MULTISTAGE MULTIRATE ADAPTIVE FILTERS

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ABSTRACT

A procedure for the adaptation of multistage multirate filters is developed in the context of a system identification perspective. The system to be identified is the cascade of a multistage decimator followed by a mulistage interpolator. For this structure, an identifiability condition is established which is sufficient to guarantee that the stagewise systems may be uniquely determined from input-output data in the ideal, exact model order case. An algorithm is then described that achieves exponential convergence of the system parameter estimates to their desired values given satisfaction of these identifiability conditions.

1. INTRODUCTION

Multistage implementations of sample rate converters and narrow-band filtering operations are known to enable significant computational efficiencies [1, 2]. However, current design procedures for such multistage, multirate filters approach specification of each stage on an individual basis, rather than simultaneously optimizing all filter stages [3]. In this paper, we formulate a simultaneous optimization by casting the design problem in the framework of adaptive filtering or equivalently system identification. We establish fundamental identifiability conditions within this framework, and define a globally convergent algorithm for achieving the identification. This work is perhaps the first effort in adaptively optimizing these multirate operations, though [4] has considered adaptive techniques for optimizing lossless filter banks. Two points to note are that our identification procedure minimizes a mean-square error, and that we do not address selection of decimation and interpolation factors nor the distribution of the filter orders among the stages.

2. MODEL AND NOTATION

We pose the multistage multirate filter adaptation in terms of a system identification problem in which the "true system" to be identified has the structure shown in Fig. 1, with ℓ_i and m_i denoting the time indices at the various sample rates.

Assumption 1 *Each $G_i(q^{-1})$, $H_i(q^{-1})$ is FIR. Treated as polynomials in q^{-1}, each is monic and of known respective degrees n_{g_i} and n_{h_i}.*

We use the polyphase decompositions

$$G_i(q^{-1}) = \sum_{j=0}^{M_i-1} q^{-j} G_{ij}(q^{-M_i}) \tag{1}$$

and

$$H_i(q^{-1}) = \sum_{j=0}^{L_i-1} q^{-j} H_{ij}(q^{-L_i}). \tag{2}$$

to describe various manipulations of the blocks in Fig. 1. We first illustrate these manipulations with two simple examples.

To wit, the system depicted in Fig. 2 is equivalent to the system in Fig. 3, with appropriate definition of the subsystems $\widetilde{G}_{2j}$. Points to note are first that

$$\widetilde{G}_{20}(q^{-2}) + q^{-1}\widetilde{G}_{21}(q^{-2}) = G_{10}(q^{-1})G_2(q^{-1}) \tag{3}$$

$$\widetilde{G}_{22}(q^{-2}) + q^{-1}\widetilde{G}_{23}(q^{-2}) = G_{11}(q^{-1})G_2(q^{-1}) \tag{4}$$

and second that for each ℓ, $\{u_i^{(2)}(\ell)\}_{i=0}^{3}$ represents a collection of *disjoint* samples of $\{u(k)\}$.[1]

Likewise, Fig. 4 is equivalent to Fig. 5. In this case,

$$\widetilde{H}_{20}(q^{-2}) + q^{-1}\widetilde{H}_{21}(q^{-2}) = H_{10}(q^{-1})H_2(q^{-1}) \tag{5}$$

$$\widetilde{H}_{22}(q^{-2}) + q^{-1}\widetilde{H}_{23}(q^{-2}) = H_{11}(q^{-1})H_2(q^{-1}) \tag{6}$$

and for each m, $\{y_i^{(2)}(m)\}_{i=0}^{3}$ represents a collection of *disjoint* samples of $\{y(k)\}$.

