2015 IEEE Conference on Control Applications (CCA)
Part of 2015 IEEE Multi-Conference on Systems and Control
September 21-23, 2015. Sydney, Australia

Consensus Problems in Networks of Agents with Communication Delay*

Zhenhua Wang[1], Huanshui Zhang[2†] and Minyue Fu[3]

Abstract— **This paper studies the consensus problem for high-order multi-agent systems with constant communication delay. Under the assumption that the delay is known and the agent dynamics has unstable poles, an allowable delay bound for consensus is provided. The key technique is the adoption of the unique positive solution of a parametric algebra Riccati equation(ARE) in designing the control gain. Finally, a numerical example is given to show the validity of the theoretical results.**

I. INTRODUCTION

During the last decade, consensus problems have been widely investigated due to their broad applications, such as flocking [1], sensor fusion [2], spacecraft formation [3] and many other areas. Consensus means that a group of agents reach an agreement via local information exchange with their neighbors. In reference [4], consensus for single-integrator multi-agent systems was studied. By studying the joint impact of agent dynamics and network topology, [5] derived a necessary and sufficient condition for a linear multi-agent system to achieve consensus.

In many applications, time delay may inevitably occur due to communication congestion or finite transmission speed. It is known from [6] that delay usually degrades the closed-loop performance and stability characteristics.Thus, it is of considerable importance to study the effect of delay on consensus. Indeed, many efforts have been devoted to this issue. For single-integrator multi-agent systems with constant communication delay, a necessary and sufficient consensus condition was established in [7]. Average consensus problem in undirected networks with time-varying communication delay was dealt with in [8]. In reference [9], state consensus for discrete-time multi-agent systems with changing topologies and time-varying communication delays was investigated. For the high-order multi-agent systems that are at most critically unstable, the consensus problem with constant communication delay was solved in [10] and [11]. Without using the explicit information of delay, consensus of multi-agent systems subjected to time-varying delay was studied in [12]. Furthermore, for the multi-agent systems with a single unstable open-loop pole, [13] gave the maximum margin of the input delay for consensus. With agent's own historical input information in the protocol design, a consensus condition related to the network topology was constructed in [14] for first-order multi-agent systems with communication delay.

This paper considers the consensus problem of high-order continuous-time multi-agent systems in the presence of communication delay. The agents are assumed to have unstable poles, i.e., each agent dynamics has eigenvalues in the closed right-half plane. Thus, the system studied in this paper is more general than those in ([10]-[14]). Furthermore, regarding the system may have exponentially unstable poles, thus the low gain feedback method dealing with the consensus of critically unstable system in ([10]-[12]) is invalid. On the other hand, the techniques of analyzing the roots of scalar equations ([13], [14]) are also invalid for high-order multi-agent systems. To achieve consensus, the truncated predictor feedback method in [15] is employed, i.e., the solution of a parametric algebraic Riccati equation(ARE) and the delay information are used in designing the control gain. Comparing with reference [15], an extra parameter in the control gain is needed in order to simultaneously stabilize a number of delayed systems, which leads the work in this paper is more challenging than that in [15]. In this paper, an allowable delay bound related to the network topology and agent dynamics is provided for consensus. For fixed agent dynamics, the delay bound is maximized when the network topology is complete. As a special case, any large yet bounded communication delay is allowed for consensus if the agent dynamics is critically unstable, which is corresponding to the results in [11].

The rest of this paper is organized as follows. Section 2 reviews some preliminary results of graph theory. In Section 3, we address the problem to be dealt with. In Section 4, delay-dependent consensus conditions are derived. The effectiveness of the approach is illustrated via numerical examples in Section 5. Section 6 concludes the paper.

The following notations will be used throughout the paper. The set of real numbers is represented by $\mathcal{R}$. For any positive integer N, define $\mathcal{N} \triangleq \{1, 2, \ldots, N\}$ and use $\mathbf{1}_N$ to represent a column vector with all entries equal to 1. Let $\|\cdot\|$ stand for the 2-norm of a vector. Denote $\text{diag}\{a_1, a_2, \ldots, a_N\}$ as a diagonal matrix with its i-th main element $a_i, i \in \mathcal{N}$. For an given matrix A, the eigenvalues of A is denoted by $\lambda(A)$ and the real part of $\lambda(A)$ is represented as $Re(\lambda(A))$. We use $A \otimes B$ to denote the *Kronecker* product [16] between matrix A and B.

