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Brief Paper

Robust Nonlinear $\mathcal{H}_\infty$ Filtering

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Key Words—$\mathcal{H}_\infty$ filtering; robust estimation; nonlinear filters.

Abstract—This paper investigates the robust nonlinear $\mathcal{H}_\infty$ filtering problem for nonlinear systems with uncertainties which are described by integral functional constraints. The objective is to design a dynamic filter such that the $\mathcal{L}_2$-gain from an exogenous input to an estimation error is minimized or guaranteed to be less or equal to a prescribed value for all admissable uncertainties. We establish the interconnection between the robust nonlinear $\mathcal{H}_\infty$ filtering problem and the nonlinear $\mathcal{H}_\infty$ filtering problem for known systems, i.e. systems without uncertainties. Using the existing nonlinear $\mathcal{H}_\infty$ filtering results for known systems, we solve the robust nonlinear $\mathcal{H}_\infty$ filtering problem in terms of Hamilton–Jacobi inequalities. Copyright ©1996 Elsevier Science Ltd.

1. *Introduction*

Over the past several years, the problem of $\mathcal{H}_\infty$ filtering for linear systems has received considerable attention. A number of approaches have been proposed (for example, Kwakernaak, 1986; Grimble, 1988; Limebeer and Shaked, 1991). This problem can be stated as follows: given a dynamic system with exogenous input and measured output, design a filter to estimate an unmeasured output such that the mapping from the exogenous input to the estimation error is minimized or no larger than some prescribed level in terms of the $\mathcal{H}_\infty$ norm. In Nagpal and Khargonekar (1991) and Basar (1991), it has been shown that the existence of a solution to the $\mathcal{H}_\infty$ filtering problem is in fact related to the solvability of an appropriate algebraic Riccati equation. This result is then extended in Fu *et al.* (1992) to a class of linear systems which are subject to parametric uncertainty. A sufficient condition for the existence of a solution is derived also via algebraic Riccati equations.

At the same time, the problem of the nonlinear $\mathcal{H}_\infty$ control problem has been studied by a number of authors (see, for example, Ball and Helton, 1989; Basar and Olsder, 1982; van der Schaft, 1991; Isidori and Astolfi, 1992; Isidori, 1991). There are two commonly used approaches for providing solutions to nonlinear $\mathcal{H}_\infty$ control problems. One is based on the dissipativity theory and theory of differential games (see Basar and Bernhard, 1991; Ball and Helton, 1989). Another is based on the nonlinear version of the classical Bounded Real Lemma as developed by Willems (1972) and Hill and Moylan (1980) (see, for example, van der Schaft, 1991; Isidori and Astolfi, 1992; Isidori, 1991). Both of these approaches convert the problem of nonlinear $\mathcal{H}_\infty$ control to the solvability of the so-called Hamilton–Jacobi equation (HJE). A nice feature of these results is that they are parallel to the linear $\mathcal{H}_\infty$ results. Further research along the line of the dissipativity theory and theory of differential games has been attempted (see, e.g. Ball *et al.*, 1993; Isidori and Astolfi, 1992; Isidori, 1991) where results on disturbance attenuation for nonlinear systems via state feedback and/or output feedback have been provided. In Nguang and Fu (1994) and Berman and Shaked (1995) solutions to the nonlinear $\mathcal{H}_\infty$ filtering problem have been obtained.

The motivation of this paper stems from the fact that all the nonlinear $\mathcal{H}_\infty$ results cited above assume that the model is perfectly known (without uncertainty). We consider nonlinear systems subject to uncertainties that are described by an integral functional constraint and input disturbance. The problem addressed here is to design a nonlinear dynamic estimator, such that the estimation error dynamics are Lyapunov stable, and achieve a prescribed level of disturbance attenuation for all admissiable uncertainties.

