $(t+1)$ is the next switching instant, and controller K_j is discarded. From (35) we have $j \notin \hat{I}(t+1)$, equivalently,

$$\theta_j^T \phi(t) > y(t+1) + (r_j + q)||\phi(t)|| + \bar{\xi} + \bar{\eta}(t+1) \tag{45}$$

or

$$\theta_j^T \phi(t) < y(t+1) - (r_j + q)||\phi(t)|| - \bar{\xi} - \bar{\eta}(t+1) \tag{46}$$

Taking $z = -\phi(t)/||\phi(t)||$ for (45), or $z = \phi(t)/||\phi(t)||$ for (46) and using (43) we see that there are $\lambda(z, j, \Theta)$ number of controller indices which do not belong to $\hat{I}(t+1)$. We note that $\phi(t) \neq 0$, because otherwise it is easy to see from (23) that there exists no element $\theta_j \in \Theta$ satisfying (45) or (46), and, consequently, switching is not possible. Since $\text{ind}(\theta_j, \Theta) \leq \lambda(z, j, \Theta)$, we conclude that there are at least $\text{ind}(\theta_j, \Theta)$ number of controllers to be discarded at the switching instant $(t+1)$. $\diamondsuit$

In terms of (44) the index of localization of the discrete set Θ is defined as

$$\text{ind } \Theta = \max_j \{\text{ind}(\theta_j, \Theta) : \ j \in I\} \tag{47}$$

That is, ind Θ, is the largest attainable lower bound on the number of controllers eliminated at the time of switching, assuming that the regressor vector can take any value. The structure of an optimal switching controller is described by

$$u(t) = K_{i(t)} x(t) \ , \tag{48}$$

$$i(t) = \begin{cases} i(t-1) \ \text{ if } \ i(t-1) \in I(t); \\ i_{opt}(t) = \arg\max_j \{\text{ind}(\theta_j, \Theta(t)) : j \in I(t)\} \text{ otherwise} \end{cases} \tag{49}$$

The problem of optimal localization reduces to determining the optimal control law, that is, specifying the switching index $i_{opt}(t)$ at each time instant when switching has to be made. To solve this problem we introduce the notion of *separable sets.*

Definition 3.3 *Given a finite set $\Theta \subset \mathbf{R}^n$ and a subset $J \subset \Theta$; J is called a separable set of order k if*

(i) $|J| = k$;

(ii) $co \ \{J\} \bigcap co \ \{\Theta - J\} = \{\}$ *where $co \ \{\cdot\}$ stands for the convex hull of a set.*

The main properties of separable sets are listed below:

(a) A vertex of $co \ \{\Theta\}$ is a separable set of order 1;

(b) The order of a separable set $k \leq |\Theta|$;

(c) For each separable set J of order k, $k > 1$, there exists a set $J' \subset J$ such that J' is a separable set of order $(k-1)$.

Proof: (a), (b) are obvious. To prove (c), we note that for each separable set J, there exists a hyperplane $\mathbf{P}$ separating J and $\Theta - J$. Let $\vec{n}$ be the normal direction of $\mathbf{P}$. Move $\mathbf{P}$ along $\vec{n}$ towards J until it hits J. Two cases are possible.
Case 1: One vertex is in contact. In this case move $\mathbf{P}$ a bit further to pass the vertex. The remaining points in J form J'.
Case 2: Multiple vertices are in contact. One can always change $\vec{n}$ slightly so that $\mathbf{P}$ still separates J and $\Theta - J$, but there is only one vertex in contact with $\mathbf{P}$, and we are back to Case 1. $\diamondsuit$

Lemma 3.3 *Let $\mathbf{\Theta}^k$ be the set of all separable sets of order k and* $\Xi^k = \bigcup_{J_k \in \mathbf{\Theta}^k} J_k$. *Then,*

$$\text{ind } \Theta = 1 + \arg \max_k \{k : \Xi^k \neq \Theta\} \tag{50}$$

Proof: Follows immediately from Definition 3.3 and the property of separable sets (c). Indeed, suppose that the index of localization satisfies the relation

$$\text{ind } \Theta \; = m > 1 + \arg \max_k \{k : \Xi^k \neq \Theta\} \tag{51}$$

then there must exist an element $\theta_j \in \Theta$, such that $\text{ind}(\theta_j, \Theta) = m$, moreover,

$$\theta_j \notin \Xi^{m-1}, \; \theta_j \in \Theta - \Xi^{m-1} \tag{52}$$

since otherwise, by definition of separable sets $\text{ind}(\theta_j, \Theta) \leq m - 1$. But it follows from (51) that $\Theta - \Xi^{m-1} = \{\}$. On the other hand by Definition 3.3 and the properties of separable sets (b),(c) the index of localization of the set Θ can not be smaller than that given by (50). This concludes the proof. $\diamondsuit$
Denote by $V(\cdot)$ the set of vertices of $co\ (\cdot)$. The complete solution to the problem is given by the following iterative algorithm.
Algorithm A
Step 1. Initialize $k = 1$. Compute $\mathbf{\Theta}^1 = \{\{\theta\} : \theta \in V(\Theta)\}$
Step 2. Set $k = k + 1$. Compute
$\mathbf{\Theta}^k = \{J_{k-1} \bigcup \theta_i : J_{k-1} \in \mathbf{\Theta}^{k-1}, \; \theta_i \in V(\Theta - J_{k-1}), \; J_{k-1} \bigcup \theta_i \text{ is separable}\}$.
Step 3. If $\Xi^k = \Theta$, then ind $\Theta = k$, and stop, otherwise go to Step 2.
The properties of localization based switching control are summarized in the following theorem. Let sub$\{\cdot\}$ denote the set of subscripts of all the elements in $\{\cdot\}$.

Theorem 3.2

(i) *the solution to the problem of optimal localization may not be unique and is given by the set*

$$\mathbf{I}_{opt} = \text{sub}\{\Theta - \Xi^{m-1}\} \tag{53}$$

where

$$m = \text{ind } \Theta = 1 + \arg \max_k \{k : \Xi^k \neq \Theta\} \tag{54}$$

(ii) *for any $\bar{\xi} \geq 0,\ \epsilon \geq 0$, the total number of switchings l made by the optimal switching controller (48),(49) applied to LTI plant (14) satisfies the relation*

$$l \leq L - \sum_{p=0}^{l-1} \text{ind } \Theta(t_p) - 2 \tag{55}$$

where $t_p,\ p = 0, 1, \cdots, l-1$ denote the switching instants.

Proof. The proof of (i) follows directly from Lemma 3.3. To prove (ii) we note that

$|\Theta(t_1)| \leq L - \text{ind } \Theta(t_0) + 1;$

$|\Theta(t_2)| \leq |\Theta(t_1)| - \text{ind } \Theta(t_1) + 1 \leq L - \text{ind } \Theta(t_0) - \text{ind } \Theta(t_1) + 2;$

. . .

then

$|\Theta(t_l)| \leq \nu_l = L - \sum_{i=0}^{l-1} \text{ind } \Theta(t_i) + l - 1.$

Since $\nu_l \geq 1$ the result follows. $\diamondsuit$

Algorithm A applied to an arbitrary localization set Θ indicates that except for a very special case, namely, $\{\theta_j\}_{j \in I} = V(\Theta)$, localization with any choice of the switching index $i(t)$ such that $\theta_{i(t)} \notin V(\Theta)$ will always result in elimination of more than one controller at any switching instant. This is a remarkable feature distinguishing localization based switching controllers from conventional switching controllers. Moreover, a simple geometrical analysis (see, e.g., Fig. 2) indicates that for "nicely" shaped uncertainty sets (for example, a convex Ω) and large L the index of localization is typically large, that is, $\text{ind}(\theta) >> 1$. Theorem 3.2 gives a complete theoretical solution to the problem of optimal localization formulated above in terms of indices of localization. However, it must be pointed out that the search for optimality in general is involved and may be computationally demanding. To alleviate potential computational difficulties we propose one possible way of constructing a suboptimal switching controller.