*This work was supported by National Key Technology Support Program of China(2014BAF07B00, 2014BAF07B03)

[1]Zhenhua Wang is with the School of Control Science and Engineering, Shandong University, Jinan, Shandong, 250061, P. R. China wzhua111@126.com

[2]Huanshui Zhang is with the School of Control Science and Engineering, Shandong University, Jinan, Shandong, 250061, P. R. China hszhang@sdu.edu.cn

[3]Minyue Fu is with the School of Electrical Engineering and Computer Science, University of Newcastle, NSW, 2308, Australia, also holds a Qian-ren Scholar position in School of Control Science and Engineering, Zhejiang University, Hangzhou, Zhejiang, P. R. China minyue.fu@newcastle.edu.au

†Corresponding author

II. PRELIMINARIES

The network topology of the multi-agent systems is modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ represent N agents, edges $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\} \subseteq \mathcal{V} \times \mathcal{V}$ represent the communication links among agents and $\mathcal{A} = [a_{ij}]_{N\times N}$ corresponds to the adjacency matrix. The edges of the diagraph are assumed to be positive, i.e., $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. Moreover, assume $a_{ii} = 0$ for all $i \in \mathcal{V}$. The set of neighbors of node i is represented by $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Define $d_i \triangleq \sum_{j=1}^{N} a_{ij}$ as the degree of node i and $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$ be the degree matrix of $\mathcal{G}$. The Lapalacian matrix of $\mathcal{G}$ is defined as $L_{\mathcal{G}} \triangleq D - \mathcal{A}$. Suppose that $\mathcal{G}$ is connected, then $L_{\mathcal{G}}$ has a simple zero eigenvalue and other eigenvalues are positive [17]. In this case, we order the eigenvalues of $L_{\mathcal{G}}$ as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. Furthermore, the eigenratio $\frac{\lambda_2}{\lambda_N}$ is known as the synchronizability, an important index of the graph[18]. Especially, $\lambda_2 = \lambda_N$ if the undirected graph $\mathcal{G}$ is complete, i.e., every two nodes in the graph are connected.

III. PROBLEM STATEMENT

Consider a network with N agents, each agent has identical continuous-time linear dynamics:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathcal{N}, \tag{1}$$

where $x_i(t) \in \mathcal{R}^n, u_i(t) \in \mathcal{R}^m$ denote the state and input of agent i, respectively; (A, B) are constant matrices with appropriate dimensions.

It should be noted that communication delay is ubiquitous in the network. In this paper, suppose that agent i receives a message sent by its neighbor j after a time-delay of τ. Making use of the delayed relative information, the control protocol is designed as

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}[x_j(t-\tau) - x_i(t-\tau)], \tag{2}$$

where $\tau > 0$ is the communication delay and $K \in \mathcal{R}^{n\times m}$ is the control gain to be designed.

We additionally take the initial values $x_i(\theta) = 0$ for any $\theta < 0$ and $i \in \mathcal{N}$. Now, the definition of consensus is given as follows.

Definition: Given an undirected graph $\mathcal{G}$, the continuous-time multi-agent system (1) is said to achieve consensus under protocol (2) if for any given initial values, the states of all agents satisfy

$$\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0, \quad i, j \in \mathcal{N},$$

or equivalently,

$$\lim_{t\to\infty} \|x_i(t) - \overline{x}(t)\| = 0, \quad i \in \mathcal{N},$$

where $\overline{x}(t) \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the average state.

From reference [12], we know that the stable eigenvalues of matrix A have no effect on consensus. Thus, we give the following assumption:

Assumption 1: All the eigenvalues of matrix A are in the closed right-half plane.

Problem Statement: Provide conditions such that multi-agent system (1) achieves consensus under protocol (2).

IV. MAIN RESULTS

Before presenting our main results, we first show the following technical lemmas.

A. Lemmas

Lemma 1: [19] Let (A, B) be controllable and $\gamma > 0$ be such that $\gamma > -2\min\{Re(\lambda(A))\}$. Then, the parametric ARE

$$A^T P(\gamma) + P(\gamma)A - P(\gamma)BB^T P(\gamma) = -\gamma P(\gamma) \tag{3}$$

has a unique positive-definite solution $P(\gamma)$ with

$$P^{-1}(\gamma) = \int_0^{\infty} e^{-(A+\frac{\gamma}{2}I)t} BB^T e^{-(A+\frac{\gamma}{2}I)^T t} dt.$$

In the sequel, denote $P \triangleq P(\gamma)$ for short. In addition, for the fixed matrix A and B in Lemma 1, the solution of equation (3) is only related to parameter γ.

