Some New Results on Robust Nonlinear Forwarding

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Abstract. This paper considers the problem of robust stabilization for a class of uncertain nonlinear systems which involve a base system with a control input and a forwarding structure. Both parts of the system are allowed to be nonlinear, multidimensional, and containing uncertain parameters. We present a new approach to design stabilizing controllers which assure both robust global asymptotic stability and local quadratic stability. The main assumptions required for such a robust stabilizing controller to exist are quadratic stabilizability for the local linearized model of the system, global asymptotic stabilizability for the base system and some mild conditions on the forwarding structure and the nonlinearity of the system.

Keywords: Nonlinear control; Forwarding; Robust control; Quadratic stabilization.

1 Introduction

In this paper, we address the robust stabilization problem for a class of cascaded nonlinear systems with the so-called *forwarding* or *feedforwarding* structure. More precisely, these systems have the following model:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, q)\\ \dot{x}_2 &= f_2(x_2, q) + B_2(x_2, q)u\end{aligned} \tag{1}$$

where $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ are state variables, $u \in \mathbf{R}^m$ is a control input, q is an uncertain parameter vector belonging to a compact set $Q \subset \mathcal{R}^p$, the nonlinear functions $f_1(x_1, x_2, q)$, $f_2(x_2, q)$ and $B_2(x_2, q) \in \mathbf{R}^{n_2 \times m}$ are smooth in x_1, x_2 and continuous in q.

Many similar cases of (1) have been studied; see, e.g., [1, 2, 3, 4, 6]. The work in these papers have led to several design control methods. In [6], a saturation function and input-output method are used to design a global asymptotic stabilizer for an upper-triangular system. This method allows some uncertainties in high order (nonlinear) terms in a system but the accurate knowledge of a local linearized model is required in control design. In [1, 2], cascaded systems similar to that in (1) are studied and two Lyapunov function based design methods are proposed. However, the aforementioned design methods rely on fairly accurate knowledge of the system as well. More precisely, [1] requires accurate knowledge of the system to design a cross term in a Lyapunov function, whereas [2] requires an accurate knowledge of a local linearized model to design a coordinate transformation that is used in control design.

In [3] and [4], a robust control design approach is developed for an upper-triangular nonlinear system with large size uncertainties in both linearized model and the nonlinear part of the system. Apart from providing a robust control design method, the results in [3] and [4] also bridge a gap between linear robust control theory and nonlinear robust control theory in the sense that the design method coincides with the seminal work of Wei [7] on quadratic stabilization of linear systems.

The aim of this paper is to generalize the design approach proposed in [3] and [4] to the system (1). The key difference between (1) and the upper triangular structure studied in [3] and [4] is that both the forwarding state x_1 and the control input u are allowed to be multidimensional in this paper. Note that multidimensional forwarding state and control input are also considered in [2]. However this work has very strict requirements on the admissible uncertainties in the system model. Indeed, [2] requires $f_1(x_1, x_2, q)$ to be the form $f_1(x_1, x_2, q) = A_1(q)x_1 + F_{12}(x_1, x_2, q)x_2$ with a constant matrix $A_1(q)$ and $F_{12}(x_1, 0, q) \equiv 0$. In our design, conditions on $f_1(x_1, x_2, q)$ will be much more relaxed.

As we will indicate later, we also require weaker assumptions on the system but provide a simpler design and proof of stabilizability. Our design method considers two notions of stability: global asymptotic stability and local quadratic stability. The latter requires the existence of a locally quadratic Lyapunov function for the system (1) for all admissible uncertain parameters $q \in Q$. Our stabilizing controller involves two steps: The first controller brings the state of the base system, x_2, to a small neighbourhood of the origin, and the second controller is used to bring both x_1 and x_2 to the origin. A non-quadratic Lyapunov function is used to establish the stability properties.

2 Nonlinear Forwarding System

Denote the local linearized model of the system (1) as

$$\dot{x} = A(q)x + B(q)u \tag{2}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};\ A(q) = \begin{bmatrix} A_1(q) & A_{12}(q) \\ 0 & A_2(q) \end{bmatrix};\ B(q) = \begin{bmatrix} B_1(q) \\ B_2(q) \end{bmatrix}$$

with $B_1(q) = 0$.

