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EXACT, OPTIMAL, AND PARTIAL LOOP TRANSFER RECOVERY

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Abstract. In this paper, a new approach is developed for loop transfer recovery (LTR). This approach employs a general feedback/feedforward structure for model matching control which includes the observer-based state feedback control as a special case. Under this framework, the problem of LTR is both generalized and simplified. First, the necessary and sufficient condition for the *exact* LTR is trivially obtained for both minimum phase and non-minimum phase systems and the corresponding controller is simply computed without any asymptotic tuning procedure. Secondly, for the non-minimum phase systems for which the exact LTR may not be possible, we address the optimal LTR problem and show that it can be formulated as an H_∞ optimization problem and can be solved by using either the Nevanlinna–Pick theory or the Ricatti equation approach. In the case where a good rejection to measurement noise is needed, the problem of partial LTR is addressed. Our approach applies to both continuous time and discrete time systems.

1 Introduction

It is well known that an observer-based state feedback design, if not carefully designed, could render poor robustness for the closed loop stability and performance against disturbance and modelling errors. An example of this non-robustness behaviour can be found in [1] where it is shown that a standard LQG design may yield an arbitrarily small gain margin. In order for an observer-based state feedback design to have good robustness, the so-called loop transfer recovery (LTR) is developed; see, for example, [2, 3, 4, 5, 6, 7, 8]. The existing LTR techniques, however, have the following problems.

- They apply mainly to minimum phase systems. Although some attempts are made on non-minimum phase systems (see, for example, [3, 8]), the applications are limited or perhaps complicated.
- The solutions to LTR are computed by asymptotic tuning procedures which may be numerically intensive. Moreover, the relationship between the tuning parameter and the degree of recovery is usually implicit.
- When the exact or asymptotic LTR is not possible, it is not clear how much recovery can be achieved.
- The existing LTR techniques are applicable only to observer-based state feedback designs and limited to recovering state feedback control.

Some attention has been paid to these problems. For example, Moore and Tay [9] proposed formulating the LTR problem as an H^∞/H^2 sensitivity recovery problem which can be solved via the standard H^∞ optimization techniques. For minimum phase systems, this method provides either exact or asymptotic sensitivity recovery. For non-minimum phase systems, it gives an optimal (called partial in [9]) sensitivity recovery.

In this paper, we develop a new approach for LTR. This approach employs a general feedback/feedforward control structure for model matching which includes the observer-based state feedback control as a special case. Under this framework, the problem of LTR is both generalized and simplified. First, the necessary and sufficient condition for the *exact* LTR is trivially obtained for both minimum phase and non-minimum phase systems and the corresponding controller is simply computed without any asymptotic tuning procedure. In fact, the controller for the exact LTR is unique when the plant is invertible, which implies that the controller obtained by any asymptotic LTR procedure will converge to the same solution. This property obviates the necessity of asymptotic procedures for the exact LTR. Secondly, for the non-minimum phase systems for which the exact LTR may not be possible, we address the optimal LTR problem and show that it can be formulated as a classical Nevanlinna–Pick problem in the single input case or a directional interpolation problem in the multi-input case both can be solved using the Nevanlinna–Pick theory. The solutions are provided and an design example is illustrated. In the special case of an observer-based state feedback design, we show that the optimal LTR problem can be solved via an algebraic Ricatti equation. Because LTR often results in poor measurement noise rejection, we address the problem of partial LTR which aims at making a compromise with measurement noise attenuation. In addition, we show that the H^∞/H^2 sensitivity recovery problem proposed in [9] is a special case of the LTR problem treated in this paper. Our approach applies to both continuous time systems and discrete time systems.

2 Problem Formulation

Consider the linear time-invariant (LTI) plant Σ (either continuous-time or discrete-time) modelled by

$$\begin{aligned} px(t) &= Ax(t) + Bu(t) + Bw(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the *state*, $u \in \mathbf{R}^m$ is the *control*, $y \in \mathbf{R}^r$ is the *output*, $w \in \mathbf{R}^m$ is the *control disturbance*, and p is either the differentiating, the differencing operator or the Delta operator proposed in [10]. The output is assumed to be noise free. Measurement errors will be considered later in the discussion of partial LTR. The transfer function of Σ is given by

$$G(s) = C(sI - A)^{-1}B + D \ \in \mathbf{R}(s)^{r\times m}. \tag{2}$$

Here s is the operator of either the Laplace transform, the z transform or the Delta transform; $\mathbf{R}(s)^{a\times b}$ denotes the set of $a \times b$ rational matrices in s. Throughout the paper, we assume
A1: Σ is stabilizable and detectable.
A2: $r \geq m$ and $G(s)$ is left invertible.