Algorithm B
Step 1. Initialize $k = 1$. Compute $\mathbf{\Gamma}^1 = V(\Theta)$.
Step 2. Set $k = k + 1$. Compute

$$\mathbf{\Gamma}^k = \mathbf{\Gamma}^{k-1} \bigcup V(\theta - \mathbf{\Gamma}^{k-1}).$$

Step 3. If $\mathbf{\Gamma}^k = \{\}$, then ind $\Theta \geq k$, and stop, otherwise go to Step 2.

Algorithm B allows for a simple geometrical interpretation, namely, at each step a new set $\mathbf{\Gamma}^k$ is obtained recursively by adding the set of vertices of $(\Theta - \mathbf{\Gamma}^{k-1})$. The simplicity of the proposed algorithm is explained by the fact that we no longer need to check the property of separability (see Step 2 in Algorithm A).
The main property of the Algorithm B is presented in the following proposition.

Proposition 3.3 *The index of localization ind* Θ *satisfies the inequality*

$$\text{ind } \Theta \geq 1 + \arg \max_k \{k : \mathbf{\Gamma}^k \neq \Theta\} \tag{56}$$

Proof: The proof is very simple and follows from the fact that for any $\theta \in \Theta$, such that $\theta \notin V(\Theta)$ it is true that ind $(\theta, \Theta) \geq 2$. By applying this rule recursively we obtain (56). $\diamondsuit$

Example 3.1 To illustrate the idea of optimal (suboptimal) localization we consider a simple localization set $\Theta = \{\theta_j\}_{j=1}^5$ depicted in Fig. 3.
We note that the point θ_5 is located exactly in the centre of the square $(\theta_1,\ \theta_2,\ \theta_4,\ \theta_3)$. Applying Algorithm A to the set Θ we obtain

$$\begin{aligned}
\mathbf{\Theta}^1 &= \{\{\theta_1\},\ \{\theta_2\},\ \{\theta_3\},\ \{\theta_4\}\},\\
\mathbf{\Theta}^2 &= \{\{\theta_1,\ \theta_2\}, \{\theta_1,\ \theta_3\}, \{\theta_2,\ \theta_4\}, \{\theta_3,\ \theta_4\}\},\\
\mathbf{\Theta}^3 &= \{\{\theta_1,\ \theta_2,\ \theta_5\}, \{\theta_1,\ \theta_3,\ \theta_5\}, \{\theta_3,\ \theta_4,\ \theta_5\}, \{\theta_2,\ \theta_4,\ \theta_5\}\}
\end{aligned}$$

Since $\bigcup_{J \in \mathbf{\Theta}^3} J = \Theta$ we conclude that ind $\Theta = 3$ and the optimal switching index is given by $i(t) = 5$. To compute a guaranteed lower bound on the index of localizaton ind Θ Algorithm B is used. We have

$$\begin{aligned}
\mathbf{\Gamma}^1 &= \{\theta_1,\ \theta_2,\ \theta_3,\ \theta_4\},\\
\mathbf{\Gamma}^2 &= \{\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5\} = \Theta,
\end{aligned}$$

therefore, ind $\Theta \geq 2$. We note that in this particular example the optimal solution, that is, $i(t) = 5$ coincides with the suboptimal one. $\diamondsuit$

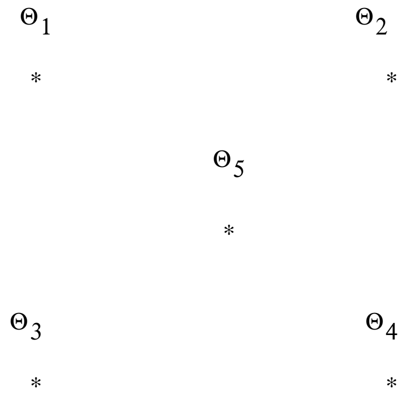

Figure 3: Example of Optimal Localization

Remark 3.2 *To deal with the problem of optimal (suboptimal) localization different simple heuristic procedures can be envisioned. For example, the following "geometric mean" algorithm of computing a new switching index is likely to perform well in practice, though it is quite difficult in general to obtain any guaranteed lower bounds on the indices of localization. At any switching instant t we choose*

$$i(t) = \arg \min_j \ ||\theta_j - \sum_{i \in I(t)} \theta_i / |\Theta(t)| \ ||. \tag{57}$$

♢

3.2 Localization in the Presence of Unknown Disturbance Bound

In this section we further relax Assumption (A4) to allow the disturbance bound $\bar{\xi}$ to be unknown. That is, we replace (A4) with

(A4') The exogenous disturbance ξ is uniformly bounded,

$$\sup_{t \geq t_0} |\xi(t)| \leq \bar{\xi} \tag{58}$$

for some *unknown* constant $\bar{\xi}$.

We further relax assumptions (A1)-(A5) by allowing parameters to be slowly varying. To this end we introduce the following additional assumption

(A7) The uncertain parameters are allowed to have slow drifting described by

$$||\theta(t) - \theta(t-1)|| \leq \alpha, \ \forall \ t > t_0 \tag{59}$$

for some constant $\alpha > 0$.

Following the results presented in previous sections, we introduce a generalized localization algorithm to tackle the new difficulty. The key feature of the algorithm is the use of an on-line estimate of $\bar{\xi}$. This estimate starts with a small (or zero) initial value, and is gradually increased when it is invalidated by the observations of the output. With the tradeoff between a larger number of switchings and a higher complexity, the new localization algorithm guarantees qualitatively similar properties for the closed-loop system as for the case of known disturbance bound.
Let $\bar{\xi}(t)$ be the estimate for $\bar{\xi}$ at time t. Define

$$\hat{I}(t,\bar{\xi}(t)) = \{j : |\theta_j^T\phi(t-1)-y(t)| \le (r_i+q)\|\phi(t-1)\|+\bar{\xi}(t-1)+\bar{\eta}(t-1),\ j=1,\cdots,L\} \quad (60)$$

That is, $\hat{I}(t,\bar{\xi}(t))$ is the index set of parameter subsets which can not be falsified by any exogenous disturbance $\sup_{t\ge t_0}|\xi(t)| \le \bar{\xi}(t-1)$.
Denote the most recent switching instant by $s(t)$. We define $s(t)$ and $\bar{\xi}(t)$ as follows:

$$\begin{aligned}
s(t_0) &= t_0 && (61)\\
\bar{\xi}(t_0) &= 0 && (62)\\
s(t) &= \begin{cases} t & \text{if } \cap_{k=s(t-1)}^{t}\hat{I}(k,\bar{\xi}(k)) = \{\} \text{ and } t-s(t)\ge t_d \\ s(t-1) & \text{otherwise}\end{cases} && (63)\\
\bar{\xi}(t) &= \begin{cases} \bar{\xi}(t-1)+\delta(t)\mu & \text{if } \cap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)) = \{\} \text{ and } t-s(t) < t_d \\ \bar{\xi}(t-1) & \text{otherwise}\end{cases} && (64)
\end{aligned}$$

where t_d is some positive integer representing a length of a moving time interval over which validation of a new estimate $\bar{\xi}(t)$ is conducted ; μ is any small positive constant representing a steady state residual (to be clarified later), and $\delta(t)$ is an integer function defined as follows:

$$\delta(t) = \min\left\{\delta :\ \cap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)+\delta\mu) \ne \{\}, \delta\in\mathbf{N}\right\} \quad (65)$$

The main idea behind the estimation scheme presented above is as follows. At each time instant when the estimate $\bar{\xi}(t-1)$ is invalidated, that is, $\cap_{k=s(t)}^{t}\hat{I}(k,\bar{\xi}(k-1)) = \{\}$ we determine the least possible value $\delta\in\mathbf{N}$ which guarantees that no exogenous disturbance $\sup_{t\ge t_0}|\xi(t)| \le (\bar{\xi}(t-1)+\delta\mu)$ would have caused the falsification of all the indices in the current localization set. This is done by recomputing the sequence of localization sets over the finite period of time $[s(t),t]$ whose length is bounded from above by t_d. Since the total number of switchings caused by the "wrong" estimate $\bar{\xi}(t)$ is finite and for every sufficiently large interval of time the number of switchings due to slow parameter drifting can be made arbitrary small by decreasing the rate of slow parameter drifting α it is always possible to choose a sufficiently large t_d which would guarantee global stability of the system.