Suppose the system (1) satisfies assumptions as below:

Assumption 2.1 (Local Quadratic Stabilizibility): There exist a linear state feedback matrix

$$K = [K_1\ K_2]$$

and a symmetric and positive-definite matrix P_0 such that

$$P_0[A(q) + B(q)K] + [A(q) + B(q)K]^T P_0 < 0,\ \forall q \in Q \tag{3}$$

Without loss of generality, we let

$$P_0 = \begin{bmatrix} P_1 & -P_1 W \\ -W^T P_1 & P_2 + W^T P_1 W \end{bmatrix} \tag{4}$$

for some $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ and W. Equivalently, the quadratic Lyapunov function is given by

$$V_0(x_1, x_2) = (x_1 - Wx_2)^T P_1 (x_1 - Wx_2) + x_2^T P_2 x_2 \tag{5}$$

Assumption 2.2 (Global Asymptotic Stabilizability of the Base System): There exists a locally smooth controller $u_0(x_2)$ such that the system below

$$\dot{x_2} = f_2(x_2, q) + B_2(x_2, q)u_0(x_2) \tag{6}$$

is globally asymptotically stable.

Assumption 2.3 (Smoothness Conditions): We require

$$f_1(x_1, x_2, q) = A_1(q)x_1 + A_{12}(q)x_2 + F_{12}(x, q)x_2 \tag{7}$$

$$f_2(x_2, q) = A_2(q)x_2 + F_2(x_2, q)x_2 \tag{8}$$

for some $F_2(x_2, q)$ which is continuous in q and smooth in x_2 with $F_2(0, q) = 0$, $\forall q \in Q$, and $F_{12}(x, q)$ which is continuous in q and smooth in x and satisfies

$$\max_{q \in Q} \|F_{12}(x, q)\| \leq \gamma_1(x_2)\|x_1\| + \gamma_2(x_2) \tag{9}$$

with some smooth functions $\gamma_i(x_2)$ and $\gamma_i(0) = 0$, $i = 1, 2$.

Assumption 2.4 (Boundedness of the Forwarding State): The matrix P_1 in Assumption 2.1 is such that

$$P_1 A_1(q) + A_1^T(q) P_1 \leq 0; \quad \forall q \in Q. \tag{10}$$

Remark 2.1: It is obvious that Assumption 2.1 is necessary for local quadratic stabilization. Since the quadratic stabilization theory for linear uncertain system is well established, we will not discuss methods of solving the linear quadratic control $u = Kx$ and quadratic Lyapunov function $V_0(x_1, x_2)$.

However, the condition in (10) is required for technical reasons (in the proof of Theorem 3.1). To justify this condition, we note several points: 1) When the forwarding state x_1 is a scalar, this condition merely requires $A_1(q)$ to be non-positive. In fact, $A_1(q) = 0$ in the upper-triangular structure. 2) When x_1 is not a scalar, a condition similar to (10) is often used; see [1, 2]. 3) To show that stabilizability may be impossible without (10), we consider the following example:

$$\begin{array}{rcl} \dot{x}_1 & = & \epsilon x_1 + 2x_2 + x_2^2, \quad \epsilon > 0 \\ \dot{x}_2 & = & u \end{array} \tag{11}$$

It is easy to verify that its local linearized model is stabilizable. However, $2x_2 + x_2^2 \geq -1$, implying that $x_1(t)$ will diverge if $x_1(0) > 1/\epsilon$ regardless what control is used.

Finally, we point out that Assumption 2.4, (7) and (9) guarantee that x_1 is bounded for bounded x_2 (see [1]). □

3 Robust Forwarding

In this section, we design a robust controller for the system (1) under Assumptions 2.1-2.4. The closed-loop system is required to have both global asymptotic stability and local quadratic stability.

The desired controller is a two-step controller. In the first step, the controller $u_0(x_2)$ is applied to drive the state x_2 into a given region Ω. This is achieved in some finite time T. During this period, the forwarding state x_1 is not regulated. But $x_1(t)$, $t \in [0, T]$ is bounded (see Remark 2.1). Once x_2 is inside Ω, a local (nonlinear) controller is applied to drive both x_1 and x_2 to zero while maintaining $x_2 \in \Omega$.