Lemma 2: [15] Suppose that (A, B) is controllable, and all the eigenvalues of matrix A are in the closed right-half plane. Then, for any $t > 0$, the following relations hold,

$$tr(B^T PB) = 2tr(A + \frac{\gamma}{2}I),$$
$$PBB^T P \leq 2tr(A + \frac{\gamma}{2}I)P,$$
$$e^{A^T t} Pe^{At} \leq e^{w\gamma t}P,$$

where $w = \frac{2tr(A)}{\gamma} + n > 0$, and γ is as given in Lemma 1.

Lemma 3: [20] For any scalars γ_1 and γ_2 with $\gamma_2 > \gamma_1$, vector function $\omega : [\gamma_1, \gamma_2] \to \mathcal{R}^n$ such that the integrations in the following are well-defined, then

$$\int_{\gamma_1}^{\gamma_2} \omega^T(\beta)d\beta P \int_{\gamma_1}^{\gamma_2} \omega(\beta)d\beta \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^T(\beta)P\omega(\beta)d\beta.$$

Lemma 4: Let $0 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$ be non-zero constants and $\delta > 0$ be a constant to be designed, then the maximum of constant h satisfying formulas

$$1 - 2(\delta\lambda_i) + (\delta\lambda_i)^2 h \leq 0, \quad i = 2, 3, \ldots, N$$

is $h^* = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$. In particular, $h = h^*$ if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$.

Proof: It yields from $1 - 2(\delta\lambda_i) + (\delta\lambda_i)^2 h \leq 0$ for $i = 2, 3, \ldots, N$ that $h \leq -\frac{1}{(\delta\lambda_i)^2} + \frac{2}{(\delta\lambda_i)}$ and $\delta \geq \frac{1}{2\lambda_2}$. Thus, the problem is equivalent to that

$$\begin{aligned} h &= h^* = \max_{\delta \geq \frac{1}{2\lambda_2}} \min\{-\frac{1}{(\delta\lambda_i)^2} + \frac{2}{\delta\lambda_i}, i = 2, 3, \ldots, N\} \\ &= \frac{4\lambda_N\lambda_2}{[\lambda_N + \lambda_2]^2} \end{aligned}$$

if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$.

From $\delta \geq \frac{1}{2\lambda_2}$, we know $0 < \frac{1}{\delta\lambda_i} \leq 2$. Denote $f(\delta\lambda_i) \triangleq -\frac{1}{(\delta\lambda_i)^2} + \frac{2}{\delta\lambda_i}$. In case of $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$, it is easy to get $f(\delta^*\lambda_2) = f(\delta^*\lambda_N) = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$, and

$$f(\delta^*\lambda_i) - f(\delta^*\lambda_N) = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}(\lambda_N - \lambda_i)(\lambda_i - \lambda_2) \geq 0.$$

Thus, $f(\delta^*\lambda_i) \geq f(\delta^*\lambda_N)$ holds for $i = 2, 3, \ldots, N$.

In the following, we show that, for any $\delta \neq \delta^*$, there holds $\min\{f(\delta\lambda_i), i = 2, 3, \ldots, N\} < h^*$. We divide the proof into two cases.

1) If $\delta > \delta^*$, it follows

$$\frac{1}{\delta\lambda_N} < \frac{1}{\delta^*\lambda_N} = \frac{2\lambda_2}{\lambda_N + \lambda_2} \leq 1.$$

Thus,

$$\begin{aligned} \min\{f(\delta\lambda_i), i = 2, 3, \ldots, N\} &\leq f(\delta\lambda_N) \\ &= -\frac{1}{(\delta\lambda_N)^2} + \frac{2}{\delta\lambda_N} \\ &< -\frac{1}{(\delta^*\lambda_N)^2} + \frac{2}{\delta^*\lambda_N} \\ &= h^*. \end{aligned}$$

2) If $\frac{1}{2\lambda_2} \leq \delta < \delta^*$, it yields

$$\frac{1}{\delta\lambda_2} > \frac{1}{\delta^*\lambda_2} = \frac{2\lambda_N}{\lambda_N + \lambda_2} \geq 1.$$

Thus,

$$\begin{aligned} \min\{f(\delta\lambda_i), i = 2, 3, \ldots, N\} &\leq f(\delta\lambda_2) \\ &= -\frac{1}{(\delta\lambda_2)^2} + \frac{2}{(\delta\lambda_2)} \\ &< -\frac{1}{(\delta^*\lambda_2)^2} + \frac{2}{\delta^*\lambda_2} \\ &= h^*. \end{aligned}$$

Thus, h take the maximum $h^* = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$ if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$ and the proof is completed. □

B. Consensus Results

In the following, the control gain in protocol (2) is designed based on ARE (3). From Lemma 1, the key is giving appropriate parameter γ in (3) to guarantee consensus. We first investigate the case when the matrix A has eigenvalues in the right-half plane.