The algorithm of localization is modified as follows:

$$I(t) = \cap_{k=s(t-1)}^{t} \hat{I}(k, \bar{\xi}(k)) \tag{66}$$

But the switching index $i(t)$ is still defined as in (37).

The key properties of the algorithm above are given as follows:

Theorem 3.4 *For any constant $\mu > 0$, there exist a parameter drifting bound $\alpha > 0$, a "size" of unmodelled dynamics $\epsilon > 0$ (both sufficiently small), and an integer t_d (sufficiently large), such that the localization algorithm described above, when applied to the plant (14) with Assumptions (A1)-(A3), (A4') and (A7), possesses the following properties:*

(1) $I(t) \neq \{\}$ *for all* $t \geq t_0$;

(2) $\sup_{t \geq t_0} \bar{\xi}(t) \leq \bar{\xi} + \mu$.

Subsequently, the following properties hold:

(3) *The closed-loop system is globally $(\bar{\xi} + \mu)$-exponentially stable, i.e., there exists constants $M_1 > 0$, $0 < \rho < 1$, and a function $M_2(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ with $M_2(0) = 0$ such that*

$$||x(t)|| \leq M_1 \rho^{(t-t_0)} ||x(t_0)|| + M_2(\bar{\xi} + \mu) \tag{67}$$

holds for all $t \geq t_0$ and $x(t_0)$;

(4) *The switching sequence $\{i(t_0), i(t_0 + 1), \cdots\}$ is finitely convergent, i.e., $i(t) =$const, $\forall\ t \geq t^*$ for some t^* if the uncertain parameters are constant.*

Proof. See Appendix B. $\diamondsuit$

We note that even though the value μ can be arbitrarily chosen, the estimate of the disturbance bound, $\bar{\xi}(t)$, can theoretically be larger than $\bar{\xi}$ by the margin μ. Consequently, the state is only guaranteed to converge to a residual set slightly larger than what is given in Theorem 3.4. Our simulation results indicate that $\bar{\xi}(t)$ very likely converges to a value substantially smaller than $\bar{\xi}$. Nevertheless, there are cases where $\bar{\xi}(t)$ exceeds $\bar{\xi}$. One possible solution is to reduce the value of μ. However, a small μ may imply a large number of potential switchings.

4 Indirect localization Principle

The idea of indirect localization was first proposed in [39] and is based on the use of a specially constructed performance criterion as opposed to direct localization considered in the previous section. To this end the output of the plant is replaced by some auxiliary output observation of which is subsequently used for the purpose of model falsification. The notion of "stabilizing sets" introduced below is central in the proposed indirect localization scheme.

We first define an auxiliary output, $z(t)$, as

$$z(t) = Cx(t), \quad C^T \in \mathbf{R}^{2n-1}, \tag{68}$$

and the inclusion:

$$\mathcal{I}_t : \quad |z(t)| \leq \Delta \|x(t-1)\| + c_0 \tag{69}$$

Definition 4.1 *$\mathcal{I}_t$ is said to be a stabilizing inclusion of the system (18) if $\mathcal{I}_t$ being satisfied for all $t > t_0$ and boundedness of $\xi(t), (\xi(t) \in \ell_\infty)$, implies boundedness of the state, $x(t)$, and in particular, there exist α_0, β_0 and $\sigma \in (0,1)$ such that:*

$$\|x(t)\| \leq \alpha_0 \sigma^{t-t_0} \|x(t_0)\| + \beta_0 \|\xi_t\|_{\ell_\infty}$$

Remark 4.1 *Note that the inclusion, $\mathcal{I}_t$ is transformed into a discrete-time sliding hyperplane [11] as $\Delta \to 0, c_0 \to 0$. In contrast with conventional discrete-time sliding mode control we explicitly define an admissible vicinity around the sliding hyperplane by specifying the values $\Delta \geq 0$ and $c_0 \geq 0$.* $\diamondsuit$

Definition 4.2 *The uncertain system (18) is said to be globally (C, Δ)-stabilizable if*

1. *$\mathcal{I}_t$ is a stabilizing inclusion of the system (18), and*
2. *there exists a control, $u(t) = -Kx(t)$, such that after a finite time, $\mathcal{I}_t$ is satisfied.*

We will show below that stabilizing sets can be effectively used in the process of localization. Before we proceed further we need some preliminary results. Assume for simplicity that $\eta(t) \equiv 0$. The case $\eta(t) \neq 0$ is analysed similarly, provided ϵ is sufficiently small.

Lemma 4.1 *Let $\sup_{t \geq t_0} |\xi(t)| < \infty$, $CB > 0$. Then there exists a c_0 such that the system (18) is globally $(C, 0)$-stabilizable if and only if*

$$|\lambda_{\max}(PA)| < 1 \tag{70}$$

where

$$P = \Big(I - (CB)^{-1} BC\Big) \tag{71}$$

Proof: First, suppose that (70) is violated, that is

$$|\lambda_{\max}(PA)| \geq 1 \tag{72}$$

We now show that $\mathcal{I}_t$ is not a stabilizing inclusion for any $c_0 > 0$. To do this, we take $\xi(t) \equiv 0$, and $u(t) = -(CB)^{-1}CAx(t)$. With this control we note from (18) that $z(t) = 0$ for $t > 0$, and so for any $c_0 > 0, z(t)$ satisfies (69). The equation for the closed-loop system takes the form

$$x(t+1) = Ax(t) + Bu(t) = PAx(t) \tag{73}$$

which is not exponentially stable. Therefore, (72) implies that there is no c_0 such that $\mathcal{I}_t$ is a stabilizing inclusion. We now establish the converse. Suppose (70) is satisfied. Then we can rewrite (18) as:

$$\begin{aligned} x(t+1) &= PAx(t) + Bu(t) + \frac{1}{CB}(BCA)x(t) + E\xi(t) \\ &= PAx(t) + \frac{B}{CB}(z(t+1) - CE\xi(t)) + E\xi(t) \end{aligned} \tag{74}$$

From (74) it is clear that if z and ξ are bounded, then in view of (70), $x(t)$ is bounded. Therefore, $\mathcal{I}_t$ is a stabilizing inclusion for any c_0. Finally, we take the control

$$u(t) = -\frac{1}{CB}(CA)x(t) \tag{75}$$

which gives

$$z(t+1) = CE\xi(t) \tag{76}$$

Therefore, for $c_0 \geq |CE| \sup_t |\xi(t)|,\ \mathcal{I}_t$ is satisfied for all $t > 0$, and the proof is complete. $\diamondsuit$

Remark 4.2 *The control, (75), is a 'one step ahead' control on the auxiliary output, $z(t)$. It then follows that the stability condition (70), (71) is equivalent to the condition that $C(zI - A)^{-1}B$ be relative degree 1, and minimum phase.* $\diamondsuit$