From Assumption 2.1, we choose

$$V_2(x_2) = x_2^T P_2 x_2 \tag{12}$$

as a local quadratic Lyapunov function for the base system and define a local region Ω as:

$$\Omega = \{x_2\ |V_2(x_2) < \mu\} \tag{13}$$

where $\mu > 0$ is to be specified. Denote

$$A(x, q) = \begin{bmatrix} A_1(q) & A_{12}(q) + F_{12}(x, q) \\ 0 & A_2(q) + F_2(x_2, q) \end{bmatrix}. \tag{14}$$

From Assumptions 2.1 and 2.3, we know that the following holds for sufficiently small $\mu > 0$ and $\varepsilon > 0$:

$$P_0[A(x, q) + B(x_2, q)K] + [A(x, q) + B(x_2, q)K]^T P_0 < -\varepsilon I; \tag{15}$$

for all $x_2 \in \Omega$.

Assumptions 2.2-2.4 guarantees that the first controller $u_0(x_2)$ can drive x_2 into Ω in finite time while keeping x_1 bounded. Hence, we assume in the sequel that $x_2(0) \in \Omega$, where $x_2(0)$ is the initial value of $x_2(t)$. Choose

$$V(x_1, x_2) = (x_1 - Wx_2)^T P_1 (x_1 - Wx_2) + \int_0^{V_2(x_2)} s(w)dw \tag{16}$$

as a local Lyapunov function for the system (1), where $s(w)$ is a positive, smooth, and monotonically non-decreasing function for $w \in [0, \mu)$, with

$$\int_0^{V_2} s(w)dw \to \infty; \ \ as \ \ V_2 \to \mu$$

and $s(0) > 1$.

Remark 3.1 A particular choice of $s(\cdot)$ is given by

$$s(w) = \frac{\mu}{\mu - w}.$$

In general, the Lyapunov function (16) is non-quadratic. However, as $x \to 0$, $V(x)$ becomes quadratic in x because $s(0) > 0$ and the smoothness of the function $s(w)$. We also note that the function $\int_0^{V_2(x_2)} s(w)dw$ resembles a "potential barrier" and the Lyapunov function (16) is valid only for $x_2 \in \Omega$, i.e.,

$$V(x_1, x_2) \to \infty \quad as \quad V_2(x_2) \to \mu. \tag{17}$$

This implies that future $x_2 \in \Omega$ as long as that $V(x_1, x_2)$ remains bounded. □

For $x_2 \in \Omega$, the derivative of $V(x_1, x_2)$ along the trajectory of the system (1) is given by

$$\dot{V}(x_1, x_2) = x^T [PA(x, q) + A^T(x, q)P]x + 2x^T PB(x_2, q)u \tag{18}$$

where

$$P = \begin{bmatrix} P_1 & -P_1 W \\ -W^T P_1 & s(V_2)P_2 + W^T P_1 W \end{bmatrix} \tag{19}$$

with its inverse given by

$$S = \begin{bmatrix} P_1^{-1} + s^{-1}(V_2)WP_2^{-1}W^T & s^{-1}(V_2)WP_2^{-1} \\ s^{-1}(V_2)P_2^{-1}W^T & s^{-1}(V_2)P_2^{-1} \end{bmatrix}. \tag{20}$$

Consider the following local controller:

$$\begin{aligned} u_l(x_1, x_2) = {} & s^{-1}(V_2(x_2))K_1 x_1 \\ & + [K_2 + (1 - s^{-1}(V_2(x_2)))K_1 W]x_2 \end{aligned} \tag{21}$$

Then we have the following main result:

Theorem 3.1: Suppose the system (1) satisfies Assumptions 2.1-2.4. Then the closed-loop system controlled by the following controller

$$u = \begin{cases} u_0(x_2), & x_2 \notin \Omega \\ u_l(x_1, x_2), & x_2 \in \Omega \end{cases} \tag{22}$$

is robustly globally asymptotically stable and locally quadratically stable.