Theorem 1: Under Assumption 1 and system matrix A has eigenvalues in the right-half plane. If (A, B) is controllable, design the control gain in (2) as $K = \delta B^T P e^{A\tau}$ with $\delta = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$. Then, for any connected undirected graph $\mathcal{G}$, the multi-agent system (1) achieves consensus under protocol (2) if the communication delay τ satisfies

$$\tau < \tau^* = \max_{q>0} \frac{q[1 - \frac{(\lambda_N+\lambda_2)^2}{4\lambda_N\lambda_2} nq^2 e^{2q}]}{2tr(A)}.$$

Furthermore, for $0 < \forall\tau_1 \leq \tau^*$, there exists set $\Omega = [\gamma^*, \gamma_1]$ such that the consensus is achieved if $\gamma \in \Omega$.

Proof: Inserting protocol (2) into system (1) follows

$$\dot{x}_i(t) = Ax_i(t) + \delta BB^T P e^{A\tau} \sum_{j=1}^{N} a_{ij}[x_j(t-\tau) - x_i(t-\tau)].$$

Thus, there holds

$$\dot{x}(t) = I_N \otimes Ax(t) - \delta L_{\mathcal{G}} \otimes BB^T P e^{A\tau} x(t-\tau),$$

where $x(t) \triangleq [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$. In view of $\mathbf{1}_N^T L_{\mathcal{G}} = \mathbf{0}^T$ and the average state $\overline{x}(t) \triangleq \frac{1}{N}\sum_{i=1}^N x_i(t) = \frac{1}{N}[\mathbf{1}_N^T \otimes I_n]x(t)$, it follows $\dot{\overline{x}}(t) = A\overline{x}(t)$. Represent the deviation of each agent from the average state by $\delta_i(t) \triangleq x_i(t) - \overline{x}(t)$ and stack $\delta_i(t)$ to obtain a new vector $\delta(t) = [\delta_1^T(t), \delta_2^T(t), \ldots, \delta_N^T(t)]^T$, then

$$\dot{\delta}(t) = I_N \otimes A\delta(t) - \delta L_{\mathcal{G}} \otimes BB^T P e^{A\tau}\delta(t-\tau).$$

Take vector $\phi_i \in \mathcal{R}^N$ that satisfies $\phi_i^T L_{\mathcal{G}} = \lambda_i\phi_i^T$, and construct unitary matrix $\Phi = [\frac{\mathbf{1}_N}{\sqrt{N}}, \phi_2, \ldots, \phi_N]$ to transform $L_{\mathcal{G}}$ into a diagonal form, i.e., $\Phi^T L_{\mathcal{G}}\Phi = diag\{0, \lambda_2, \ldots, \lambda_N\}$. Let $\tilde{\delta}(t) \triangleq [\Phi \otimes I_n]^T\delta(t)$ and partition $\tilde{\delta}(t) \in \mathcal{R}^{nN}$ into N parts $\tilde{\delta}(t) = [\tilde{\delta}_1^T(t), \ldots, \tilde{\delta}_N^T(t)]^T$, then $\tilde{\delta}_1(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \delta_i(t) \equiv 0$, and

$$\dot{\tilde{\delta}}_i(t) = A\tilde{\delta}_i(t) - \sigma_i BB^T P e^{A\tau}\tilde{\delta}_i(t-\tau), \quad (4)$$

where $i = 2, 3, \ldots, N$ and $\sigma_i = \delta\lambda_i$. Thus, the consensus is reached if and only if $\lim_{t\to\infty}\tilde{\delta}_i(t) = 0$ holds simultaneously for $i = 2, 3, \ldots, N$.