In general, the output of $G_i(q^{-1})$ can be expressed as

$$v_i(\ell_i) = \widetilde{G}_i(q^{-1})U^{(i)}(\ell_i) \tag{7}$$

where

$$\widetilde{G}_i(q^{-1}) = \left[\ \widetilde{G}_{i0}(q^{-1}) \ \cdots \ \widetilde{G}_{i\widetilde{M}_i}(q^{-1})\ \right], \tag{8}$$

where $U^{(i)}(\ell_i)$ is a column vector whose elements are certain disjoint samples of $u(k)$, and

$$\widetilde{M}_i = \prod_{j=1}^{i} M_j. \tag{9}$$

Further,

$$\widetilde{G}_{1j}(q^{-1}) = G_{1j}(q^{-1}), \tag{10}$$

and for each $0 \le j \le \widetilde{M}_i - 1$,

$$\widetilde{G}_{ij}(q^{-1})G_{i+1}(q^{-1}) = \sum_{k=0}^{M_{i+1}-1} q^{-k}\widetilde{G}_{i+1,jM_{i+1}+1}(q^{-M_{i+1}}). \tag{11}$$

[1] In fact, for k a multiple of four, $\{u(k), u(k-1), u(k-2), u(k-3)\} = \{u_0^{(2)}(k/4), u_1^{(2)}(k/4), u_2^{(2)}(k/4), u_3^{(2)}(k/4)\}$.

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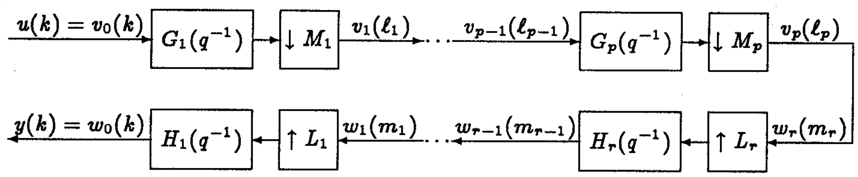

Figure 1. General multistage, multirate filter.

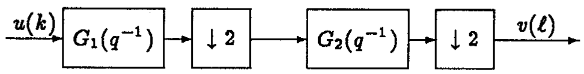

Figure 2. Simple two-stage decimator.

In a similar vein, with $Y^{(i)}(m_i)$ a column vector with elements that are disjoint samples of $y(k)$,

$$Y^{(i)}(m_i) = \widetilde{H}_i(q^{-1})w_i(m_i) \tag{12}$$

where

$$\widetilde{H}_i(q^{-1}) = \left[\begin{array}{ccc} \widetilde{H}_{i0}(q^{-1}) & \cdots & \widetilde{H}_{i\widetilde{L}_i}(q^{-1}) \end{array} \right]^T, \tag{13}$$

where

$$\widetilde{L}_i = \prod_{j=1}^{i} L_j. \tag{14}$$

The vectors $\widetilde{H}_i(q^{-1})$ follow recursions very similar to (10) and (11).

Remark 1 *Because of Assumption 1, the highest degree element of each $\widetilde{H}_i(q^{-1})$ and $\widetilde{G}_i(q^{-1})$ is monic and of known degree, greater than zero.*

Finally observe that one can write

$$Y^{(r)}(m_r) = \widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})U^{(p)}(\ell_p) \tag{15}$$

where $\widetilde{H}_r$ is a column vector and $\widetilde{G}_p$ is a row vector. We note here that m_r and ℓ_p are time indices at the same sample rate, henceforth both denoted n, and that $\widetilde{M}_p = \widetilde{L}_r \stackrel{\text{def}}{=} N$.

3. IDENTIFIABILITY

For the identification algorithm of the next section to offer globally convergent parameter estimates it is essential that the "true" parameters can be uniquely identified, given a suitable choice of input sequence. In this section we give a sufficient condition for the system in Fig. 1 to be identifiable. To this end we first formally define lack of identifiability.

Definition 1 *The system in Fig. 1 is* unidentifiable *if there exist monic polynomials $\widehat{G}_i(q^{-1})$, $i = 1, \ldots p$ and $\widehat{H}_i(q^{-1})$, $i = 1, \ldots r$ with respective degrees n_{g_i} and n_{h_i} such that*

(i) $[\widehat{G}_1 \cdots \widehat{G}_p \ \widehat{H}_1 \cdots \widehat{H}_r] \neq [G_1 \cdots G_p \ H_1 \cdots H_r]$;

(ii) the input output behavior of the system obtained by replacing $\{G_i, H_i\}$ by $\{\widehat{G}_i, \widehat{H}_i\}$ in Fig. 1 is identical to the actual system.