Our first main result establishes the equivalence between a robust nonlinear $\mathcal{H}_\infty$ filtering problem and nonlinear $\mathcal{H}_\infty$ filtering for a system without uncertainty. This allows us to solve the robust nonlinear $\mathcal{H}_\infty$ filtering problem via existing nonlinear $\mathcal{H}_\infty$ filtering techniques (Nguang and Fu, 1994; Berman and Shaked, 1995). The second main result provides a complete solution to the robust nonlinear $\mathcal{H}_\infty$ filtering problem in terms of two 'scaled' HJEs. This result can be viewed as a generalization of robust linear $\mathcal{H}_\infty$ filtering results in Fu *et al.* (1992) to a class of nonlinear systems with uncertainties.

2. *System description and problem formulation*

Consider a smooth uncertain nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x}(t) &= A(x) + \Delta A(x) + B(x)w(t), \quad x(0) = 0, \\ y(t) &= C(x) + \Delta C(x) + D(x)w(t), \\ z(t) &= L(x), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^m$ is the measured output, $w(t) \in \mathbb{R}^m$ is the exogenous input noise, $z(t) \in \mathbb{R}^p$ is the signal to be estimated, $A(x), B(x), C(x), D(x)$ and $L(x)$ are known C^2 matrix functions with appropriate dimensions, $A(0) = 0, C(0) = 0$, and $L(0) = 0$. $\Delta A(x)$ and $\Delta C(x)$ represent the uncertainties in the system.

Assumption 1.

$$\begin{bmatrix} \Delta A(x) \\ \Delta C(x) \end{bmatrix} = \begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix} F(x,t)E(x), \tag{2}$$

where $H_1(x), H_2(x)$ and $E(x)$ are known matrix functions that characterize the structure of the uncertainties and $E(0) = 0$. Further, the following integral functional constraint

$$\int_0^\infty (\|E(x)\|^2 - \|F(x,t)E(x)\|^2)\mathrm{d}t \geqslant 0 \tag{3}$$

holds and $[D(x) \;\; H_2(x)][D(x) \;\; H_2(x)]' > 0, \; \forall x \in \mathbb{R}^n$.

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We use the same definition of finite $\mathcal{L}_2$-gain as in van der Schaft (1992)

Definition 1. Given any $\gamma > 0$, the mapping from $\omega(t)$ to $z(t)$ is said to have $\mathcal{L}_2$-gain less than or equal to γ if

$$\int_0^T \|z(t)\|^2 \mathrm{d}t \le \gamma^2 \int_0^T \|w(t)\|^2 \mathrm{d}t, \tag{4}$$

for all $T \ge 0$ and all $w \in \mathcal{L}_2(0, T)$, where $\|\cdot\|$ denotes the Euclidean norm.

In analogy to the robust linear $\mathcal{H}_\infty$ filtering theory (see, e.g. Fu *et al.*, 1992), we define the robust nonlinear $\mathcal{H}_\infty$ filtering problem as follows.

Robust nonlinear $\mathcal{H}_\infty$ filtering problem. Given any $\gamma > 0$, find a filter of the form

$$\begin{aligned} \dot{\xi}(t) &= a(\xi) + b(\xi)y(t), \quad \xi(t) \in \mathbb{R}^n, \quad \xi(0) = 0, \\ \hat{z}(t) &= l(\xi), \quad \hat{z}(t) \in \mathbb{R}^p, \end{aligned} \tag{5}$$

where $\xi(t) \in \mathbb{R}^n$ is the state of the filter, $\hat{z}(t) \in \mathbb{R}^p$ is the estimate of $z(t)$, $a(\xi), b(\xi)$ and $l(\xi)$ are C^2 matrix functions with appropriate dimensions, $a(0) = 0$ and $l(0) = 0$. The objective is such that the $\mathcal{L}_2$-gain from the disturbance $w(t)$ to the estimation error $z(t) - \hat{z}(t)$ for the augmented system (1) with (5) is less than or equal to γ, i.e.

$$\int_0^T \|z(t) - \hat{z}(t)\|^2 \mathrm{d}t \le \gamma^2 \int_0^T \|w(t)\|^2 \mathrm{d}t, \tag{6}$$

for all $T \ge 0$, all $w \in \mathcal{L}_2(0, T)$ and all admissible uncertainties. Without loss of generality, we assume $\gamma = 1$ in the sequel.