Proof: As discussed before, we only need to consider the case when $x_2(0) \in \Omega$ and $u_l(x_1, x_2)$ is applied. Denoting

$$z = [z_1^T \ z_2^T]^T = Px, \tag{23}$$

the equation (18) becomes

$$\dot{V}(x_1, x_2) = z^T [A(x, q)S + SA^T(x, q)]z + 2z^T B(x_2, q)u. \tag{24}$$

Denote $s^{-1}(V_2(x_2))$ by s^{-1} and let

$$u = s^{-1}K_1 x_1 + K_2 x_2 + v. \tag{25}$$

Then,

$$\begin{aligned} \dot{V}(x_1, x_2) = {} & 2z^T (A(x, q) + B(x_2, q)K)Sz \\ & -2(1 - s^{-1})z^T B(x_2, q)[K_1 \ 0]Sz + 2z^T B(x_2, q)v \end{aligned} \tag{26}$$

Rewrite (15) as

$$[A(x, q) + B(x_2, q)K]S_0 + S_0[A(x, q) + B(x_2, q)K]^T < -\epsilon S_0^2, \quad \forall x \in \mathbf{R}^{n_1} \times \Omega; \ \ \forall q \in Q \tag{27}$$

where $S_0 = P_0^{-1}$. Rewriting S as

$$S = s^{-1}S_0 + (1 - s^{-1}) \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

we simplify (26) to

$$\begin{aligned} & \dot{V}(x_1, x_2) \\ = \ & 2s^{-1}z^T (A(x, q) + B(x_2, q)K)S_0 z \\ & +2(1 - s^{-1})z_1^T [A_1(q)P_1^{-1}]z_1 \\ & -2(1 - s^{-1})z^T B(x_2, q)K_1 W x_2 + 2z^T B(x_2, q)v \end{aligned}$$

Applying Assumption 2.4 and choosing

$$v = (1 - s^{-1})K_1 W x_2, \tag{28}$$

we have

$$\begin{aligned} \dot{V}(x_1, x_2) &= 2s^{-1}z^T (A(x, q) + B(x_2, q)K)S_0 z \\ &= 2s^{-1}(S_0 Px)^T P_0 (A(x, q) + B(x_2, q)K)(S_0 Px) \end{aligned} \tag{29}$$

From Assumption 2.1, we can see that

$$\dot{V}(x_1, x_2) < 0; \ \ \forall x \in \mathbf{R}^{n_1} \times \Omega - \{0, 0\}. \tag{30}$$

This implies that

$$V(x_1(t), x_2(t)) \le V(x_1(0), x_2(0)), \ \forall t \ge 0.$$

Using (16) and monotonicity of $s(\cdot)$, we have

$$V_2(x_2(t)) \le V(x_1(0), x_2(0))/s(0) =: \rho$$

Hence, (29) leads to

$$\dot{V}(x_1, x_2) \le -\hat{\varepsilon} s^{-1}(\rho) x^T x \tag{31}$$

with

$$\hat{\varepsilon} = \varepsilon \min_{V_2(x_2) \le \rho} \lambda_{min}^2(S_0 P).$$

Therefore, the system (1) is robustly globally asymptotically stabilizible. Finally, the robust local quadratic stability property follows from (31) and the fact that $V(x_1, x_2)$ becomes quadratic as $x_2 \to 0$. ∇∇∇

4 Block Upper-Triangular Systems

In this section, we provide some extensions to Theorem 3.1. The first extension is concerned with the following nonlinear system:

$$\begin{array}{l} \dot{x}_1 = f_1(x_1, x_2, u, q) \\ \dot{x}_2 = f_2(x_2, u, q) \end{array} \qquad (32)$$

where x_1, x_2 and q are as in (1), $f_1(x_1, x_2, u, q)$ and $f_2(x_2, u, q)$ are nonlinear functions. The local linearized model is given by (2) with $B_1(q) \neq 0$ in general.

The assumptions required for robust stabilization are the same as in Assumptions except for the following simple changes: (6) should read as

$$\dot{x}_2 = f_2(x_2, u_0(x_2), q)$$

and the x_2 in (7) and (9) should be interpreted as $[x_2^T, u^T]^T$.