From equation (4) and the variation of constants formula, we obtain

$$\tilde{\delta}_i(t) = e^{A\tau}\tilde{\delta}_i(t-\tau) - \sigma_i\int_{t-\tau}^{t} e^{A(t-s)}BB^T P e^{A\tau}\tilde{\delta}_i(s-\tau)ds.$$

Thus, there holds

$$\begin{aligned} \dot{\tilde{\delta}}_i(t) &= [A - \sigma_i BB^T P]\tilde{\delta}_i(t) \\ &\quad -\sigma_i^2 BB^T P\int_{t-\tau}^{t} e^{A(t-s)}BB^T P e^{A\tau}\tilde{\delta}_i(s-\tau)ds \\ &\triangleq [A - \sigma_i BB^T P]\tilde{\delta}_i(t) - \sigma_i^2 BB^T P\Pi_i(t). \end{aligned} \quad (5)$$

Take Lyapunov function $V(\tilde{\delta}_i(t)) = \tilde{\delta}_i^T(t)P\tilde{\delta}_i(t)$, where P is the unique positive definite solution of equation (3). Then, the derivative of the Lyapunov function along the trajectories (5) can be expressed as

$$\begin{aligned} &\dot{V}(\tilde{\delta}_i(t)) \\ =\ &\dot{\tilde{\delta}}_i^T(t)P\tilde{\delta}_i(t) + \tilde{\delta}_i^T(t)P\dot{\tilde{\delta}}_i(t) \\ =\ &\tilde{\delta}_i^T(t)[A^T P + PA - 2\sigma_i PBB^T P]\tilde{\delta}_i(t) \\ &-\sigma_i^2\Pi_i^T(t)PBB^T P\tilde{\delta}_i(t) - \sigma_i^2\tilde{\delta}_i^T(t)PBB^T P\Pi_i(t) \\ \leq\ &-\gamma\tilde{\delta}_i^T(t)P\tilde{\delta}_i(t) + [1 - 2\sigma_i + \sigma_i^2 h]\tilde{\delta}_i^T(t)PBB^T P\tilde{\delta}_i(t) \\ &+\frac{1}{h}\Pi_i^T(t)PBB^T P\Pi_i(t). \end{aligned}$$

From Lemma 2 and Lemma 3, we yield

$$\begin{aligned}
&\Pi_i^T(t)P\Pi_i(t)\\
=\ &[\int_{t-\tau}^{t} e^{A(t-s)}BB^TPe^{A\tau}\tilde{\delta}_i(s-\tau)ds]^TP\\
&\times[\int_{t-\tau}^{t} e^{A(t-s)}BB^TPe^{A\tau}\tilde{\delta}_i(s-\tau)ds]\\
\leq\ &4tr(A+\frac{\gamma}{2}I)^2\tau\int_{t-\tau}^{t} e^{w\gamma(t-s+\tau)}\tilde{\delta}_i(s-\tau)P\tilde{\delta}(s-\tau)ds.
\end{aligned}$$

Assume $V(\tilde{\delta}_i(t+\theta)) < \eta V(\tilde{\delta}_i(t))$ for $\forall\theta\in[-\tau,0]$ and $\forall t\geq\tau$, where $\eta>1$ is a constant to be designed, then

$$\begin{aligned}
&\Pi_i^T(t)P\Pi_i(t)\\
\leq\ &4tr(A+\frac{\gamma}{2}I)^2\tau\eta\int_{t-\tau}^{t} e^{w\gamma(t-s+\tau)}dsV(\tilde{\delta}_i(t))\\
=\ &\frac{4}{w\gamma}tr(A+\frac{\gamma}{2}I)^2\tau\eta e^{w\gamma\tau}[e^{w\gamma\tau}-1]V(\tilde{\delta}_i(t)).
\end{aligned}$$

From $e^{w\gamma\tau}-1\leq w\gamma\tau e^{w\gamma\tau}$, we obtain

$$\Pi_i^T(t)P\Pi_i(t)\leq 4tr(A+\frac{\gamma}{2}I)^2\eta\tau^2e^{2w\gamma\tau}V(\tilde{\delta}_i(t)),$$

which leads to

$$\begin{aligned}
&\dot{V}(\tilde{\delta}_i(t))\\
\leq\ &-\gamma V(\tilde{\delta}_i(t))+[1-2\sigma_i+\sigma_i^2h]\tilde{\delta}_i^T(t)PBB^TP\tilde{\delta}_i(t)\\
&+\frac{8tr(A+\frac{\gamma}{2}I)^3\eta\tau^2}{h}e^{2w\gamma\tau}V(\tilde{\delta}_i(t)). \qquad (6)
\end{aligned}$$