Remark 4.3 *If the original plant transfer function, (14), is known to be minimum phase, and relative degree 1 then it suffices to take $C = E^T$, and the system is then c_0 stabilizable for any $c_0 \geq 0$.*
If the original plant transfer function is nonminimum phase, then let:

$$C = [f_0, f_1 \ \cdots \ f_{n-2}, g_0, g_1 \ \cdots \ g_{n-1}] \tag{77}$$

The transfer function from $u(t)$, via (14) to $z(t)$ is then:

$$\begin{aligned} z(t) &= F(q)u(t) + G(q)y(t) \\ &= \left(\frac{D(q)F(q) + G(q)N(q)}{D(q)} \right) u(t) \end{aligned} \tag{78}$$

where $F(q) = (f_0 + f_1 q + \cdots + f_{n-2}q^{n-2})$ and $G(q) = (g_0 + g_1 q + \cdots + g_{n-1}q^{n-1})$. Therefore, for a nonminimum phase plant, knowledge of a C such that $\mathcal{I}_t$ is a stabilizing inclusion is equivalent to knowledge of a (possible improper) controller $\{u(t) = -G(q)/F(q)y(t)\}$ which stabilizes the system. Because we are dealing with discrete time systems, it is not clear whether this corresponds to knowledge of a proper, stabilizing controller for the set. $\diamondsuit$

Remark 4.4 *Because of the robustness properties of exponentially stable linear time invariant systems, Lemma 4.1 can easily be generalized to include non-zero, but sufficiently small Δ.* $\diamondsuit$

Lemma 4.2 *Any Ω which satisfies Assumptions (A2) and (A3) has a finite decomposition into compact sets:*

$$\Omega = \bigcup_{\ell=1}^{L} \Omega^{\ell} \tag{79}$$

such that for each ℓ, there exists a C_ℓ, Δ_ℓ and $c_{0,\ell}$ such that, for all $(A, B) \in \Omega^\ell, \mathcal{I}_t$ is a stabilizing inclusion, and $C_\ell B$ has constant sign.

Proof: (Outline)
It is well known that (see, for example, [8]) that Ω has a finite decomposition into sets stabilized by a fixed controller. From Remark 4.3, the requirements for knowledge of a C_ℓ such that $\mathcal{I}(C_\ell, \cdot, \cdot)$ is a stabilizing set on Ω are less stringent than knowledge of a stabilizing controller for the set Ω^ℓ. $\diamondsuit$
We now introduce our control method, including the method of localization for determining which controller to use. The first case we consider, is the simplest case where there is a single set to consider.
Case 1: $L = 1$ (sign of CB known)
This case covers a class of minimum phase plants, plus also certain classes of nonminimum phase plants.
For $L = 1$ we have:

$$\Omega = \Omega^1 = \bigcup_{i=1}^{s} \Omega_i \tag{80}$$

For $i = 1 \dots s$ we define a control law:

$$u(t)^i = -K_i x(t) \triangleq -\frac{1}{CB_i} CA_i x(t) \tag{81}$$

where the plant model, A_i, B_i is in the set Ω_i. We require knowledge of a Δ such that:

$$\|C(A - A_i\left(\frac{CB}{CB_i}\right))\| \leq \Delta;\ \forall i, \forall (A,B) \in \Omega_i \tag{82}$$

and $\mathcal{I}_t$ is a stabilizing inclusion on Ω_i for all i. Note that for any bounded Ω, for which we can find a single C which gives $C(zI - A)^{-1}B$ minimum phase and relative degree 1 we can always find, for s large enough, a Δ with the required properties (see, for example, [8]).

At any time $t > 0$, the auxiliary output $z(t+1)^i$ which would have resulted if we applied $u(t)^i = -K_i x(t)$ to the true plant is, using (18),

$$\begin{aligned} z(t+1)^i &\triangleq CAx(t) + CBu(t)^i + CE\xi(t) \\ &= z(t+1) - CB\left(u(t) - u(t)^i\right) \end{aligned} \tag{83}$$

Note that if the true plant is in the set Ω_i, then from (83) and (81)

$$z(t+1)^i = C\left(A - A_i\left(\frac{CB}{CB_i}\right)\right)x(t) + CE\xi(t) \tag{84}$$

and therefore, if the true plant is in Ω_i, then from (82), and with $c_0 = |CE\bar{\xi}|$

$$|z(t+1)^i| \leq \Delta\|x(t)\| + c_0 \tag{85}$$

Our proposed control algorithm for Case 1 is as follows (where, without loss of generality, we take $CB > 0$).

Algorithm C

1.1 Initialisation

Define

$$S_0 = \{1, 2, \cdots s\} \tag{86}$$

1.2 If $t > 0$,

If $z(t) > \Delta|x(t-1)| + c_0$ then set $S_t = S_{t-1} - \{k,\ \dots\ j_{s-1},\ j_s\}$

If $-z(t) > \Delta|x(t-1)| + c_0$ then set $S_t = S_{t-1} - \{j_1, j_2,\ \dots\ , k\}$

otherwise, $S_t = S_{t-1}$.

where $k, j_1\ \dots\ j_s$ and s are integers from the previous time instant (see 1.4, 1.5).

1.3 If $t > 0$,

For all $i \in S_t$, compute $u(t)^i$ as in (81).

1.4 Order $u(t)^i, \ i \in S_t$ such that:

$$u(t)^{j_1} \leq u(t)^{j_2} \leq \ \dots \ \leq u(t)^{j_s} \tag{87}$$

1.5 Apply the "median" control:

$$u(t) = u(t)^k \tag{88}$$

where $k = j_{\lfloor s/2 \rfloor}$,

1.6 Then wait for the next sample and return to 1.2.

We then have the following stability result for this control algorithm.

Theorem 4.1 *The control algorithm, (86) - (88), applied to a plant where C is known, and where the decomposition (80) has the properties that (82) is satisfied and $\mathcal{I}_t$ is a stabilizing inclusion, has the following properties:*

(a) *The inclusion:*

$$\mathcal{I}_t : |z(t)| \leq \Delta \|x(t-1)\| + c_0 \tag{89}$$

is violated no more than $N = \lfloor \log_2(s) \rfloor$ times, and

(b) *All signals in the closed loop system are bounded. In particular, there exist constants $\alpha, \beta < \infty, \sigma \in (0,1)$ such that all trajectories satisfy, for any $t_0, T > 0$,*

$$\|x(t_0 + T)\| \leq \alpha \sigma^T \|x(t_0)\| + \beta \tag{90}$$

Proof:

(a) Suppose at time $(t+1)$, (89) is violated. This can occur in one of two ways which we consider separately:

(i)

$$z(t+1) = z(t+1)^{i_t} > \Delta \|x(t)\| + c_0 \tag{91}$$

In this case, because of the ordering of $u(t)^i$ in (87), and the definition of $z(t+1)^i$ in (83), then

$$z(t+1)^i > \Delta \|x(t)\| + c_0 \tag{92}$$

for all $i \in \{k, \ \cdots \ j_{s-1}, j_s\}$

(ii)

$$z(t+1) = z(t+1)^{i_t} < -\left(\Delta\|x(t)\| + c_0\right) \tag{93}$$

In this case

$$z(t+1)^i < -\left(\Delta\|x(t)\| + c_0\right) \tag{94}$$

for all $i = \{j_1, j_2, \cdots k\}$. In either case, we see that if (91) is violated at time t, then

$$s_{t+1} \leq \frac{1}{2} s_t \tag{95}$$

from which the result follows.

(b) Firstly, we note that the control is well defined, that is, S_t is never empty. This follows since there is at least one index, namely the index of the set Ω_i which contains the true plant, which is always an element of S_t.

Next, we note that although we cannot guarantee that we converge to the correct control, from (a) we know (85) is satisfied all but a finite number of times.

Since $\mathcal{I}_T$ is a stabilizing inclusion, then by definition the states and all signals will be bounded.