In order to solve the robust stabilization problem for (32), we introduce a new state vector x_3

$$x_3 = u \qquad (33)$$

and a new control input

$$v = \dot{x}_3 \qquad (34)$$

Substituting (33) and (34) into (32), the system can be rewritten as

$$\begin{array}{l} \dot{x}_1 = f_1(x_1, x_2, x_3, q) \\ \dot{x}_2 = f_2(x_2, x_3, q) \\ \dot{x}_3 = v \end{array} . \qquad (35)$$

Its local linearized model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1(q) & A_{12}(q) & B_1(q) \\ 0 & A_2(q) & B_2(q) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} v \qquad (36)$$

Lemma 4.1 Suppose Assumption 2.1 holds. Then, the local linearized model (36) is quadratically stabilizable by

$$v = -\gamma(x_3 - Kx) \qquad (37)$$

for a sufficiently large $\gamma > 0$ with $x = [x_1, x_2]^T$. Furthermore, P_1 can still be the $(1,1)$-block of the Lyapunov matrix for (36).

Proof From Assumption 2.1, we know that

$$[(A(q) + B(q)K]^T(P_0 + \delta P_0) + (P_0 + \delta P_0)[A(q) + B(q)K] < 0$$

for all $q \in Q$, provided that δP_0 is sufficiently small. Choose a Lyapunov function for (36) as follows:

$$V_0(x_1, x_2, x_3) = x^T(P_0 + \delta P_0)x + \epsilon(x_3 - Kx)^T(x_3 - Kx)$$

for some $\epsilon > 0$. Its derivative along the trajectory of (36) is

$$\begin{array}{rcl} \dot{V}_0 & = & 2x^T(P_0 + \delta P_0)(A(q)x + B(q)x_3) \\ & & +2\epsilon(x_3 - Kx)^T(v - K(A(q)x + B(q)x_3)) \end{array}$$

Let $\bar{x}_3 = x_3 - Kx$, we have

$$\begin{array}{rcl} \dot{V}_0 & = & 2x^T(P_0 + \delta P_0)(A(q) + B(q)K)x \\ & & +2x^T(P_0 + \delta P_0)B(q)\bar{x}_3 \\ & & -2\epsilon\bar{x}_3^T K(A(q) + B(q)K)x \\ & & +2\epsilon\bar{x}_3^T(v - KB(q)\bar{x}_3) \end{array}$$

It is easy to see that

$$v = -\gamma\bar{x}_3 = -\gamma(x_3 - Kx)$$

will make $\dot{V}_0 < 0$, $\forall q \in Q$ when $\gamma > 0$ is sufficiently large. Hence, the system (36) is quadratically stabilizable. To verify that P_1 can still be the $(1,1)$-block of the Lyapunov matrix for (36), we simply take $\delta P_0 = -\epsilon K^T K$ and ϵ sufficiently small. This gives $V_0(x_1, 0, 0) = x_1^T P_1 x_1$. $\nabla\nabla\nabla$

The stabilizing controller for (32) again consists of two parts: First, the global controller $u_0(x_2)$ is applied to (32) until x_2 is sufficiently small in some finite time T. Then, consider the system (35) with initial $x_3 = 0$ and treat the sub-system with x_2 and x_3 as the base system. Since $u_0(x_2)$ is locally smooth, $x_3(T)$ should be sufficiently small. Now apply Theorem 3.1 to come up with a local controller $v_l(x_1, x_2, x_3)$. The resulting controller

$$u(t) = x_3(t) = \int_0^t v_l(x_1, x_2, x_3)dt$$

will robustly stabilize the system (32). This is summarized below:

Theorem 4.1 Suppose the system (32) satisfies Assumptions 2.1-2.4 with the modifications mentioned earlier in this section. Then the system is robustly globally asymptotically stabilizible and locally quadratically stabilizable.