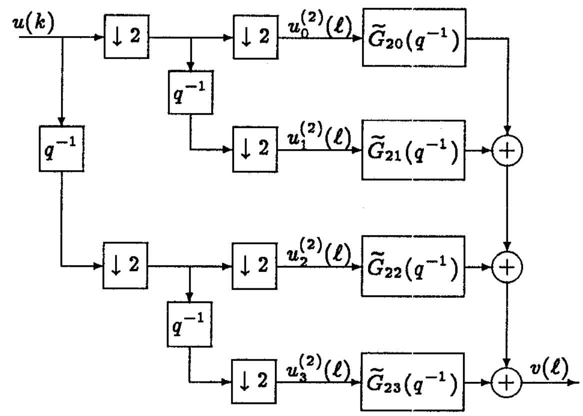

Figure 3. Decomposed simple two-stage decimator.

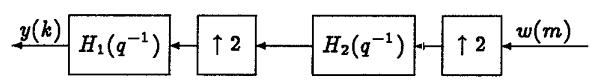

Figure 4. Simple two-stage interpolator.

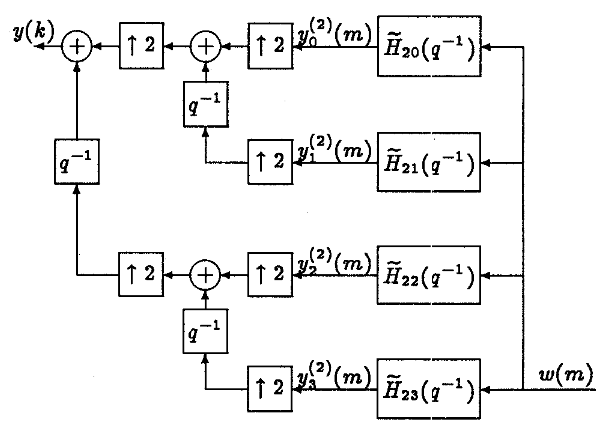

Figure 5. Decomposed simple two-stage interpolator.

The sequel makes extensive reference to the concept of coprimeness, which we now define.

Definition 2 *The set of polynomials* $\{A_1(q^{-1}),\dots,A_N(q^{-1})\}$ *is* coprime *if there is no complex* α *for which* $A_i(\alpha)=0$ *for all* $1\le i\le N$.

We are now ready to state the main result of this section.

Theorem 1 *Under Assumption 1, the system in Fig. 1 is identifiable if the following hold:*

(i) for all $1\le i\le p-1$, *the set*

$$\{G_{i0}(q^{-1}),\dots,G_{i,M_i-1}(q^{-1})\} \tag{16}$$

is coprime;

(ii) for all $1\le i\le r-1$, *the set*

$$\{H_{i0}(q^{-1}),\dots,H_{i,L_i-1}(q^{-1})\} \tag{17}$$

is coprime;

(iii) at least one of the sets

$$\{G_{p0}(q^{-1}),\dots,G_{p,M_p-1}(q^{-1})\} \tag{18}$$

$$\{H_{r0}(q^{-1}),\dots,H_{r,L_r-1}(q^{-1})\} \tag{19}$$

is coprime.

We will prove the theorem as follows. From (15) and the fact that the elements of $Y^{(r)}(m_r)$ and $U^{(p)}(\ell_p)$ can be independently manipulated, it will follow that all elements of the rank 1 matrix polynomial $\widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})$ can be uniquely estimated. Subject to Assumption 1 and (3), we then show that $\widetilde{H}_r(q^{-1})$ and $\widetilde{G}_p(q^{-1})$ are uniquely fixed. Following this, through an inductive argument we show that (1) and (2) respectively fix the $H_i(q^{-1})$ and $G_i(q^{-1})$ uniquely. We need the following lemmas.