To motivate the problem of model matching and our general feedback/feedforward control structure, let us first discuss state feedback control and observer-based state feedback control.

State Feedback Control: If the state $x(t)$ is measurable, the design problem is simply to find a feedback matrix $F \in \mathbf{R}^{m\times n}$ such that the state feedback control below stabilizes the plant (1) and the closed-loop system enjoys the desired performance and robustness.

$$u(t) = r(t) - v(t) \tag{3}$$

$$v(t) = Fx(t) \tag{4}$$

where $r(t)$ is the *system command* and $v(t)$ is *the feedback signal*. The corresponding closed loop output is given by

$$y(s) = H_s(s)[r(s) + w(s)] \tag{5}$$

where

$$H_s(s) := G(s)[I + L_s(s)]^{-1} \in \mathbf{R}(s)^{r\times m} \tag{6}$$

is the closed loop input output transfer function and

$$L_s(s) := F(sI - A)^{-1}B \in \mathbf{R}(s)^{m\times m} \tag{7}$$

is the loop transfer function.

Observer–based State Feedback Control: When the state is not available, the observer-based feedback control below is often applied. For simplicity, we only discuss the full state observer case and assume $D = 0$.

$$p\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \tag{8}$$

$$v(t) = F\hat{x}(t) \tag{9}$$

with stable $A - KC$. This leads to

$$v(s) = M_{\text{obs}}(s)y(s) + N_{\text{obs}}(s)u(s) \tag{10}$$

where

$$\begin{aligned} M_{\text{obs}}(s) &:= F(sI - A + KC)^{-1}K \in \mathbf{R}(s)^{m\times r} \\ N_{\text{obs}}(s) &:= F(sI - A + KC)^{-1}B \in \mathbf{R}(s)^{m\times m} . \end{aligned} \tag{11}$$

It is easy to show that the output is governed by

$$y(s) = H_s(s)[r(s) + (I + N_{\text{obs}}(s))w(s)] . \tag{12}$$

As shown in (12), the observer–based design guarantees that the closed loop system has the same (required) input output transfer function as in the state feedback case (separation principle). However, the transfer function from the control disturbance to the output is further amplified by $I + N_{\text{obs}}(s)$. This could seriously contaminate the output if $N_{\text{obs}}(s)$ is large. We will also show later the effect of this term to the robustness of the closed loop stability.

The capability of the state feedback control (4) is, however, limited. This is due to the pole placement nature of (4), i.e., no stable zeros of $G(s)$ can be replaced or added and no more than n stable poles can be assigned. Naturally, the observer–based design suffers from a similar problem. For this reason, it is more general to consider the following control approach.

Model Matching with Feedback/Feedforward Control: Consider the feedback/feedforward control (3) with

$$v(s) = M(s)y(s) + N(s)u(s) \ , \ I + N(s) \text{ is invetible} \tag{13}$$

with stable $N(s) \in \mathbf{R}(s)^{m\times m}$ and $M(s) \in \mathbf{R}(s)^{m\times r}$, such that the closed loop system has input output transfer function equal to $H_d(s)$; see fig. 1. We call this control a feedback/feedforward law because the resulting control $u(s)$ involves both $y(s)$ and $r(s)$ terms. The output is then governed by

$$y(s) = H(s)[r(s) + (I + N(s))w(s)] \tag{14}$$

where

$$H(s) := G(s)[I + L(s)]^{-1} \in \mathbf{R}(s)^{r\times m} \tag{15}$$

is the *closed loop input output transfer function* and

$$L(s) := N(s) + M(s)G(s) \in \mathbf{R}(s)^{m\times m} \tag{16}$$

is the associated *loop transfer function*.

Note that (3) and (13) include all the stabilizing controllers of $G(s)$ because, by seting $r(s) = 0$,

$$u(s) = -(I + N(s))^{-1}M(s)y(s) \tag{17}$$

which describes the left coprime factorization of every possible controller, including the stablizing controllers.

Given a desired *closed loop input output model* $H_d(s)$ expressed in the following form:

$$H_d(s) = G(s)[I + L_d(s)]^{-1} \in \mathbf{R}(s)^{r\times m} \tag{18}$$

where $L_d(s)$ is the *desired loop transfer function*, the problem of model matching is to design stable $M(s)$ and $N(s)$ such that

$$L(s) = L_d(s) \ , \tag{19}$$

which leads to $H(s) = H_d(s)$.