Furthermore, since $\mathcal{I}_t$ is a stabilizing inclusion, there exist α_0, β_0 and $\sigma \in (0,1)$ such that if the inclusion, (89) is satisfied, for $t \in [t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \leq \alpha_0 \sigma^T \|x(t_0)\| + \beta_0 \tag{96}$$

(Note that if this is not the case, then from the definition, $\mathcal{I}_t$ is not a stabilizing inclusion). Also, there exist $\bar{\alpha}$ and $\bar{\beta}$ such that when (89) is violated:

$$\|x(t+1)\| \leq \bar{\alpha}\|x(t)\| + \bar{\beta} \tag{97}$$

If we define $\alpha_1 = \frac{\alpha_0^2 \bar{\alpha}}{\sigma}$ and $\beta_1 = (\alpha_0 \bar{\alpha} \beta_0 + \beta_0 + \alpha_0 \bar{\beta})$, then after some algebraic manipulations we can show that for any $t_0, T > 0$ such that (89) is violated not more than once in the interval, $(t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \leq \alpha_1 \sigma^T \|x(t_0)\| + \beta_1 \tag{98}$$

Also, we can show that with $\alpha_2 = \frac{\bar{\alpha}\alpha_0\alpha_1}{\sigma} = \frac{\bar{\alpha}\alpha_0^3}{\sigma^2}$, and $\beta_2 = \beta_0 + \bar{\beta} + \alpha_0\bar{\alpha}\beta_1 = (1 + \alpha_0\bar{\alpha} + (\alpha_0\bar{\alpha})^2)\beta_0(1+\alpha_0\bar{\alpha})\bar{\beta}$, provided (89) is not violated more than twice in the interval $[t_0, t_0+T)$, then

$$\|x(t_0 + T)\| \leq \alpha_2 \sigma^T \|x(t_0)\| + \beta_2 \tag{99}$$

Repeating this style of argument leads to the conclusion that with

$$\alpha_N = (\bar{\alpha}\alpha_0)\left(\frac{\alpha_0}{\sigma}\right)^N, \ \beta_N = (\alpha_0\bar{\alpha})^N\beta_0 + \left[\frac{(\alpha_0\bar{\alpha})^N - 1}{(\alpha_0\bar{\alpha}) - 1}\right](\beta_0 + \bar{\beta})$$

then if there are not more than N switches in $[t_0, t_0 + T)$, then

$$\|x(t_0 + T)\| \leq \alpha_N \sigma^T \|x(t_0)\| + \beta_N \tag{100}$$

The desired result follows from (a) since we know that there are at most $N = \lfloor \log_2(s) \rfloor$ times at which (89) is violated. $\diamondsuit$

Case 2: $L > 1$

Suppose that we do not know a single C such that $\mathcal{I}_t$ is a stabilizing inclusion, and CB is of known sign, then using finite covering ideas [8], as in Remark 4.3 let

$$\Omega = \bigcup_{\ell=1}^{L} \Omega^\ell = \bigcup_{\ell=1}^{L} \bigcup_{m=1}^{s^\ell} \Omega_m^\ell \tag{101}$$

where for each ℓ, we know $C_\ell, \Delta_\ell, c_0^\ell$ such that $\mathcal{I}_t$ is a stabilizing inclusion on Ω^ℓ and the sign of $(C_\ell B)$ is constant for all plants in Ω^ℓ.

At this point, one might be tempted to apply localization, as previously defined, on the sets Ω^ℓ individually and switch from Ω^ℓ should the set of valid indices, S^ℓ, become empty. Unfortunately, this procedure cannot be guaranteed to work. In particular, if Ω^ℓ does not contain the true plant, $\mathcal{I}_t$ needs not be a stabilizing inclusion, and so divergence of the states may occur without violating (89). To alleviate this problem, we use the exponential stability result, (90), in our subsequent development.

Algorithm D

We initialise $t(i) = 0$, $R_0 = \{1, 2, \ \ldots \ L\}$ and take any $\ell_0 \in R_0$.

We then perform localization on Ω^ℓ, with the following additional[2] steps:

If at any time

$$\|x(t)\| > \alpha\sigma^{t-t(i)}\|x(t(i))\| + \beta \tag{102}$$

(where α, σ, β are the appropriate constants for Ω^ℓ from Theorem 4.1), then we set $S^\ell = \{\}$.

If at any time t, S^ℓ becomes empty, we set $R_t = R_{t-1} - \{\ell\}, t(i) = t$, and we take a new ℓ from R_t.

With these modifications, it is clear that Theorem 4.1 can be extended to cover this case as well:

[2]In fact, we can localize simultaneously within other $\Omega^i, i \neq \ell$, however for simplicity and brevity we analyse only the case where we localize in one set at a time.

Corollary 4.1 *The control algorithm (86)-(88) with the above modifications applied to a plant with decomposition as in (101) satisfies:*

(a) There are no more than: $L-1+\sum_{\ell=1}^{L}\lfloor\log_2(s_\ell)\rfloor$ *instants such that*

$$|z(t+1)^{\ell_t}| \geq \Delta_{\ell_t}\|x(t)\| + c_{0,\ell_t} \tag{103}$$

(where ℓ_t *denotes the value of* ℓ *at time* t*),*

(b) All signals in the closed loop are bounded. In particular, there exist constants $\bar{\alpha}, \bar{\beta} < \infty, \bar{\sigma} \in (0,1)$ *such that for any* $t_0, T > 0$

$$\|x(t_0+T)\| \leq \bar{\alpha}\bar{\sigma}^T\|x(t_0)\| + \bar{\beta} \tag{104}$$

Proof: Follows from Theorem 4.1. $\diamondsuit$

4.1 Localization in the presence of unknown disturbance

In the previous section the problem of indirect localization based switching control for linear uncertain plants was considered assuming that the level of the generalized exogenous disturbance $\xi(t)$ was known. This is equivalent to knowing some upper bound on $\xi(t)$. The flexibility of the proposed adaptive scheme allows for simple extension covering the case of exogenous disturbances of unknown magnitude. This can be done in the way similar to that considered in Section 2.2.2. Omitting the details we just make the following useful observation. The control law described by Algorithms C, D is well defined, that is, $R_t \neq \{\}$ for all $t \geq t_0$ if $c_0^\ell \geq \sup_{t\geq t_0}|C_\ell E\xi(t)|,\ \forall\ell = 1,\ \ldots\ L$. This is the key point allowing us to construct an algorithm of on-line identification of the parameters $c_0^\ell, \ell = 1,\ \ldots\ L$.

5 Simulation Examples

Extensive simulations conducted for a wide range of LTI, LTV and nonlinear systems demonstrate the rapid falsification capabilities of the proposed method. We summarize some interesting features of the localization technique observed in simulations which are of great practical importance.

(i) falsification capabilities of the algorithm of localization do not appear to be sensitive to the switching index update rule. One potential implication of this observation is as follows. If not otherwise specified any choice of a new switching index is admissible and will most likely lead to good transient performance;

(ii) the speed of localization does not appear to be closely related to the total number of fixed controllers obtained as a result of decomposition. The practical implication of this observation (combined with the quadratic stability assumption) is that decomposition of the uncertainty set Ω can be conducted in a straightforward way employing, for example, a uniform lattice which produces subsets Ω_i, $i = 1, 2, \cdots L$ of an equal size

Example 5.1 Consider the following family of unstable (possibly nonminimum phase) LTV plants:

$$y(t) = 1.2y(t-1) - 1.22y(t-2) + b_1(t)u(t-1) + b_2(t)u(t-2) + \xi(t) \tag{105}$$

where the exogenous disturbance $\xi(t)$ is uniformly distributed on the interval $[-0.1,\ 0.1]$, and $b_1(t)$ and $b_2(t)$ are uncertain parameters. We deal with two cases which correspond to constant parameters and large-size jumps in the values of the parameters.