Proof. The proof simply follows from Theorem 3.1 and Lemma 4.1. $\nabla\nabla\nabla$

The second extension involves a more general class of uncertain nonlinear systems that our approach can deal with. These systems are given by the following block forwarding structure:

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, \cdots, x_k, u, q) \\ & & \cdots \\ \dot{x}_{k-1} & = & f_{l-1}(x_{k-1}, x_k, u, q) \\ \dot{x}_k & = & f_l(x_k, u, q) \end{array} \qquad (38)$$

where $x_1 \in \mathbf{R}^{n_1}, \cdots, x_k \in \mathbf{R}^{n_k}$ are the state vectors, and other terms are defined similarly as before. It is tedious but straightforward to determined the conditions required for robust control design. The details are thus omitted.

5 Extension of Base Systems

In this section, we consider the stabilization problem for the system (1) in which the base system is in a more general form as below:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, q)\\ \dot{x}_2 &= f_{21}(x_1, x_2, q) + f_2(x_2, q) + B_2(x_2, q)u\end{aligned} \quad (39)$$

Suppose the system (39) satisfies the following assumptions:

Assumption 5.1 (Matching Condition of the Base Systems) For the base system of the system (39), there exist $B_2(x_2)$, $d_{21}(x_1, x_2, q)$ and $\Delta_b(x_1, x_2, q)$ smooth in x_1, x_2 and continuous in q such that, for all $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$ and $q \in Q$,

$$f_{21}(x_1, x_2, q) = B_2(x_2)d_{21}(x_1, x_2, q). \quad (40)$$

with $d_{21}(0, 0, q) \equiv 0$, $\forall q \in Q$, and

$$B_2(x_2, q) = B_2(x_2)[I + \Delta_b(x_1, x_2, q)] \quad (41)$$

with $\|\Delta_b(x_1, x_2, q)\| < \delta_b$ for a positive constant $\delta_b \in (0,\ 1)$.

Assumption 5.2 (Global Asymptotic Stabilizability of the Base System): There exist a smooth controller $u_0(x_2)$ and a Lyapunov function $W_2(x_2)$ such that

$$\frac{\partial W_2(x_2)}{\partial x_2}[f_2(x_2, q) + B_2(x_2, q)u_0(x_2)] < -\varepsilon_2\|x_2\|^2 \quad (42)$$

for some position constant $\varepsilon_2 > 0$.

Lemma 5.1 Suppose the base system of the system (39) satisfies Assumptions 5.1 and 5.2. Then, for any bounded state x_1, the controller

$$\begin{aligned}\bar{u}_0(x_1, x_2) &= u_0(x_2)\\ &-\frac{\lambda_{max}(P_2)}{4\varepsilon_2\mu(1-\delta_b)}B_2^T(x_2)\frac{\partial^T W_2}{\partial x_2}(\delta_{21}^2(x_1, x_2) + 1)\end{aligned} \quad (43)$$

where $\|d_{21}(x_1, x_2, q)\| \le \delta_{21}(x_1, x_2)$ and $\lambda_{max}(\cdot)$ denotes as the maximum eigenvalue, drives the state x_2 into a local region Ω after a finite time T which is independent of x_1. Further the state x_2 is bounded by a constant $\Gamma(x_2(0))$, i.e., $\|x_2(t)\| \le \Gamma(x_2(0))$, $\forall t > 0$.

Proof : Omitted.

Lemma 5.2 Suppose the x_1-subsystem of the system (39) satisfies Assumptions 2.3 and 2.4. For any positive constant Γ and T, if the state $x_2(t)$, $t \le [0, T]$ is bounded by Γ, i.e., $\|x_2(t)\| \le \Gamma$, $t \le [0, T]$, the state $x_1(t)$, $t \in [0, T]$ is bounded by smooth function

$$\|x_1(t)\| \le \left(\frac{V_1(x_1(0))}{\lambda_{min}(P_1)}\right)^{1/2} e^{\alpha T/2}; \quad \forall t \in [0, T] \quad (44)$$

where

$$\begin{aligned}\alpha = \frac{2\|P_1\|}{\lambda_{min}(P_1)} \max_{\substack{\|x_2\| \le \Gamma\\ q \in Q}} &[\|A_{12}(q)x_2\|\\ &+[\gamma_1(x_2) + \gamma_2(x_2)]\|x_2\|]\end{aligned} \quad (45)$$

and $\lambda_{min}(\cdot)$ denotes as the minimum eigenvalue.