Before imposing the second assumption on system (1) with (5), we need the following definition.

Definition 2. A system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad x(0) = 0, \\ \eta(t) &= h(x(t)) \end{aligned} \tag{7}$$

is said to be *responsive* if there exists some $u(\cdot)$ such that $\eta(T) \neq \eta(0)$ for some $T > 0$.

Assumption 2. The following two systems generated from (1) with (5) are responsive:

$$\begin{aligned} \dot{x} &= A(x) + B(x)w(t), \quad x(0) = 0, \\ \eta &= E(x) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \dot{x} &= A(x) + H_1(x)v, \quad x(0) = 0, \\ \dot{\xi} &= a(\xi) + b(\xi)C(x) + b(\xi)H_2(x)v, \quad \xi(0) = 0, \\ \eta &= l(\xi) - L(x). \end{aligned} \tag{9}$$

Remark 1. Note that the assumption above is very mild. In particular, (8) being responsive is natural because otherwise the disturbance terms in (1) vanishes. Also, (9) being responsive roughly means that v can influence the state (x, ξ) through $H_1(x)$ and $H_2(x)$. Note that v represents the uncertainty. If (9) is not responsive then the uncertainty terms in (1) will vanish.

3. *Review of nonlinear $\mathcal{H}_\infty$ filtering*

The following nonlinear system has been considered in Nguang and Fu (1994):

$$\begin{aligned} \dot{x}(t) &= A(x) + B(x)w(t), \quad x(0) = 0, \\ y(t) &= C(x) + D(x)w(t), \\ z(t) &= L(x), \end{aligned} \tag{10}$$

which is the same as (1) except that $\Delta A(x)$ and $\Delta C(x)$ are void here. The following theorem provides sufficient condition for the existence of a solution to nonlinear $\mathcal{H}_\infty$ filtering problem.

Theorem 1. Consider the system (10), then the nonlinear $\mathcal{H}_\infty$ filtering problem has a solution if there exist a nonnegative scalar function $\epsilon(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ and a matrix function $b(\xi) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$ for (5) satisfying the following condition:

$$HJ(x, \xi) \triangleq AHJ(x, \xi) + b_e(x, \xi)e(x)b_e^t(x, \xi) \le 0, \tag{11}$$

for all $x, \xi \in \mathbb{R}^n$, where $AHJ(x, \xi)$ is defined as

$$\begin{aligned} AHJ(x, \xi) &\triangleq [\nabla_x^t \epsilon(x, \xi) \;\; \nabla_\xi^t \epsilon(x, \xi)]\{A_e(x, \xi) \\ &\quad + F_e(x, \xi)\nabla_x \epsilon(\xi, \xi)\} \\ &\quad + \frac{1}{4}[\nabla_x^t \epsilon(x, \xi) \;\; \nabla_\xi^t \epsilon(x, \xi)]B_e(x, \xi)B_e^t(x, \xi)\begin{bmatrix} \nabla_x \epsilon(x, \xi) \\ \nabla_\xi \epsilon(x, \xi) \end{bmatrix} \\ &\quad - \frac{1}{4}\nabla_x^t \epsilon(\xi, \xi)B(\xi)D^t(\xi)e^{-1}(x)D(\xi)B^t(\xi)\nabla_x \epsilon(\xi, \xi) \\ &\quad - C_e^t(x, \xi)e^{-1}(x)C_e(x, \xi) \\ &\quad + L_e^t(x, \xi)L_e(x, \xi) + \frac{1}{2}\nabla_x^t \epsilon(\xi, \xi)B(\xi)D^t(\xi)e^{-1}(x)[C(x) - C(\xi)] \end{aligned} \tag{12}$$