It is straightforward to verify that the observer–based state feedback control (3)–(4) is a special case of the general feedback/feedforward control with $H_d = H_s$, $L_d = L_s$, $M = M_{\text{obs}}$ and $N = N_{\text{obs}}$. Obviously, it is not possible for the feedback/feedforward control above to match an arbitrary closed loop transfer function mainly because of the stability constraint. For this reason, we assume the following.

A3: $L_d(s)$ has the same unstable poles as $G(s)$, multiplicities included; in addition, $[I + L_d(s)]^{-1}$, $G(s)[I + L_d(s)]^{-1}$ and $[I + L_d(s)]^{-1}L_d(s)$ are all stable.

From the robustness point of view, it is a good practice to select $H_d(s)$ such that the resulting sensitivity $[I + L_d(s)]^{-1}$ is small. This additional condition, however, does not technically effect the results to come.

It is clear from (14) that the general feedback/feedfoward control may also yield a nonrobustness problem due to a possible large $N(s)$, as discussed in the observer–based design case. To gain further insight about the non–robustness potential of the feedback/feedforward control, we suppose there is some mismatch between $G(s)$ and the true plant $G_\Delta(s)$, where $G_\Delta(s)$ may be of higher order than $G(s)$, allowing unmodelled dynamics. Then, assuming $G(s)$ is square and invertible, the system output (14) becomes

$$\begin{aligned} y &= G_\Delta[I + N + MG_\Delta]^{-1}[r + (I + N)w] \\ &= G_\Delta[I + LR + N(I - R)]^{-1}[r + (I + N)w] \end{aligned} \tag{20}$$

where

$$R(s) := G^{-1}(s)G_\Delta(s) \ . \tag{21}$$

This shows that a large $N(s)$ not only amplifies the control disturbance but also jeopardizes the stability and the closed loop transfer because of the term $N(s)(I - R(s))$ in (20). Notice that this term could be significant when $N(s)$ is very large even if $G(s)$ and $G_\Delta(s)$ are close. That is, the stability margin of the closed loop system could be arbitrarily small and the closed loop transfer function could be seriously distorted.

Now let us define the problem of LTR. Denote by $L_v(s)$ the open loop transfer function from w to v (breaking the loop at $\times$ in Figure 1), i.e.,

$$L_v(s) := [I + N(s)]^{-1}M(s)G(s) \in \mathbf{R}(s)^{m\times m} \ . \tag{22}$$

Equation (14) with $L(s) = L_d(s)$ can be rewritten as follows:

$$y(s) = G(s)[I + L_d(s)]^{-1}r(s) + G[I + L_v(s)]^{-1}w(s) .$$

The idea of LTR is to make the loop transfer function $L_v(s)$ to be as close to $L_d(s)$ as possible so that the closed loop system will recover the robustness of $L_d(s)$. More precisely, the problem of LTR is as follows: find stable $N(s)$ and $M(s)$ such that they satisfy (19) and that the difference between $L_v(s)$ and $L_d(s)$ is minimized. Hence, the exact LTR requires

$$[I + N(s)]^{-1}M(s)G(s) = L_d(s) . \quad (23)$$

If the exact LTR is not achievable, we would like to propose an optimal LTR problem. Notice that LTR requires the closed loop transfer function from w to y to be as close to $H_d(s)$ as possible. Due to (14), the difference between these two transfer functions is caused by a nonzero $N(s)$. Hence, the optimal LTR problem is proposed as follows: find

$$\lambda_m := \inf ||N(s)||_\infty , \quad (24)$$

subject to (19) and stable $M(s)$, and parameterize all $N(s)$ attaining the infimum (24).

Remark 2.1 Once λ_m is computed, the gain of the transfer function from the control disturbance to the output can be simply estimated. In fact, due to (14), this gain can be made no larger than $1 + \lambda_m$ times the gain of $H_d(s)$.

Remark 2.2 It is possible to incorporate frequency weightings in (24) as is normally done in a standard H_∞ optimization problem. By doing so, the equation (24) will become

$$\lambda_m := \inf ||W_1(s)N(s)W_2(s)||_\infty \quad (25)$$

where the weighting matrices $W_1(s), W_2(s) \in \mathbf{R}(s)^{m\times m}$ are stable with stable inverse. This new problem, however, can be modified to (24) by defining

$$\begin{aligned} \hat{N}(s) &:= W_1(s)N(s)W_2(s) \ , \ \hat{M}(s) := W_1(s)M(s) \ , \\ \hat{L}_d(s) &:= W_1(s)L_d(s)W_2(s) \ , \ \hat{G}(s) := G(s)W_2(s) \ . \quad (26) \end{aligned}$$