To weaken the effect of the latter two parts on the stability of inequality (6), take parameter $h=\frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$. From Lemma 4, we know $h=\frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$ is the maximum of h such that $1-2\sigma_i+\sigma_i^2h\leq 0$ for $i=2,3,\ldots,N$. Then, equation (6) becomes

$$\begin{aligned}
&\dot{V}(\tilde{\delta}_i(t))\\
\leq&-\left(\gamma-\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}\times 8tr(A+\frac{\gamma}{2}I)^3\eta\tau^2e^{2w\gamma\tau}\right)V(\tilde{\delta}_i(t)).
\end{aligned}$$

In the following, we are to find appropriate $\tau>0,\gamma>0$ and $\eta>1$, such that

$$\gamma-\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}\times 8tr(A+\frac{\gamma}{2}I)^3\eta\tau^2e^{2w\gamma\tau}>0. \qquad (7)$$

Let $q\triangleq w\gamma\tau=2tr(A+\frac{\gamma}{2}I)\tau$ and $R(\lambda_2,\lambda_N)\triangleq\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}$. Then, with the help of $\gamma=\frac{2tr(A+\frac{\gamma}{2}I)}{n}-\frac{2tr(A)}{n}$, equation (7) becomes

$$2tr(A)\tau<q[1-R(\lambda_2,\lambda_N)\eta nq^2e^{2q}].$$

Denote $g(q)\triangleq q[1-R(\lambda_2,\lambda_N)nq^2e^{2q}]$. If $2tr(A)<g(q)$, we can take constant $\eta=\frac{R(\lambda_2,\lambda_N)nq^3e^{2q}+q-2tr(A)\tau}{2R(\lambda_2,\lambda_N)nq^3e^{2q}}>1$, such that

$$\begin{aligned}
2tr(A)\tau\ &<\ tr(A)\tau+\frac{1}{2}q[1-R(\lambda_2,\lambda_N)nq^2e^{2q}]\\
&=\ q[1-R(\lambda_2,\lambda_N)\eta nq^2e^{2q}].
\end{aligned}$$

Thus, based on the Razumikhin Stability Theorem[21], we yield that the consensus is achieved if $2tr(A)\tau<g(q)$, i.e., $\tau<\frac{g(q)}{2tr(A)}$.

Now, we are to give the maximum value of function $g(q)$. It is easy to know $g(0)=0$, and $g(q)<0$ for any $q\geq 1$. Besides, we can also obtain

$$g'(q)=1-R(\lambda_2,\lambda_N)nq^2e^{2q}[3+2q]$$

and

$$g''(q)=-2R(\lambda_2,\lambda_N)nqe^{2q}[2q^2+6q+3].$$

From $g''(q)<0$, it yields that $g'(q)$ is monotone decreasing. Due to $g'(0)=1>0$ and $g'(1)<0$, there exists constant $0<q^*<1$ such that $g(q)$ is monotone increasing in $[0,q^*]$ and monotone decreasing in $[q^*,1]$. Thus, $\max_{q>0}g(q)=g(q^*)$.

Let τ^* satisfy $g(q^*)=2tr(A)\tau^*$. In the following, the allowable delay for consensus will be proved as $\tau\in[0,\tau^*)$. We first show that there exists γ^* such that $q^*=2tr(A+\frac{\gamma^*}{2})\tau^*$. In fact, assume $q^*<2tr(A+\frac{\gamma}{2})\tau^*$ for any $\gamma>0$, then $q^*\leq 2tr(A)\tau^*$ is follows. Thus,

$$q^*\leq 2tr(A)\tau^*=g(q^*)=q^*[1-R(\lambda_2,\lambda_N)n(q^*)^2e^{2q^*}],$$

which is a contradiction.

Next, we are to show that for $0<\tau_1<\tau^*$, there exists $\gamma_1>\gamma^*$ such that the consensus is guaranteed for any $\gamma\in[\gamma^*,\gamma_1]$. In fact, take γ_1 satisfying $w_1\gamma_1\tau_1=2tr(A+\frac{\gamma_1}{2}I)\tau_1=q^*$. From $q^*=2tr(A+\frac{\gamma^*}{2}I)\tau^*=w^*\gamma^*\tau^*$ and $\tau_1<\tau^*$, we know $\gamma^*<\gamma_1$ and $g(w^*\gamma^*\tau_1)<g(q^*)$. In light of

$$\begin{aligned}
&2tr(A)\tau^*\\
=\ &g(q^*)\\
=\ &(w^*\gamma^*\tau^*)[1-R(\lambda_2,\lambda_N)n(w^*\gamma^*\tau^*)^2e^{2w^*\gamma^*\tau^*}],
\end{aligned}$$