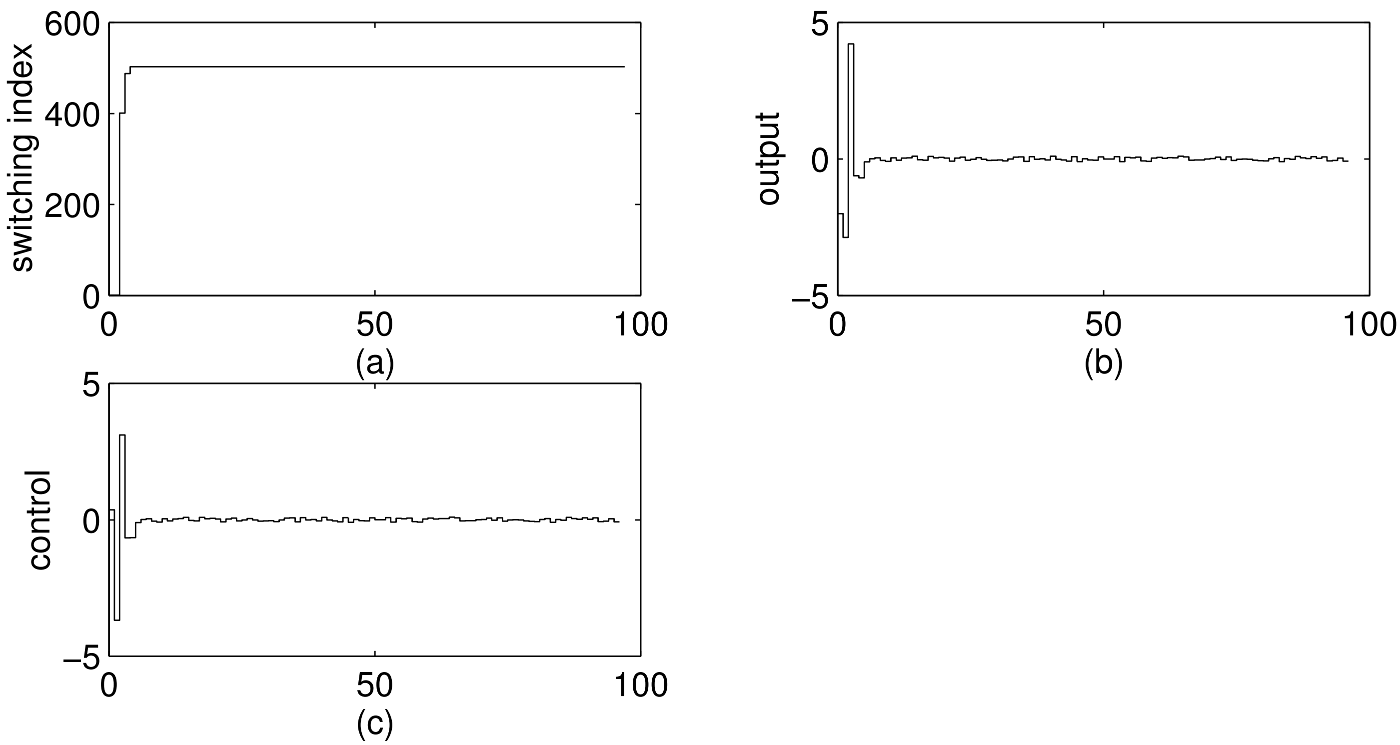

Figure 4: Example of Localization: Constant Parameters

Case 1: Constant Parameters

The *a priori* uncertainty bounds are given by

$$b_1(t) \in [-1.6,\ -0.15] \cup [0.15,\ 1.6], \qquad b_2(t) \in [-2,\ -1] \cup [1,\ 2] \tag{106}$$

i.e., $\Omega = \{[-1.6,\ -0.15] \cup [0.15,\ 1.6] \times [-2,\ -1] \cup [1,\ 2]\}$. To meet the requirements of the localization technique, we decompose Ω into 600 non-intersecting subsets with their centres $\theta_i = (b_{1i}, b_{2i}), i = 1, \cdots, 600$ corresponding to

$$b_{1i} \in \{-1.6,\ -1.5, \cdots, -0.3,\ -0.2,\ 0.2,\ 0.3, \cdots, 1.5,\ 1.6\}$$

$$b_{2i} \in \{-2,\ -1.9, \cdots, -1.1,\ -1,\ 1,\ 1.1, \cdots, 1.9,\ 2\}$$

respectively.
Figures 4(a)-(c) illustrate the case where θ is constant. The switching sequence $\{i(1), i(2), \cdots\}$ depicted in Fig. 4(a) indicates a remarkable speed of localization.

Case 2: Parameter Jumps

The results of localization on the finite set $\{\theta_i\}_{i=1}^{600}$ are presented in Fig. 5(a)-(e). Random abrupt changes in the values of the plant parameters occur every 7 steps. In both cases above the algorithm of localization in Section 2 is used. However, in the latter case the algorithm of localization is appropriately modified. Namely, $I(t)$ is updated as follows

$$I(t) = \begin{cases} I(t-1) \bigcap \hat{I}(t) & \text{if } I(t-1) \bigcap \hat{I}(t) \neq \{\ \} \\ \hat{I}(t) & \text{otherwise} \end{cases} \tag{107}$$

Once the switching controller, based on (107) has falsified every index in the localization set it disregards all the previous measurements, and the process of localization continues (see, [40] for details). In the example above a pole placement technique was used to compute the set of the controller gains $\{K_i\}_{i=1}^{600}$. The poles of the nominal closed loop system were chosen to be (0, 0.07, 0.1). $\diamondsuit$

Example 5.2 Here we present an example of indirect localization considered in Section 4. The model of a third order unstable discrete time system is given by

$$y(t+1) = a_1 y(t) + a_2 y(t-1) + a_3 y(t-2) + u(t) + \xi(t) \tag{108}$$

where a_1, a_2, a_3 are unknown constant parameters, and $\xi(t) = \xi_0 sin(0.9t)$ represents exogenous disturbance. The *a priori* uncertainty bounds are given by

$$a_1 \in [-1.6,\ -0.1] \bigcup [0.1,\ 1.6], \quad b_2 \in [-1.6,\ -0.1] \bigcup [0.1,\ 1.6], \quad a_3 \in [0.1, 1.6] \tag{109}$$

i.e., $\Omega = \{[-1.6,\ -0.1] \bigcup [0.1,\ 1.6] \times [-1.6,\ -0.1] \bigcup [0.1,\ 1.6] \times [0.1, 1.6]\}$. Choosing the vector C and the stabilizing set $\mathcal{I}$ as prescribed in Section 4, we obtain

$$\mathcal{I} : |z(t+1)| \leq \Delta||x(t)|| + c_0 \tag{110}$$

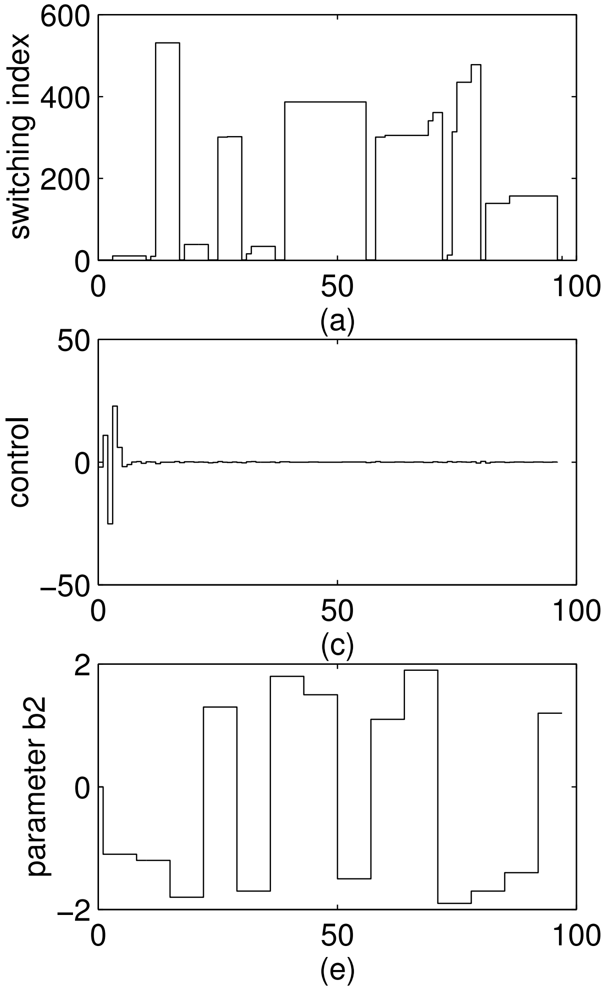

Figure 5: Example of Localization: Parameters Jump Every 7 Steps

where $C = (0,\ 0,\ 1)$ and $\Delta = 0.6$. We decompose Ω into 256 non-intersecting subsets with their centres $\theta_i = (a_{1i}, a_{2i}, a_{3i}),\ i = 1, \cdots, 256$ corresponding to