Remark 2.3 Following Remark 2.2, we now show that the H^∞/H^2 sensitivity recovery problem proposed in [9] is a speical case of the weighted optimal LTR problem (25). Note that in [9] the output dynamic feedback is used to recover the state feedback. With our notation, the input sensitivity function given by the state feedabck (3-4) is

$$S^i_{SF}(s) = [I + L_s(s)]^{-1}$$

and the input sensitivity function given by the feedback/feedfoward control (3) and (13) is

$$S^i(s) = [I + (I + N(s))^{-1}M(s)G(s)]^{-1} .$$

Therefore, the sensitivity difference is given by

$$\begin{aligned} \epsilon^i(s) &= S^i_{SF}(s) - S^i(s) \\ &= [I + L_s(s)]^{-1} - [I + (I + N(s))^{-1}M(s)G(s)]^{-1} \\ &= [I + L_s(s)]^{-1} - [I + N(s) + M(s)G(s)]^{-1}(I + N(s)) \\ &= -[I + L_s(s)]^{-1}N(s) \quad (27) \end{aligned}$$

In the above, $N(s)+M(s)G(s) = L_s(s)$ is used; see (19). Hence, it is clear that $\epsilon^i(s)$ is the weighted $N(s)$ with weighting matrices $W_1(s) = -[I + L_s(s)]^{-1}$ and $W_2(s) = I$. As we will see, the solution to the exact LTR problem is very simple, and the optimal LTR problem becomes a Nevalinna-Pick interpolation problem which seems simpler to solve than the H^∞ optimization problem given in [9]. It should also be pointed out that not every weighted $N(s)$ can be regarded as a (weighted) sensitivity error because of the possible unstable zeros imposed by $S^i_{SF}(s)$. The case of output sensitivivy recovery studied in [9] can be treated by considering a dual system.

Now we end this section with the following lemma; the proof is omitted due to space limit.

Lemma 2.1 *Consider the system (1) and the desired loop transfer function $L_d(s)$ satisfying assumptions A1–A3. Let the feedback/feedforward control (3) with (13) with stable $N(s) \in \mathbf{R}(s)^{m\times m}$ and $M(s) \in \mathbf{R}(s)^{m\times r}$ satisfy (19). Then,*

i) *$M(s)$ has no unstable zero identical to any unstable pole of $G(s)$;*

ii) *$L_d(s) - N(s)$ contains all unstable zeros of $G(s)$, multiplicities included; and*

iii) *the closed loop system is internally stable.*

Conversely. if $N(s)$ is stable, and $N(s)$ and $M(s)$ satisfy (19) and condition ii) above, then $M(s)$ must be stable.

3 Exact LTR

The condition for the exact LTR is easily obtained by solving (19) and (23):

$$M(s) = L_d(s)G^+(s) \ ; \quad N(s) = 0 \quad (28)$$

where $G^+(s)$ is any left inverse of $G(s)$. In order for the closed loop system to have internal stability, $M(s)$ and $G(s)$ should have no unstable zero pole cancellation. Since $L_d(s)$ and $G(s)$ have the identical unstable poles (assumption A3), the internal stability can be guaranteed if and only if $L_d(s)$ contains all the unstable zeros of $G(s)$. Thus, we have the following necessary and sufficient condition for the exact LTR.

Theorem 3.1 *Consider the system (1) and the given desired loop transfer function $L_d(s)$ satisfying assumptions A1–A3. Then, the exact LTR can be achieved by the feedback/feedforward control (3) with (13) if and only if $L_d(s)$ contain all the unstable zeros of $G(s)$, if any, multiplicities included. When this condition holds, the solution to the feedback/feedforward control is given by (28). In particular, this solution implies that the exact LTR requires no feedforward control ($N(s) = 0$).*

Remark 3.1 The recoverability condition above has been realized by a number of researchers under the framework of observer-based state feedback design and asymptotic LTR ([2] for minimum phase systems, [3] and [8] for non–minimum phase systems). What is new here is that the same recoverability condition holds for the *exact* LTR under the framework of model matching and a general feedback/feedforward control structure.

Remark 3.2 The recoverability condition given in Theorem 3.1 is automatically satisfied by minimum phase systems. For non-minimum phase systems, the exact LTR requires $L_d(s)$ to preserve all unstable zeros of $G(s)$. It should be pointed out that this constraint can not be satisfied generically. In particular, the zeros of $L_d(s)$ will be all stable if the state feedback law is the solution of an LQ design [3].