with

$$\begin{aligned} A_e(x, \xi) &= \begin{bmatrix} A(x) - B(x)D^t(x)e^{-1}(x)[C(x) - C(\xi)] \\ A(\xi) \end{bmatrix}, \\ L_e(x, \xi) &= L(x) - L(\xi), \\ F_e(x, \xi) &= \begin{bmatrix} \frac{1}{2}B(\xi)D^t(\xi)e^{-1}(x)D(\xi)B^t(\xi) \\ \frac{1}{2}B(\xi)B^t(\xi) \end{bmatrix}, \\ e(x) &= D(x)D^t(x), \\ C_e(x, \xi) &= C(x) - C(\xi), \\ B_e^t(x, \xi) &= \begin{bmatrix} B(x)[I - D^t(x)e^{-1}(x)D(x)]^{\frac{1}{2}} & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} b_e^t(x, \xi) &= \frac{1}{2}\nabla_\xi^t \epsilon(x, \xi)b(\xi) + (\frac{1}{2}\nabla_x^t \epsilon(x, \xi)B(x)D^t(x) \\ &\quad + C^t(x) - C^t(\xi))e^{-1}(x). \end{aligned}$$

If this is the case, then a suitable filter of the form (5) is given by

$$a(\xi) = A(\xi) - b(\xi)C(\xi), \tag{13}$$

$$l(\xi) = L(\xi). \tag{14}$$

Remark 2. Under some mild assumptions on $\epsilon(x, \xi)$, $AHJ(x, \xi) \le 0 \;\; \forall x, \xi \in \mathbb{R}^n$ is a necessary condition for the existence of a solution to nonlinear $\mathcal{H}_\infty$ filter problem, where $AHJ(x, \xi)$ is defined in (12) (for details, see Nguang and Fu, 1994). The gap between the necessary condition and sufficient condition is well recognized and is due to the nature of nonlinear systems. This gap disappears for linear systems.

4. *Main result*

In this section, we show that the problem of the robust nonlinear $\mathcal{H}_\infty$ filtering problem is solvable if and only if the nonlinear $\mathcal{H}_\infty$ filtering problem for a scaled system is solvable. This in fact leads to the solvability of some scaled Hamilton–Jacobi inequalities. Our solution is obtained using a recent result on S-procedure for nonlinear systems (Savkin and Petersen, 1993).

4.1. *Review of S-procedure for nonlinear systems.* Recently Savkin and Petersen (1993) have extended the so-called S-procedure (Yakubovich, 1971) to a very general set of integral functionals defined over the space of solutions to a stable nonlinear time-invariant systems. The nonlinear time-invariant systems they consider are of the form

$$\dot{x}(t) = \phi(x(t), w(t)), \tag{15}$$

where $x(t) \in \mathbb{R}^n$ is the state and $w(t) \in \mathbb{R}^m$ is the input. Associated with (15) is the following set of integral functionals:

$$f_s(x(\cdot), w(\cdot)) = \int_0^\infty \mu_s(x(t), w(t))dt, \quad s = 0, \ldots, k \tag{16}$$

which satisfy the following assumptions:

A.1 The function $\phi(\cdot, \cdot)$ is continuous.

A.2 For all $\{x(\cdot), w(\cdot)\} \in \mathcal{L}_2(0, \infty)$, the corresponding integral functionals defined in (16) are finite.

A.3 For any given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $w(\cdot) \in \mathcal{L}_2(0, \infty)$ and any initial condition $x_0 \in \{x_0 \in \mathbb{R}^n : \|x_0\| \leq \delta\}$, the following condition holds:

$$|f_s(x_1(\cdot), w(\cdot)) - f_s(x_2(\cdot), w(\cdot))| < \varepsilon, \quad s = 0, 1, 2, \ldots, k, \tag{17}$$

where $x_1(\cdot)$ and $x_2(\cdot)$ are the trajectories of the system (15) corresponding to the initial conditions $x(0) = x_0$ and $\xi(0) = 0$, respectively.