we yield

$$2tr(A)=(w^*\gamma^*)[1-R(\lambda_2,\lambda_N)n(w^*\gamma^*\tau^*)^2e^{2w^*\gamma^*\tau^*}],$$

and

$$\begin{aligned}
&2tr(A)\tau_1\\
=\ &(w^*\gamma^*\tau_1)[1-R(\lambda_2,\lambda_N)n(w^*\gamma^*\tau^*)^2e^{2w^*\gamma^*\tau^*}]\\
<\ &(w^*\gamma^*\tau_1)[1-R(\lambda_2,\lambda_N)n(w^*\gamma^*\tau_1)^2e^{2w^*\gamma^*\tau_1}]\\
=\ &g(w^*\gamma^*\tau_1).
\end{aligned}$$

Due to $g(w^*\gamma^*\tau_1)<g(w\gamma\tau_1)$ for any $\gamma\in[\gamma^*,\gamma_1]$, it follows $2tr(A)\tau_1<g(w\gamma\tau_1)$, which means the consensus can be guaranteed. Therefore, the proof is completed. □

Remark 2: In Theorem 1, the allowable delay bound is related to the eigenratio $\frac{\lambda_2}{\lambda_N}$, i.e., the synchronizability of undirected graph $\mathcal{G}$. Especially, the delay bound is maximized when the network topology $\mathcal{G}$ is complete, i.e., $\lambda_N=\lambda_2$.

Next, we will see that for the special case, where the system poles are constraint on the imaginary axis, the consensus condition is reduced to the same results presented in [11].

Corollary 1: Under the assumption in Theorem 1, if all the eigenvalues of A are on the imaginary axis, the multi-agent system (1) under protocol (2) will achieve consensus for any large yet bounded communication delay.

Proof: All the eigenvalues of matrix A on the imaginary axis and $A \in \mathcal{R}^{n\times n}$ lead to $tr(A) = 0$. On the other hand, Theorem 1 shows that the consensus is reached if

$$2tr(A)\tau < g(q) = q[1 - R(\lambda_2, \lambda_N)nq^2e^{2q}].$$

Select $\gamma > 0$ such that $q[1 - R(\lambda_2, \lambda_N)nq^2e^{2q}] > 0$. Then the consensus problem will be solved from $tr(A) = 0$. □

V. Simulation

This section illustrates the validity of the analysis results with simulations. Consider 3 agents in the network with dynamics

$$\begin{bmatrix} \dot{x}_{i1}(t) \\ \dot{x}_{i2}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t),$$

where $i = 1, 2, 3$. Obviously, the above system is controllable and exponentially unstable. After simple computation, the unique positive definite solution of the parametric algebra *Riicati* equation (3) is

$$P(\gamma) = \begin{bmatrix} \frac{(\gamma+1)(2\gamma+1)^2}{4} & \frac{(\gamma+1)(2\gamma+1)}{2} \\ \frac{(\gamma+1)(2\gamma+1)}{2} & 2\gamma + 1 \end{bmatrix}.$$

The communication among agents is described in Fig. 1. It

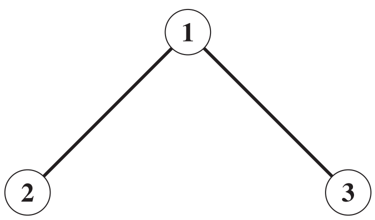

Fig. 1. Network Topology

is easy to get that the nonzero eigenvalues of the Laplacian matrix are $\lambda_2 = 1$ and $\lambda_3 = 3$. Based on Theorem 1, the allowable delay bound for consensus is $\tau^* = 0.1813$. Take $\tau = 0.15$ in this example, then the consensus can be achieved if $\gamma \in (0.1994, 0.3456)$. Select $\gamma = 0.25, \delta = \frac{2}{3}$, and the initial values $x_1(0) = [-8, 4]^T, x_2(0) = [2, 12]^T$ and $x_3(0) = [16, 9]^T$, then Fig. 2 shows the error states $x_{ij}(t) - \overline{x}_j(t)$ converge to zero asymptotically.

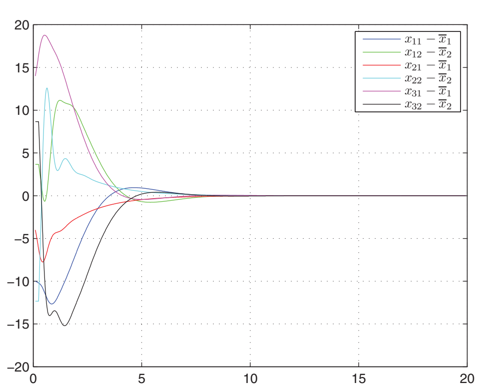

Fig. 2 Error states

VI. Conclusion

In this paper, the consensus problem for unstable multi-agent systems that are subjected to communication delay is investigated. With the aid of the parametric algebra Riccati equation technique, delay-dependent consensus conditions in terms of agent dynamics, network topology and communication delay are derived. For the future work, the results in this paper are expected to extend to the time-varying and unknown communication delay case.