$$a_{1i} \in \{-0.3, -0.7, -1.1, -1.5,\ 0.3,\ 0.7,\ 1.1,\ 1.5\} \tag{111}$$

$$a_{2i} \in \{-0.3, -0.7, -1.1, -1.5,\ 0.3,\ 0.7,\ 1.1,\ 1.5\} \tag{112}$$

$$a_{3i} \in \{0.3,\ 0.7,\ 1.1,\ 1.5\} \tag{113}$$

respectively. This allows us to compute the set of controller gains $\{K_i\}_{i=1}^{256}$, $K_i = (k_{1i}, k_{2i}, k_{3i})$. Each element of the gain vector k_{ij}, $i \in \{1, 2, 3\}$, $j \in \{1, \cdots, 256\}$ takes values in the sets (111), (112), (113), respectively. The results of simulation with $\xi_0 = 0.1$, $a_1 = -1.1$, $a_2 = -0.7$, $a_3 = 1.4$, are presented in Fig. 6(a)-(b). Algorithm C has been used for this study. $\diamondsuit$

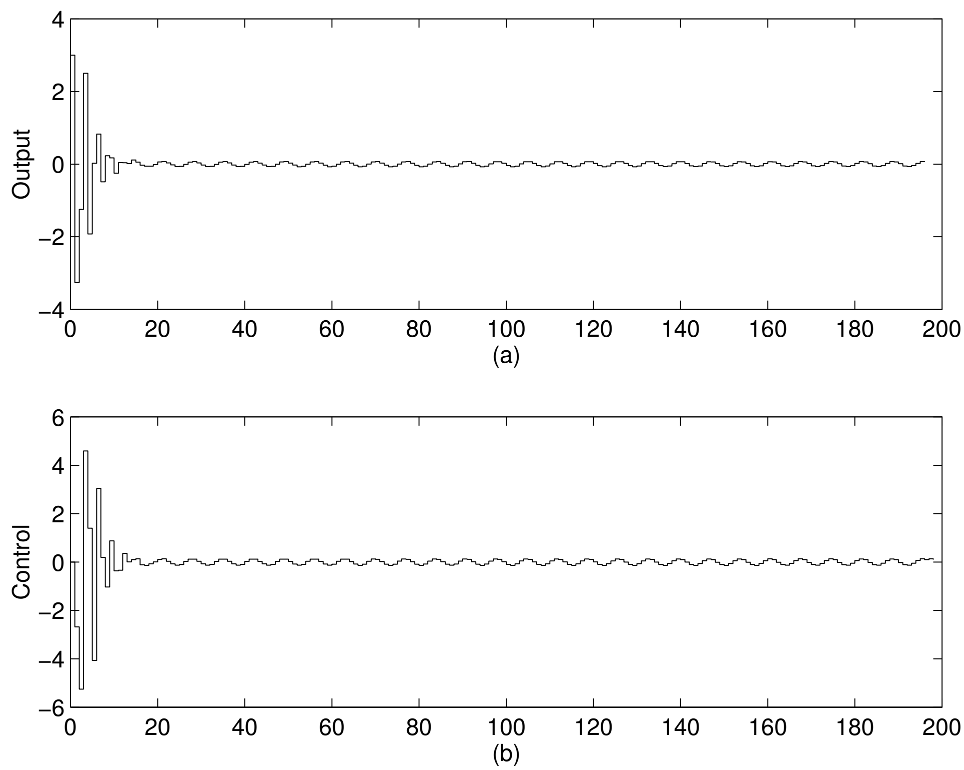

Figure 6: Example of Indirect Localization

6 Conclusions

In this chapter we have presented a new unified switching control based approach to adaptive stabilization of parametrically uncertain discrete-time systems. Our approach is based on a localization method which is conceptually different from the existing switching adaptive schemes and relies on on-line simultaneous falsification of incorrect controllers. It allows slow parameter drifting, infrequent large parameter jumps and unknown bound on exogenous disturbance. The unique feature of localization based switching adaptive control distinguishing it from conventional adaptive switching controllers is its rapid model falsification capabilities. In the LTI case this is manifested in the ability of the switching controller to quickly converge to a suitable stabilizing controller. We believe that the approach presented in this chapter is the first design of a falsification based switching controller which is applicable to a wide class of linear time invariant and time varying systems and which exhibits good transient performance.

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Appendix A

Proof of Theorem 3.1. First we note that it follows from Lemma 3.1 and the switching index update rule (37) that the total number of switchings made by the controller is finite. Let $\{t_1, t_2, \cdots, t_l\}$ be a finite set of switching instants. By virtue of (31)-(33) the behaviour of the closed-loop system between any two consecutive switching instants $t_s, t_j,\ 1 \leq s, j\ \leq\ l,\ t_j \geq t_s$ is described by

$$\begin{aligned} x(t+1) = (A(\theta) + B(\theta)K_{i(t_s)})x(t) + E(\xi(t) + \eta(t)) = \\ (A(\theta_{i(t_s)}) + B(\theta_{i(t_s)})K_{i(t_s)})x(t) + E\psi(t) \end{aligned} \tag{114}$$

where $|\psi(t)| \leq r_{i(t_s)}||\phi(t)|| + \bar{\xi} + \bar{\eta}(t)$.

Therefore, taking into account the structure of the parameter dependent matrices $A(\theta)$ and $B(\theta)$, namely the fact that only the last rows of $A(\theta)$ and $B(\theta)$ depend on θ the last equation can be rewritten as

$$x(t+1) = (A(\theta_{i(t_s)} + \Delta\theta(t)) + B(\theta_{i(t_s)} + \Delta\theta(t))K_{i(t_s)})x(t) + E\hat{\xi}(t) \tag{115}$$

for some $\Delta\theta(t) : ||\Delta\theta(t)|| \leq r_{i(t_s)} + q$ and $|\hat{\xi}(t)| \leq \bar{\xi} + \bar{\eta}(t)$. This is a direct consequence of the fact that the last equation in (114) can be rewritten as $y(t+1) = \theta^T_{i(t_s)}\phi(t) + \psi(t)$ and that $\max_{||\Delta\theta|| \leq 1} ||\Delta\theta^T\phi(t)|| = ||\phi(t)||$ holds for any $\phi(t)$. By Definition 3.2 and condition C3' the system (115) is quadratically stable with $\hat{\xi}(t) \equiv 0$ and t_s being fixed; moreover, there exists a positive definite matrix $H^T_{t_s} = H_{t_s}$ such that

$$P_{t_s} = \max_{||\Delta\theta(t)|| \leq r_{i(t_s)} + q} ||A(\theta_{i(t_s)} + \Delta\theta(t)) + B(\theta_{i(t_s)} + \Delta\theta(t))K_{i(t_s)}||_{H_{t_s}} < 1 \tag{116}$$

Here $||x||_H = (x^T Hx)^{1/2}$ and for any matrix $A \in \mathbf{R}^{n\times n}$, $||A||_H = \max_{||x||_H = 1} ||Ax||_H$ denotes the corresponding induced matrix norm. The equation (115) along with the property of quadratic stability guarantee that between any two consecutive switchings the closed loop system behaves as an exponentially stable LTI system subject to small parametric perturbations $\Delta\theta(t)$ and bounded disturbance $\hat{\xi}(t)$ and this property holds regardless of the evolution of the plant parameters. This is the key point making the rest of the proof transparent.