Remark 3.3 An alternative implication of Remark 3.2 is that we have to choose the desired model $H_d(s)$ or, equivalently, $L_d(s)$ carefully if the exact LTR is desired. This raises the following interesting question. Given a plant transfer function

$G(s)$, under what conditions can we find a stable $H_d(s)$ or, equivalently, $L_d(s)$ satisfying assumption A3 such that the exact LTR is possible. The answer concluded from Theorem 3.1 is that such a $H_d(s)$ or $L_d(s)$ exists if and only if $G(s)$ is strongly stabilizable, i.e., stabilizable by a stable $M(s)$ with $N(s) = 0$. The conditions for strong stabilization are well known [11, 12].

Remark 3.4 For square and invertible plants, the exact LTR leads to a *unique* solution to the controller. This implies that all the asymptotic LTR procedures will converge to the same controller, regardless of the parameterization method and convergence behaviour. This fact obviates every asymptotic procedure *if* the exact LTR or almost exact LTR is the objective. When the plant is only left invertible, the controller for the exact LTR will not be unique because of the non-uniqueness of $G^+(s)$. However, this has no effect on the closed loop dynamics. Again, there seems no need for asymptotic tuning procedures.

Remark 3.5 It should be noted that the exact LTR may require a non-causal controller, i.e., $M(s)$ may be improper. This can be easily fixed by cascading it a low pass filter with a sufficiently high cutoff frequency. Strictly speaking, the resulting system does not have the exact LTR, but the difference is neglectable. It is interesting to observe that an observer-based design not being able to achieve the exact LTR is simply because it enforces a strictly proper controller or a proper one in the case of reduced order observer-based design; see $M_{obs}(s)$ and $N_{obs}(s)$. Furthermore, the difference among various asymptotic LTR procedures simply lies on the question of where to place the additional high frequency poles which are tuned implicitly but have little effect on the system dynamics.

4 Optimal LTR: the General Case

If the plant is of non-minimum phase and it does not satisfy the condition in Theorem 3.1, the problem of optimal LTR (24) arises. Although this problem can be restated as finding

$$\lambda_m = \inf ||L_d(s) - M(s)G(s)||_\infty$$

subject to stable $M(s)$, it is not a standard H_∞ optimization problem because $L_d(s)$ and $G(s)$ may be unstable. Fortunately, because $L_d(s)$ and $G(s)$ have the identical unstable poles, the problem (24) can be solved using the Nevanlinna-Pick theory. For simplicity, we only consider continuous time systems. A discrete time system can be converted to a continuous time one by using, e.g., a bilinear mapping. The following analysis is built up mainly on the results in [13, 14]. It is further assumed

A4: the unstable zeros $\alpha_1, \cdots, \alpha_\ell$ of $G(s)$ are distinct and simple satisfying $\mathrm{Re}[\alpha_i] > 0$.

This assumption implies that, for each α_i, there exists a unique $\xi_i \in \mathbf{C}^m$ with $||\xi_i|| = 1$ such that

$$G(\alpha_i)\xi_i = 0 \ , \quad i = 1, 2, \cdots, \ell \ . \tag{29}$$

Note that $\xi_i = 1$ for the single input case ($m = 1$). Let

$$\eta_i := L_d(\alpha_i)\xi_i \in \mathbf{C}^m \ , \quad i = 1, 2, \cdots, \ell \ . \tag{30}$$

Then, the constraint (19) is equivalent to

$$N(\alpha_i)\xi_i = \eta_i \ , \ i = 1, 2, \cdots, \ell \ , \tag{31}$$

i.e., $L_d(s) - N(s)$ contains all unstable zeros of $G(s)$. According to Lemma 2.1, the stability of $N(s)$ and (31) automatically imply the stability of $M(s)$. Hence, we have the following result.

Theorem 4.1 *The optimization problem (24) is equivalent to finding*

$$\lambda_m = \inf\{||N(s)||_\infty : N(\alpha_i)\xi_i = \eta_i, i = 1, 2, \cdots, \ell\} \tag{32}$$

and all such $N(s)$. The corresponding stable $M(s)$ is given by

$$M(s) = (L_d(s) - N(s))G^+(s) \ . \tag{33}$$

Single Input Case: In the single input case ($m = 1$), the equivalent problem (32) is a classical Nevanlinna–Pick interpolation problem which has a *unique* solution [13, 15], solved in the following four steps. The first step is to compute λ_m. Let $N_\lambda(s) = N(s)/\lambda$, then $||N(s)||_\infty \le \lambda$ if and only if $||N_\lambda(s)||_\infty \le 1$. According to [13, 15], there exists $N_\lambda(s)$ interpolating $N_\lambda(\alpha_i) = \eta_i/\lambda, i = 1, 2, \cdots, \ell$ if and only if the following $\ell \times \ell$ *Pick matrix* is nonnegative definite:

$$P_\lambda = P_0 - \lambda^{-2} P_1 \tag{34}$$

where

$$P_0 = \{\frac{1}{\alpha_i + \bar{\alpha}_j}\} \ ; \quad P_1 = \{\frac{\bar{\eta}_i \eta_j}{\alpha_i + \bar{\alpha}_j}\} \tag{35}$$

and the accent $\bar{}$ denotes the complex conjugate. Then,

$$\lambda_m = \sup\{\lambda : \det P_\lambda = 0\} \ .$$

Equivalently, due to the positive-definiteness of P_0, we have

$$\lambda_m = \sup\{\lambda : \det[\lambda^2 I - P_0^{-1} P_1] = 0\}$$

$$\lambda_m = \sqrt{\lambda_{\max}[P_0^{-1} P_1]} \tag{36}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue. The second step is to scale $N(s)$ and η_i: set $N(s) := N(s)/\lambda_m$ and $\eta_i := \eta_i/\lambda_m$, $i = 1, 2, \cdots, \ell$. Then the problem (32) becomes to find all the stable $N(s)$ with $||N(s)||_\infty \le 1$ subject to $N(\alpha_i) = \eta_i$, $i = 1, 2, \cdots, \ell$. Step 3 is to solve this scaled problem. The final step is to reverse the scaling done in step 2. The complete procedure, which is summarized from [13, 15], is given below.

Step 0: Compute α_i and evaluate η_i according to (30), $i = 1, 2, \cdots, \ell$.

Step 1: Compute P_0 and P_1 in (35) and λ_m according to (36);

Step 2: Set $\eta_i := \eta_i/\lambda_m$, $i = 1, 2, \cdots, \ell$;

Step 3.1: Form the so-called *Fenyves array* $\beta_{i,j}$ as follows:

$$\begin{aligned} \beta_{i,1} &:= \eta_i \ , \ i = 1, 2, \cdots, \ell \ ; \\ \beta_{i,j+1} &:= \frac{(\alpha_i + \bar{\alpha}_j)(\beta_{i,j} - \beta_{j,j})}{(\alpha_i - \alpha_j)(1 - \bar{\beta}_{j,j}\beta_{i,j})} \ , \\ & \qquad 1 \le j \le i - 1 \le \ell - 1 \ ; \end{aligned} \tag{37}$$

Step 3.2: Find $k < \ell$ (which must exist) such that

$$|\beta_{i,i}| < 1, \ i = 1, \cdots, k \ ; \ \ |\beta_{k+1,k+1}| = 1 \ ; \tag{38}$$

Step 3.3: Set $N^{(k+1)}(s) = \beta_{k+1,k+1}$;

Step 3.4: For $i = k, k-1, \cdots, 1$, do

$$N^{(i)} = \frac{(s - \alpha_i)N^{(i+1)}(s) + \beta_{i,i}(s + \bar{\alpha}_i)}{(s + \bar{\alpha}_i) + \bar{\beta}_{i,i}(s - \alpha_i)N^{(i+1)}(s)} \tag{39}$$

and set $N(s) = N^{(1)}(s)$;

Step 4: Set $N(s) = \lambda_m N(s)$ and compute

$$M(s) = [L_d(s) - N(s)]G^+(s) .$$

Multi–input Case: In the multi–input case ($m > 1$), the problem (32) is known as a ***directional interpolation problem*** (DIP). Thanks to a recent paper by Kimura [14], this can be solved by using an extension of the so-called *Schur–Nevanlinna algorithm*. As in the single input case, the solution involves four steps. Step one is to compute λ_m given by (36) but with P_i defined by

$$P_0 = \{\frac{\tilde{\xi}_i \xi_j}{\alpha_i + \tilde{\alpha}_j}\} \ ; \quad P_1 = \{\frac{\tilde{\eta}_i \eta_j}{\alpha_i + \tilde{\alpha}_j}\} \tag{40}$$

where the accent $\tilde{}$ denotes the Hermitian transpose. The second step is to scale $N(s)$ and η_i, same as in the single input case. Step 3 is to solve the scaled DIP problem and the final step is to reverse the scaling done in Step 2. The details are given below.

Step 0: Compute α_i, ξ_i and η_i according to (29) and (30),$i = 1, 2, \cdots, \ell$.