References

[1] R. Olfati-Saber, Flocking for multi-agent dynamic system: Algorithms and theory, IEEE Trans. Automat. Contr., vol. 51, no. 3, pp. 401-420, 2006.

[2] Y. Shang, Fast distributed consensus seeking in large-scale sensor networks via shortcuts, Int. J. Comput. Sci. Eng., vol. 7, no. 2, pp. 121-124, 2012.

[3] Y. Cao, W. Yu, W. Ren and G. Chen, An overview of recent progress in the study of distributed multi-agent coordination, IEEE Trans. Ind. Inform., vol. 9, no. 1, pp. 427-438, 2013.

[4] W. Ren and R. W. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies, IEEE Trans. Automat. Contr., vol. 50, no. 5, pp. 655-661, 2005.

[5] C. Q. Ma and J. F. Zhang, Necessary and sufficient conditions for consensusability of linear multi-agent systems, IEEE Trans. Automat. Contr., vol. 50, no. 5, pp. 1263-1268, 2010.

[6] J. K. Hale and S. M. Verduyn Lunel, Introduction to functional difference equations. New York: Springer-Verlag, 1993.

[7] R. Olfati-Saber and R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. Automat. Contr., vol. 49, no. 9, pp. 1520-1533, 2004.

[8] Y. G. Sun, L. Wang and G. M. Xie, Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays, System & Control Letters, vol. 57, no. 2, pp. 175-183, 2008.

[9] F. Xiao and L. Wang, State consensus for multi-agent systems with switching topologies and time-varying delays, Int. J. Control, vol. 79, no. 10, pp. 1277-1284.

[10] X. Wang, A. Saberi, A. A. Stoovogel, H. F. Grip and T. Yang, Consensus in network with uniform constant communication delay, Automatica, vol. 49, no. 8, pp. 2461-2467, 2013.

[11] B. Zhou and Z. L. Lin, Consensus of high-order multi-agent systems with large input and communication delays, Automatica, vol. 50, no. 2, pp. 452-464, 2014.

[12] Z. H. Wang, J. J. Xu and H. S. Zhang, Consensusability of multi-agent systems with time-varying communication delay, System & Control Letters, vol. 65, no. 1, pp. 37-42, 2014.

[13] J. J. Xu, H. S. Zhang and L. H. Xie, Input delay margin for consensusability of multi-agent systmes, *Automatica*, vol. 49, no. 6, pp. 1816-1820, 2013.

[14] Z. H. Wang, K. Y. You, J. J. Xu and H. S. Zhang, Consensus design for continuous-time multi-agent systems with communication delay, J. Syst. Sci. Complexity, vol. 27, no. 4, pp. 701-711, 2014.

[15] S. Y. Yoon and Z. L. Lin, Truncated predictor feedback control for exponentially unstable linear systems with time-varying input delay, System & Control Letters, vol. 62, no. 10, pp. 837-844, 2013.

[16] R. A. Horn and C. R. Johnson. Topics in matrix analysis. Cambrige, U. K.: Cambrige. Univ. Press, 1991.

[17] M. Fiedler, Laplacian matrices of graph: a survey, Linear Algebra Appl., vol. 197, pp. 143-176, 1994.

[18] M. Barahona and L. M. Pecora, Synchronization in small-world systems, Physical Review Letters, vol. 89, no. 5, pp. 054101-054101, 2002.

[19] B. Zhou, G. R. Duan and Z. L. Lin, A parametric lyapunov equation approach to the design of low gain feedback, IEEE Trans. Automat. Contr., vol. 53, no. 6, pp. 1548-1554, 2008.

[20] K. Gu, An integral inequality in the stability problem of time-delay systems, In Proc. 39th IEEE Conf. Service Operations and Logistics, and Control, Sydney, 2000, pp. 2805-2810.

[21] Z. L. Lin, H. J. Fang, On asymptotic stabilizability of linear systems with delayed input, IEEE Trans. Automat. Contr., vol, 52, no. 6, pp. 998-1013, 2007.