Lemma 1 *Suppose the set of polynomials*

$$\{B_1(q^{-1}),\dots,B_M(q^{-1})\} \tag{20}$$

is coprime. Define for $1\le j\le M$

$$A(q^{-1})B_j(q^{-1})=\sum_{i=0}^{N-1}q^{-i}P_{ji}(q^{-N}) \tag{21}$$

where

$$A(q^{-1})=\sum_{i=0}^{N-1}q^{-i}A_i(q^{-N}) \tag{22}$$

for some $N>0$. *Then the set of polynomials*

$$\{P_{10}(q^{-1}),\dots,P_{1,N-1}(q^{-1}),P_{20}(q^{-1}),\dots\dots,P_{M,N-1}(q^{-1})\} \tag{23}$$

is coprime if the set

$$\{A_0(q^{-1}),\dots,A_{N-1}(q^{-1})\} \tag{24}$$

is coprime. Further, regardless of the coprimeness of (20), (23) cannot be coprime if (24) is not coprime.

Proof: Observe that each element of (23) is a linear combination of the elements of (24). Thus the lack of coprimeness of (24) implies the same for (23).

Now suppose (20) and (24) are coprime but that (23) is not. Thus, there is β such that $P_{ji}(\beta)=0$ for all $0\le i\le N-1$, $1\le j\le N$. Let $\alpha_1,\dots,\alpha_N$ be the N^{th} roots of β. Then for each $1\le k\le N$, $P_{ji}(\alpha_k^N)=0$.

Then from (21)

$$A(\alpha_k)B_j(\alpha_k)=0,\quad 1\le k\le N,1\le j\le M. \tag{25}$$

Since (20) is coprime, for all $1\le k\le N$ $A(\alpha_k)=0$. Thus $q^{-N}-\beta$ is a factor of $A(q^{-1})$, and there exists

$$a(q^{-1})=\sum_{i=0}^{N-1}q^{-i}a_i(q^{-N}) \tag{26}$$

such that

$$A(q^{-1}) = (q^{-N}-\beta)a(q^{-1}) \tag{27}$$

$$= \sum_{i=0}^{N-1}q^{-i}(q^{-N}-\beta)a_i(q^{-N}). \tag{28}$$

Comparing (28) to (22) we get

$$A_i(q^{-1})=(q^{-1}-\beta)a_i(q^{-1}),\quad 0\le i\le N-1. \tag{29}$$

Thus, (24) cannot be coprime. The contradiction proves the result. ∎

Lemma 2 *Let* $A(q^{-1})$ *be a (column or row) vector polynomial with elements* $A_1(q^{-1}),\dots,A_N(q^{-1})$, *and let* $B(q^{-1})$ *is a scalar, monic polynomial of known degree. If*

$$\{A_1(q^{-1}),\dots,A_N(q^{-1})\} \tag{30}$$

is coprime, then $A(q^{-1})B(q^{-1})$ *suffices to determine* $\{A_i(q^{-1})\}$ *and* $B(q^{-1})$ *uniquely.*

Proof: Suppose $a_i(q^{-1}) = A_i(q^{-1})B(q^{-1})$. Then as (30) is coprime, $B(q^{-1})$ is the unique common factor of $a_1(q^{-1}),\dots,a_N(q^{-1})$ with degree equaling the specified degree of $B(q^{-1})$. Hence $B(q^{-1})$ can be determined as can then be $A_i(q^{-1})$. ∎

Proof of Theorem 1: From Lemma 1, for each $1\le i\le p-1$

$$\{\widetilde{G}_{i0}(q^{-1}),\dots,\widetilde{G}_{i,\widetilde{M}_i-1}(q^{-1})\} \tag{31}$$

is coprime. Similarly, for each $1\le i\le r-1$

$$\{\widetilde{H}_{i0}(q^{-1}),\dots,\widetilde{H}_{i,\widetilde{L}_i-1}(q^{-1})\} \tag{32}$$

is coprime. Thus, the recursions given in Section 2., together with Lemma 2, prove that the unique knowledge of $\widetilde{H}_r(q^{-1})$ and $\widetilde{G}_p(q^{-1})$ uniquely fixes $G_1(q^{-1}),\dots,G_p(q^{-1})$ and $H_1(q^{-1}),\dots,H_r(q^{-1})$. Now, without loss of generality assume (18) is coprime. Then again from Lemma 2 and Remark 1, $\widetilde{H}_r(q^{-1})$ is uniquely determinable from $\widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})$. Hence, so is $\widetilde{G}_p(q^{-1})$ as $\widetilde{H}_r(q^{-1})$ is a column vector and $\widetilde{G}_p(q^{-1})$ is a row vector. Hence the result. ∎