Assume temporarily that $\bar{\eta}(t) \equiv 0$, then it follows from (115),(116) that

$$||x(t_s+1)||_{H_{t_s}} \leq P_{t_s}||x(t_s)||_{H_{t_s}} + \hat{\xi}_{t_s} \tag{117}$$

$$||x(t_s+2)||_{H_{t_s}} \leq P_{t_s}^2||x(t_s)||_{H_{t_s}} + (P_{t_s}+1)\hat{\xi}_{t_s} \tag{118}$$

$$\cdots$$

$$||x(t_s+k)||_{H_{t_s}} \leq P_{t_s}^k||x(t_s)||_{H_{t_s}} + \hat{\xi}\sum_{i=1}^{k} P_{t_s}^{i-1} \tag{119}$$

$$||x(t_s+k)|| \leq (\lambda_{max}(H_{t_s})/\lambda_{min}(H_{t_s}))^{1/2} P_{t_s}^k||x(t_s)|| + \hat{\xi}_{t_s}\sum_{i=1}^{k} P_{t_s}^{i-1}/\lambda_{min}(H_{t_s})^{1/2} \tag{120}$$

where $\hat{\xi}_{t_s} = \max_{|\xi|\leq\bar{\xi}} ||E\xi||_{H_{t_s}}$.

Denote

$$M = \max_{t_1\leq i\leq t_l} (\lambda_{max}(H_{t_s})/\lambda_{min}(H_{t_s}))^{1/2},\ \rho = \max_{t_1\leq i\leq t_l} P_i < 1, \tag{121}$$

$$M(\bar{\xi}) = \max_{t_1\leq i\leq t_l} \hat{\xi}_i/(\lambda_{min}(H_i))^{1/2}\sum_{j=1}^{\infty} P_i^{j-1} < \infty \tag{122}$$

Since $\theta_{i(t)} \in \theta(I_{t_0}),\ K_{i(t)} \in \{K_i\}_{i=1}^L$ for all $t \in \mathbf{N},\ i(t) \in I_{t_0}$ there exist constants $0 < M_0 < \infty,\ \gamma_0 = \max_{|\xi|\leq\bar{\xi}} ||E\xi|| < \infty$ such that

$$||x(t_s)|| \leq M_0||x(t_s-1)|| + \gamma_0 \tag{123}$$

for any switching instant $t_1 \leq t_s \leq t_l$.

Hence,

$$[t_0,t_l): \quad ||x(t_1)|| \leq M_0||x(t_1-1)|| + \gamma_0 \leq M_0 M\rho^{t_1-t_0-1}||x(t_0)|| + M_0 M(\bar{\xi}) + \gamma_0 \tag{124}$$

$$||x(t_2)|| \leq M_0||x(t_2-1)|| + \gamma_0 \leq M_0^2 M^2\rho^{t_2-t_0-2}||x(t_0)|| + \hat{M}_2(\bar{\xi}) \tag{125}$$

where $\hat{M}_2(\bar{\xi}) = M_0(M(M_0M(\bar{\xi})+\gamma_0) + M(\bar{\xi})) + \gamma_0$;

$$\cdots$$

$$[t_l,\infty): \quad ||x(t)|| \leq M_0^l M^l \rho^{t-t_0-l}||x(t_0)|| + \hat{M}_l(\bar{\xi}) \quad . \tag{126}$$

Having denoted $M_1 = (M_0M/\rho)^l,\ M_2(\bar{\xi}) = \hat{M}_l(\bar{\xi}) < \infty$ we obtain (42). To conclude the proof we note that the result above can be easily extended to the case $\bar{\eta}(t) \neq 0$, provided that the "size" of unmodelled dynamics ϵ is sufficiently small. Indeed, let $\eta(t) \neq 0$. First, we note that due to the term $\bar{\eta}(t)$ in the algorithm of localization (33)-(37) the process of localization can not be disrupted by the presence of small unmodelled dynamics. In view of (A5),(117)-(126) it is easy to show that provided that ϵ is sufficiently small

$$[t_l, \infty): \ ||x(t)|| \leq M_0^l M^l \rho^{t-t_0-l} ||x(t_0)|| + \hat{M}_l(\bar{\xi}) + M_\eta \epsilon ||x(t_0)|| \tag{127}$$

with M_η being a positive constant independent of $x(t_0)$. Therefore,

$$||x(t)|| \leq (M_1 \rho^{t-t_0} + M_\eta \epsilon)||x(t_0)|| + \hat{M}_l(\bar{\xi}) \tag{128}$$

is valid for all $t_0 \in \mathbf{N}$, $t \geq t_l$. From (128) and Assumption (A5) exponential stability of the closed loop system (if $\hat{M}_l(\bar{\xi}) = 0$) or exponential convergence of the states to the residual set (if $\hat{M}_l(\bar{\xi}) > 0$) can be easily established. Indeed, in this case it is always possible to specify a sufficiently large integer T such that $(M_1 \rho^T + M_\eta \epsilon) < 1$. This, in turn, trivially implies stability. The finite number of the controller switchings follows directly from the switching index update rule (37). This also implies the finite convergence of switching, however, it is quite difficult, in general, to put an upper bound on t_l. This obviously does not affect the stability properties of the closed loop. $\diamondsuit$

Appendix B

Proof of Theorem 3.4. First we note that the property (1) follows directly from the structure of the algorithm of localization (66). It is straightforward to verify that relations (60)-(65) guarantee that the sequence of localization sets $I(t)$ is well defined.

To prove (2) consider first the case $\alpha = 0$. It is clear that

$$\bigcap_{k=s(t)}^{t} \hat{I}(k, \bar{\xi}(k-1)) \neq \{\} \tag{129}$$

if $\min_{k \in [s(t), t]} \{\bar{\xi}(k+1)\} \geq \bar{\xi}$ for all $t > t_0$. Since, according to (64), the estimate $\bar{\xi}(t)$ is updated only if (129) does not hold, and taking into account the discrete nature of updating expressed by (65) we conclude that

$$\sup_{t \geq t_0} \bar{\xi}(t) \leq \bar{\xi} + \mu \tag{130}$$

Let $\alpha > 0$. Then it is easy to see that the arguments above remain valid for any finite interval of time $[s(t), s(t)+t_d)$, provided that the rate of parameter variations is sufficiently small, namely, $\alpha \leq q/t_d$. To conclude the proof of (130) it suffices to note that the estimate $\bar{\xi}(t)$ in (64) does not change if $t - s(t) \geq t_d$.

Proof of statements (3), (4) follows closely those of Theorem 3.1. Here we present a brief sketch of the proof. Consider a finite time interval $T = [s(t), s(t) + t_d), l < t_d < \infty$. Let $\bar{\xi}(s(t)) \geq \bar{\xi}$, then the total number of switchings s made by the controller over T satisfies the condition $s \leq l$ if $\alpha \leq q/t_d$. Therefore, the states are bounded by (126) with t_0 replaced by $s(t)$. Moreover, (126) is valid for any time interval $\bar{T} = [s(t), s(t) + \bar{t}),\ \bar{t} > t_d$ such that

$$\bigcap_{k=s(t)}^{\bar{t}} \hat{I}(k, \bar{\xi}(s(t))) \neq \{\} \tag{131}$$

Relying on (126) and taking into account the fact that the index $s(t)$ is reset every time when (131) is violated for $t - s(t) \geq t_d$ it is always possible to choose sufficiently large integer t_d such as to guarantee exponential stability of the closed-loop system. Let $\bar{\xi}$ be unknown, then for any $\bar{\xi}(t_0) > 0$ the inequality (126) can be possibly violated no more than $([\bar{\xi}/\mu] + 1)$ times. Relying on this fact and using standard arguments exponential stability of the closed-loop system is easily established. $\diamondsuit$