Step 1: Compute P_0 and P_1 in (40) and λ_m according to (36);

Step 2: Set $\eta_i := \eta_i/\lambda_m$, $i = 1, 2, \cdots, \ell$;

Step 3: For the scaled η_i, find $N(s)$ with $||N(s)||_\infty \le 1$ such that (31) holds. Due to space limit, the reader is referred to [14] for the algorithm. $N(s)$ may not be unique.

Step 4: Set $N(s) := \lambda_m N(s)$ and compute

$$M(s) = [L_d(s) - N(s)]G^+(s) .$$

5 Optimal LTR: the Observer Case

In this section, we show that if the observer–based state feedback design (8)-(9) is used then the optimal LTR problem can be solved via an algebraic Ricatti equation. For simplicity, the system under consideration is assumed to be continuous-time with $D = 0$. The generalization to its discrete-time counterpart can be obtained by using a discrete algebraic Ricatti equation.

For the observer-based design (8)-(9), the goal of optimal LTR is to minimize $N_{obs}(s)$. From the consideration of practical implementation, it is also desirable to avoid a high gain K. That is, the transfer function $M_{obs}(s)$ should be penalized. Therefore, the optimal LTR problem becomes solving

$$\lambda_m := \inf ||E(s)||_\infty \tag{41}$$

where

$$E(s) := [N_{obs}(s) \ \ M_{obs}(s)Q] \tag{42}$$

with $Q \in \mathbf{R}^{r\times r}$ being a symmetric positive-definite weighting matrix. Note that the constraint of stable $A - KC$ is implied in (41). If a high gain is allowed, Q should be kept small. It turns out that the optimization problem above can be recast as the following standard H_∞ optimization problem (with notation mainly adopted from [16]).

Consider the auxiliary LTI system

$$\begin{aligned} p\bar{x}(t) &= \bar{A}\bar{x}(t) + \bar{B}_1\bar{w}(t) + \bar{B}_2\bar{u}(t) \\ \bar{y}(t) &= \bar{C}_1\bar{x}(t) + \bar{D}\bar{u}(t) \end{aligned} \tag{43}$$

where $\bar{x} \in \mathbf{R}^n$ is the *state*, $\bar{u} \in \mathbf{R}^r$ is the *control*, $\bar{w} \in \mathbf{R}^r$ is the *control disturbance*, $\bar{y} \in \mathbf{R}^{(m+r)}$ is the *output* and

$$\begin{aligned} \bar{A} &:= A^T; \ \bar{B}_1 := F^T; \ \bar{B}_2 := C^T; \\ \bar{C}_1 &:= [B \ \ 0]^T; \ \bar{D}_1 := [0 \ \ Q]^T. \end{aligned} \tag{44}$$

The null submatrix in $\bar{C}_1$ and that in $\bar{D}_1$ are $r \times n$ and $m \times r$, respectively. Denoting the control weighting matrix QQ^T by R, note that R is positive-definite and

$$\bar{D}^T[\bar{C}_1 \ \ \bar{D}] = [0 \ \ R] .$$

For any given scalar $\lambda > 0$, the objective is to design a feedback control law $\bar{u}(t) = -\bar{K}\bar{x}$ such that

- The closed loop matrix $\bar{A} - \bar{B}_2\bar{K}$ is stable; and
- The closed loop transfer function from the disturbance $\bar{w}$ to the output $\bar{y}$,

$$\bar{E}(s) = (\bar{C}_1 - \bar{D}\bar{K})(sI - \bar{A} + \bar{B}_2\bar{K})^{-1}\bar{B}_1 \ , \tag{45}$$

  satisfies $||\bar{E}(s)||_\infty < \lambda$.

It is straightforward to verify that $\bar{E}^T(s) = E(s)$ with $K^T = \bar{K}$. Therefore, the optimal LTR problem (41) and the H_∞ optimization problem above are equivalent, and λ indicates the degree of LTR. The solvability of this optimization problem is known [16, 17, 18, 19, 20]: a solution exists if and only if the following algebraic Riccati equation

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} + \bar{P}(\lambda^{-2}\bar{B}_1\bar{B}_1^T - \bar{B}_2R^{-1}\bar{B}_2^T)\bar{P} + \bar{C}_1\bar{C}_1^T = 0 \tag{46}$$

has a stabilizing solution $\bar{P}$. Here, $\bar{P}$ is called a stabilizing solution if it solves (46) and $\bar{A} + (\lambda^{-2}\bar{B}_1\bar{B}_1^T - \bar{B}_2R^{-1}\bar{B}_2^T)\bar{P}$ is stable. the required control matrix is given by

$$\bar{K} = R^{-1}\bar{B}_2^T\bar{P} . \tag{47}$$

Translating this result back to the original problem (41), we obtain the following theorem.