4. ADAPTIVE ESTIMATION

This section outlines a globally convergent algorithm for adaptively estimating the coefficients of the G_i and H_i blocks in Fig. 1. The algorithm contains several stages, all implemented simultaneously. We assume that the system in Fig. 1 is identifiable.

The basic idea of the algorithm is as follows. The relationship (15) allows direct estimation from input-output data of the coefficients in the rank-1 matrix polynomial $\widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})$, using standard techniques. Label as $F_{ij}(q^{-1})$ the i,j^{th} element of $\widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})$. The first algorithm stage is this estimation of the coefficients of $F_{ij}(q^{-1})$. The next two stages are the estimation of the coefficients of $\widetilde{G}_{pi}(q^{-1})$ and $\widetilde{H}_{ri}(q^{-1})$ using the estimates for $F_{ij}(q^{-1})$. Finally, from these one recovers $G_i(q^{-1})$ and $H_i(q^{-1})$ in the last stage.

Given that the first stage is straightforward to accomplish, say using equation error approaches, we proceed now with the development of the second and third stages. From the discussion in Section 3. and our assumption of identifiability, either or both the set of elements of $\widetilde{H}_r(q^{-1})$ and the set of elements of $\widetilde{G}_p(q^{-1})$ is coprime. Without loss of generality, assume the elements of $\widetilde{H}_r(q^{-1})$ are coprime. Equally, at least one among the elements of $\widetilde{G}_p(q^{-1})$, without loss of generality $\widetilde{G}_{pk}(q^{-1})$, is monic with known degree. With this in mind, consider the set of simultaneous polynomial equations

$$\widetilde{H}_{r1}(q^{-1})F_{1k}(q^{-1}) - \widetilde{H}_{ri}(q^{-1})F_{ik}(q^{-1}) = 0;\ i = 0,\ldots,N. \quad (33)$$

Viewing the $\{F_{ik}(q^{-1})\}$ as known, and subject to the requirement that the highest degree polynomial among the $\{\widetilde{H}_{ri}(q^{-1})\}$ is monic and of known degree, the set of $\{\widetilde{H}_{ri}(q^{-1})\}$ that solve (33) is unique.

We need here some further notation. Let $\tilde{h}_{ri}$ be the coefficient vector for $\widetilde{H}_{ri}(q^{-1})$, and concatenate the $\tilde{h}_{ri}$ vectors together in $\tilde{h}_r$. Similarly define $\tilde{g}_{pi}$ and $\tilde{g}_p$. The coefficient vector for $F_{ij}(q^{-1})$ is denoted f_{ij}, with f_k the concatenation of f_{ik} as i varies. Also, let the coefficient vector for $\widetilde{H}_{ri}(q^{-1})F_{ij}(q^{-1})$ be denoted $\tilde{h}_{ri}\otimes f_{ij}$. With this notation, (33) reduces to

$$V_{rh}(\tilde{h}_r, f_k) \stackrel{\text{def}}{=} \sum_i \|\tilde{h}_{r1}\otimes f_{1k} - \tilde{h}_{ri}\otimes f_{ik}\|^2 = 0. \quad (34)$$

Assuming for the moment that the f_{ik} are known, and denoting "hatted" quantities as the estimate of their "unhatted" counterparts, one can show that

$$\hat{\tilde{h}}_r(n+1) = \hat{\tilde{h}}_r(n) - \mu\left[\frac{\partial V_{rh}(\hat{\tilde{h}}_r(n), f_k)}{\partial \hat{\tilde{h}}_r(n)}\right]^T \quad (35)$$

is globally convergent for sufficiently small μ. Of course, as only estimates of f_k would be available at a given time, we would instead implement

$$\hat{\tilde{h}}_r(n+1) = \hat{\tilde{h}}_r(n) - \mu\left[\frac{\partial V_{rh}(\hat{\tilde{h}}_r(n), \hat{f}_k(n))}{\partial \hat{\tilde{h}}_r(n)}\right]^T. \quad (36)$$

The iteration of (36) comprises the second stage of the algorithm.