Proof: Denote $V_1(x_1) = x_1^T P_1 x_1$. Since the x_1-subsystem of the system (39) satisfies Assumptions 2.3 and 2.4, there holds

$$\begin{aligned}\dot{V}_1(x_1) &= 2x_1^T P_1 A_1(q)x_1 + 2x_1^T P_1 A_{12}(q)x_2\\ &\quad +2x_1^T P_1 F_{12}(x_2, q)x_2\\ &\le 2x_1^T P_1 A_{12}(q)x_2 + 2x_1^T P_1 F_{12}(x_2, q)x_2\\ &\le 2\|x_1\|\|P_1 A_{12}(q)x_2\| + 2\|x_1\|\|P_1\|(\gamma_1(x_2)\|x_1\|\\ &\quad +\gamma_2(x_2))\|x_2\|\end{aligned} \quad (46)$$

When $\|x_1\| \ge 1$, the inequality (46) can be written as

$$\dot{V}_1(x_1) \le \alpha V_1(x_1). \quad (47)$$

Then it can be easily checked that $\left(\frac{V_1(x_1(0))}{\lambda_{min}(P_1)}\right)^{1/2} e^{\alpha T/2}$ is an upper bound of the state $x_1(t)$, $\forall t \in [0, T]$. ∇∇∇

Lemma 5.3 The controller $\bar{u}_0(x_1, x_2)$ in (43) guarantees that, for any initial state $(x_1(0), x_2(0))^T$, the state $(x_1(t), x_2(t))^T$ $t \in [0, T]$ of the system (39) is bounded by a constant $C(x_1(0), x_2(0), T)$.

Proof: Suppose $\bar{u}_0(x_1, x_2)$ is applied to the system (39). Then we claim that $(x_0(t), x(t))$ is bounded by a constant $C(x_1(0), x_2(0), T)$ for all $t \le T$. Indeed, if this is incorrect, there must be a $t_0 \le T$ such that $\|(x_1(t), x_2(t))\| < \infty$, $\forall t \in [0, t_0)$ and $\|(x_1(t), x_2(t))\| \to \infty$ as $t \to t_0$. From Lemma 5.2 we realise that $x_2(t) \to \infty$ as $t \to t_0$ because if $x_2(t)$ were bounded, so would be $x_1(t)$. Since $x_2(t)$ is continuous on t, $t \in [0, t_0)$, for any given positive constant c, there is a positive $\bar{\varepsilon}$ such that $\|x_2(t))\| > c$ when $t \in [t_0 - \bar{\varepsilon}, t_0)$ and $\max_{t\in[0,t_0-\bar{\varepsilon}]}\|x_2(t)\| > c$. Then, from the Lemma 5.2, $x_1(t)$ is bounded $t \in [0, t_0 - \bar{\varepsilon}]$. Now, we choose $c = \Gamma(x_2(0))$ in Lemma 5.1, then there is a $\bar{\varepsilon}_0$ such that $\max_{t\in[0,t_0-\bar{\varepsilon}_0]}\|x_2(t)\| > \Gamma(x_2(0))$ and $x_1(t)$ is bounded $t \in [0, t_0 - \bar{\varepsilon}_0]$. This contradicts Lemma 5.1. Therefore, $(x_1(t), x_2(t))$ is bounded for any $t \le T$. Then, from Lemma 5.1 again, $\|x_2(t)\| \le \Gamma(x_2(0))$, $\forall t \le T$. Hence, for any $t \in [0, T]$,

$$\|x_1(t)\|^2 \le \frac{V_1(x_1(0))}{\lambda_{min}(P_1)}e^{\alpha(x_2(0))T}; \quad \forall t \in [0, T] \quad (48)$$

where

$$\begin{aligned}\alpha(x_2(0)) = \frac{2\|P_1\|}{\lambda_{min}(P_1)} \max_{\substack{\|x_2\| \le \Gamma(x_2(0))\\ q \in Q}} &[\|A_{12}(q)x_2\|\\ &+(\gamma_1(x_2) + \gamma_2(x_2))\|x_2\|]\,.\end{aligned} \quad (49)$$