Lemma 1. Savkin and Petersen (1993) consider the system (15) and suppose the associated functionals (16) satisfying conditions A.1–A.3. Then $f_0(x(\cdot), w(\cdot)) \geq 0$ for all pairs $\{x(\cdot), w(\cdot)\} \in \mathbb{R}^n \times \mathbb{R}^m$ subject to $f_s(x(\cdot), w(\cdot)) \geq 0$, $s = 1, \ldots, k$, if and only if there exist $\tau_s \geq 0$, $s = 0, \ldots, k$ with $\sum_{s=0}^k \tau_s > 0$, such that

$$\begin{aligned}\tau_0 f_0 \ (x(\cdot), w(\cdot)) &\geq \tau_1 f_1(x(\cdot), w(\cdot)) \\ &+ \tau_2 f_2(x(\cdot), w(\cdot)) + \ldots + \tau_k f_k(x(\cdot), w(\cdot)),\end{aligned} \tag{18}$$

for all $w(\cdot) \in \mathcal{L}_2(0, \infty)$ and $x(\cdot)$ satisfying (15).

4.2. *Analysis for the robust nonlinear $\mathcal{H}_\infty$ filtering problem.* Given the system (1) satisfying Assumption 1, we define the following scaled system:

$$\begin{aligned}\dot{\tilde{x}}(t) &= A(\tilde{x}) + [B(\tilde{x}) \ \ \tfrac{1}{\tau}H_1(\tilde{x})]\tilde{w}(t), \ \tilde{x}(0) = 0, \\ \tilde{y}(t) &= C(\tilde{x}) + [D(\tilde{x}) \ \ \tfrac{1}{\tau}H_2(\tilde{x})]\tilde{w}(t), \\ \tilde{z}(t) &= \begin{bmatrix} L(\tilde{x}) \\ \tau E(\tilde{x}) \end{bmatrix},\end{aligned} \tag{19}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ is the state, $\tilde{w}(t) \in \mathbb{R}^{m+i}$ is the input noise, $\tau > 0$ is a constant and $\tilde{z}(t)$ is a function of the state to be estimated. The associated estimator for $\tilde{z}(t)$ is the same as in (5) except that the estimate of $\hat{\tilde{z}}(t)$ is given by

$$\hat{\tilde{z}}(t) = \begin{bmatrix} l(\xi) \\ 0 \end{bmatrix}. \tag{20}$$

Note that the second block of $\hat{\tilde{z}}(t)$ is restricted to be zero. Using Lemma 1, we obtain the following result (see Appendix A for the proof).

Theorem 2. Consider system (1) and a filter of the form (5) satisfying Assumptions 1 and 2. The property (6) holds for (1) with (5) for all admissible uncertainties if and only if there exists a positive constant τ, such that (6) holds for (19) with the same filter.

4.3. *Solution to global robust nonlinear $\mathcal{H}_\infty$ filtering problem.* In view of Theorem 2, the remaining task is to solve the scaled nonlinear $\mathcal{H}_\infty$ filtering problem given in (19). Note that this problem is not the same as the $\mathcal{H}_\infty$ filtering problem in Section 2 because of the special structure of the estimate $\hat{\tilde{z}}(t)$ of $\tilde{z}(t)$ (see (20)). Indeed, the solution to the problem requires a Hamilton–Jacobi inequality as given in the following theorem.