Now assume that the $\{\widetilde{H}_{ri}(q^{-1})\}$ are known. Then with $\widetilde{H}_{rm}(q^{-1})$ the highest degree monic polynomial among the $\{\widetilde{H}_{ri}(q^{-1})\}$, a unique set of $\tilde{g}_{pi}$ solve

$$V_{pg,i}(\tilde{h}_{rm}, \tilde{g}_{pi}, f_{mi}) \stackrel{\text{def}}{=} \|\tilde{g}_{p1}\otimes\tilde{h}_{rm} - f_{mi}\|^2 = 0. \quad (37)$$

For each i, we then implement as the steps in the third stage of the algorithm

$$\hat{\tilde{g}}_{pi}(n+1) = \hat{\tilde{g}}_{pi}(n) - \mu\left[\frac{\partial V_{pg,i}(\hat{\tilde{h}}_{rm}(n), \hat{\tilde{g}}_{pi}(n), \hat{f}_{mi}(n))}{\partial \hat{\tilde{g}}_{pi}(n)}\right]^T. \quad (38)$$

In principle, these together estimate $\widetilde{H}_r(q^{-1})$ and $\widetilde{G}_p(q^{-1})$.

The final stage of the algorithm is to extract the $G_i(q^{-1})$ and $H_i(q^{-1})$. To do this, one can use the coprimeness condition and (11) to formulate similar algorithms for estimating $\widetilde{G}_{p-1}(q^{-1})$ and $G_p(q^{-1})$. Continuing in this vein one obtains, one after the other, steps for estimating each $G_i(q^{-1})$. Similarly one formulates algorithms for estimating $H_i(q^{-1})$. Space constraints prevent the provision of further details regarding these steps.

The convergence properties of this multistep algorithm are summarized below.

Theorem 2 *Let ν be the highest degree among the polynomial elements of $\widetilde{H}_r(q^{-1})\widetilde{G}_p(q^{-1})$ and $N = \widetilde{M}_p = \widetilde{L}_r$. If there exist $\alpha_1, \alpha_2, T > 0$ such that for all n and all $0 \le i \le N-1$,*

$$\alpha_1 I \le \sum_{j=n}^{n+T} U_i(n)U_i(n)^T \le \alpha_2 I, \quad (39)$$

where

$$U_i(n) = \begin{bmatrix} u(nN+i) \\ u((n-1)N+i) \\ \vdots \\ u((n-\nu)N+i) \end{bmatrix}, \quad (40)$$

then for every initial error there exists μ^* such that the multistep algorithm given above with step size μ in each step is exponentially convergent for all $0 < \mu \le \mu^*$.

5. CONCLUSION

This paper's contributions are an an algorithm formulation that adaptively optimizes the coefficients of a multistage multirate filter, and the detailing of a sufficient condition for the filter's identifiability. Thes algorithm is exponentially convergent in the exact modelling case, given identifiability and suitable input excitation.

REFERENCES

[1] R.E. Crochiere and L.R. Rabiner, *Multirate Digital Signal Processing.* Englewood Cliffs, NJ: Prentice-Hall, 1983.

[2] P.P. Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[3] R.G. Shenoy, D. Burnside, and T.W. Parks, "Linear periodic systems and multirate filter design," *IEEE Trans. Signal Processing,* vol. 42, pp. 2242-2256, Sept. 1994.

[4] P.A. Regalia and D.-Y. Huang, "Attainable error bounds in multirate adaptive lossless FIR filters," *Proc. ICASSP 95,* vol. 2, pp. 1460-1463, May 1995.