Hence, $\|(x_1(t), x_2(t))\| \le C(x_1(0), x_2(0), T)$ if we choose $C(x_1(0), x_2(0), T) = \Gamma(x_2(0)) + \left(\frac{V_1(x_1(0))}{\lambda_{min}(P_1)}\right)^{1/2} e^{\alpha(x_2(0))T/2}$. ∇∇∇

Since $d_{21}(x_1, x_2, q)$ is smooth in x_1, x_2 and $d_{21}(0, 0, q) \equiv 0$, $\forall q \in Q$, there exists $D_{21}(x_1, x_2, q)$ such that $d_{21}(x_1, x_2, q) = D_{21}(x_1, x_2, q)x$ where $D_{21}(x_1, x_2, q)$ is bounded by a smooth function $\bar{\delta}_{21}(x_1, x_2)$, i.e., $\|D_{21}(x_1, x_2, q)\| \leq \bar{\delta}_{21}(x_1, x_2)$, $\forall x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}, q \in Q$.

Theorem 5.1 Suppose the system (39) satisfies Assumptions 5.1, 5.2, 2.3 and 2.4. Then the closed-loop system controlled by the following controller

$$u = \begin{cases} \bar{u}_0(x_1, x_2), & x_2 \notin \Omega \\ \bar{u}_l(x_1, x_2), & x_2 \in \Omega \end{cases} \tag{50}$$

where

$$\bar{u}_l(x_1, x_2) = u_l(x_1, x_2) + \bar{v} \tag{51}$$

and

$$\bar{v} = -\frac{s(\rho)}{\hat{\varepsilon}(1-\delta_b)} B^T(x_2) P x \delta_{21}^2(x_1, x_2) \tag{52}$$

where $B(x_2) = (0\ B_2^T(x_2))^T$, is robustly globally asymptotically stable and locally quadratically stable.

Proof Suppose the initial condition $x_2(0)$ is in the given local region Ω, (29) can be rewritten as

$$\begin{aligned} \dot{V}(x_1, x_2) &= 2s^{-1} x^T P_0 (A(x, q) + B(x_2, q)K)x \\ &\quad + 2z^T B(x_2, q) D_{21}(x_1, x_2, q) S z + 2z^T B(x_2, q)\bar{v} \\ &\quad < -\frac{\hat{\varepsilon}}{2}\|x\|^2. \end{aligned} \tag{53}$$

Suppose the initial condition $x_2(0)$ is not in the given local region Ω. Then applying Lemmas 5.1 and 5.2, the state x_2 is driven by the controller $\bar{u}_0(x_1, x_2)$ into the local region Ω after a finite time T and, further, the state $(x_1(t), x_2(t))$, $t \in [0, T]$ is bounded by a constant $C(x_1(0), X_2(0), T)$. Further, considering (52), the state $(x_1(t), x_2(t))$, $t > 0$ is bounded by a constant $\Gamma^+(x_1(0), x_2(0))$. Furthermore, the system (39) is robustly globally asymptotically and locally quadratically stabilizied by the controller (50). $\nabla\nabla\nabla$

Remark 5.1 Theorem 5.1 can be recursively applied to solve the robust stabilization problem for the system in a form as below:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \cdots, x_k, q) \\ &\cdots \\ \dot{x}_{k-1} &= f_{k-1}(x_{k-1}, x_k, q) \\ \dot{x}_k &= f_{k1}(x_1, \cdots, , x_k, q) + f_k(x_k, q) + B_k(x_1, \cdots, , x_k, q)u \end{aligned} \tag{54}$$

where $f_{k1}(x_1, \cdots, x_k, q)$ and $B_k(x_1, \cdots, , x_k, q)$ satisfy the matching condition. The system (54) is a general form of the system (1) in [3]. $\square$

6 Conclusions

In this paper, we have studied the robust stabilization problem for a class of uncertain nonlinear systems in a block forwarding structure. A new design method is introduced to achieve both robust global asymptotic stability and local quadratic stability. This method is generalized from [3, 4], enabling us to simplify the design process and the required assumptions. It should be noted that this method can be combined with the backstepping design method to give a recursive design for robust controllers for a much larger class of uncertain nonlinear systems involving both forwarding and backstepping structures; see [5] for details.

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