Theorem 5.1 *Consider the optimal LTR problem in (41) with a given weighting matrix $Q > 0$. For any scalar $\lambda > 0$, there exists an observer gain matrix K such that $||E(s)||_\infty < \lambda$ if and only if the following algebraic Ricatti equation*

$$AP + PA^T + P(\lambda^{-2}F^TF - C^TR^{-1}C)P + BB^T = 0 \tag{48}$$

has a stabilizing solution P, i.e., $A + P(\lambda^{-2}F^TF - C^TR^{-1}C)$ is stable. When a stabilizing solution exists, the required observer gain matrix is given by

$$K = PC^TR^{-1} . \tag{49}$$

6 Measurement Noise and Partial LTR

In our previous discussions, we assumed that the output measurement is noise free. In the case when the measurement noise is not neglectable, the LTR techniques will usually cause noisy output. Indeed, replacing the output $y(t)$ in (13) by

$$y_m(t) = y(t) + d(t) \tag{50}$$

where $d(t)$ is the measurement noise, the equation (14) becomes

$$y(s) = G(s)[I+L(s)]^{-1}[r(s) - M(s)d(s) + (I+N(s))w(s)] . \tag{51}$$

Under the exact LTR, the equation above simplifies to

$$y(s) = G(s)[I + L_d(s)]^{-1}[r(s) - L_d(s)G^{-1}(s)d(s) + w(s)]$$

for invertible plants. Since $L(s)$ is usually large at low frequencies in order to have small sensitivity, the corresponding gain from $d(s)$ to $y(s)$ will be approximately equal to 1. That is, the measurement noise is almost directly injected into the output.

Furthermore, the measurement noise may saturate the plant input $u(t)$ when $L_d(s)G^{-1}(s)$ is very large. These phenomena,investigated in [21], display a trade-off between robustness and measurement noise rejection. In order to reduce the effect of measurement noise, the degree of LTR must sacrifice. Consequently, the problem of partial LTR arises: Given a scalar $\lambda > \lambda_m$, find all stable $N(s)$ and $M(s)$ subject to (19) and $||N(s)|| \leq \lambda$ and choose among them the solution for which the transfer function from d to y is minimized. Using Assumption A4 and the analysis in section 4, the first part of the problem is equivalent to finding all stable $N(s)$ with $||N(s)|| \leq \lambda$ subject to

$$N(\alpha_i)\xi_i = \eta_i, \ i = 1, 2, \cdots, \ell \,. \tag{52}$$

The corresponding $M(s)$ are given by (33). For the single input case, the algorithm for solving the later problem is similar to the one given in section 4 except that λ_m needs to be replaced by λ in step 2; k is set to be ℓ in step 3.2 because all $|\beta_{i,i}| < 1$ after scaling; and $N^{(\ell+1)}(s)$ is an arbitrary stable rational function with $||N^{(\ell+1)}||_\infty \leq 1$. For the multi-input case, the same change in step 2 applies while other steps remain unchanged. Once all the feasible $N(s)$ are parameterized in terms of $N^{(\ell+1)}(s)$, the second stage of solution takes place which minimizes $H_d(s)M(s)$. This problem deserves further research.

7 Conclusion

In this paper, we have investigated the problem of LTR for model matching by using the general feedback/feedforward control structure (3) and (13). Under this framework, the problem of exact LTR has been solved. The solution is of closed form, requiring no parameter tuning or asymptotic converging. The uniqueness of the resulting controller for invertible plants obviates the necessity of asymptotic procedures for the exact LTR. For non-minimum phase systems where the exact LTR may not be possible, we have shown that the optimal LTR problem can be solved using the Nevanlinna-Pick theory. In the case of an observer-based state feedback design, the optimal LTR problem can be solved via an algebraic Ricatti equation. The problem of partial LTR is also discussed. We end this paper by referring the reader to a recent paper [22] by the author in which it is shown that the optimal LTR problem is equivalent to an H_∞ optimal estimation problem.

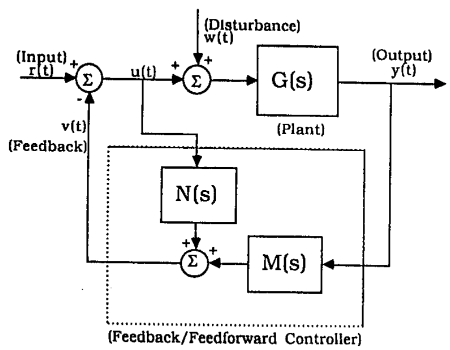

Figure 1: A General Feedback/Feedforward Control System

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