Theorem 3. Consider the uncertain system (19) satisfying Assumptions 1 and 2. Given a scaling function $\tau > 0$, suppose there exist a nonnegative function $\epsilon(x, \xi)$ and a matrix function $b(\xi)$ for (5) satisfying

$$UHJ(x, \xi) + \hat{b}(x, \xi)e(x)\hat{b}'(x, \xi) \leq 0, \tag{21}$$

for all $x, \xi \in \mathbb{R}^n$, where

$$\begin{aligned}UHJ(x, \xi) \triangleq\ & [\nabla_x^t\epsilon(x, \xi) \ \ \nabla_\xi^t\epsilon(x, \xi)]\{\hat{A}_e(x, \xi) + \hat{F}_e\nabla_x\epsilon(\xi, \xi)\} \\ & + \frac{1}{4}[\nabla_x^t\epsilon(x, \xi) \ \ \nabla_\xi^t\epsilon(x, \xi)]B_e(x, \xi)B_e^t(x, \xi)\begin{bmatrix} \nabla_x\epsilon(x, \xi) \\ \nabla_\xi\epsilon(x, \xi) \end{bmatrix} \\ & - \frac{1}{4}\nabla_x^t\epsilon(\xi, \xi)\hat{B}(\xi)\hat{D}^t(\xi)\hat{e}^{-1}(x)\hat{D}(\xi)\hat{B}^t(\xi)\nabla_x\epsilon(\xi, \xi) \\ & - C_e^t(x, \xi)\hat{e}^{-1}(x)C_e(x, \xi) \\ & + L_e^t(x, \xi)L_e(x, \xi) + \frac{1}{2}\nabla_x^t\epsilon(\xi, \xi)\hat{B}(\xi)\hat{D}^t(\xi)\hat{e}^{-1}(x)[C(x) - C(\xi)] \\ & + \tau^2E^t(x)E(x)\end{aligned} \tag{22}$$

with

$$\begin{aligned}\hat{A}_e(x, \xi) &= \begin{bmatrix} A(x) - \hat{B}(x)\hat{D}^t(x)\hat{e}^{-1}(x)[C(x) - C(\xi)] \\ A(\xi) \end{bmatrix}, \\ \hat{D}(x) &= [D(x) \ \ \tfrac{1}{\tau}H_2(x)], \\ \hat{F}_e(x, \xi) &= \begin{bmatrix} \frac{1}{2}\hat{B}(\xi)\hat{D}^t(\xi)\hat{e}^{-1}(x)\hat{D}(\xi)\hat{B}^t(\xi)\nabla_x\epsilon(\xi, \xi) \\ \frac{1}{2}\hat{B}(\xi)\hat{B}^t(\xi)\nabla_x\epsilon(\xi, \xi) \end{bmatrix}, \\ \hat{e}(x) &= \hat{D}(x)\hat{D}^t(x), \\ B_e^t(x, \xi) &= \left[\hat{B}(x)[I - \hat{D}^t(x)\hat{e}^{-1}(x)\hat{D}(x)]^{\frac{1}{2}} \ \ 0\right], \\ \hat{B}(x) &= [B(x) \ \ \tfrac{1}{\tau}H_1(x)].\end{aligned}$$

$L_e(x, \xi)$ and $C_e(x, \xi)$ are defined in Theorem 1 and

$$\begin{aligned}\hat{b}(x, \xi)^t = \ & \frac{1}{2}\nabla_\xi^t\epsilon(x, \xi)b(\xi) + (\frac{1}{2}\nabla_x^t\epsilon(x, \xi)B(x)D^t(x) \\ & - \frac{1}{2}\nabla_x\epsilon(\xi, \xi)B(\xi)D^t(\xi) + C^t(x) - C^t(\xi))e^{-1}(x).\end{aligned}$$

If this is the case, then there exists a filter of the form (5) with functions $a(\xi)$ and $l(\xi)$ defined as

$$\begin{aligned}a(\xi) &= A(\xi) + \frac{1}{2}\hat{B}(x)\hat{B}^t(x)\nabla_x\epsilon(\xi, \xi) \\ &\quad - b(\xi)[C(\xi) + \frac{1}{2}\hat{D}(x)\hat{B}^t(x)\nabla_x\epsilon(\xi, \xi)],\end{aligned} \tag{23}$$

$$l(\xi) = L(\xi), \tag{24}$$

which will render (6) for all $x, \xi \in \mathbb{R}^n$.

Proof. By Theorem 2 the robust nonlinear $\mathcal{H}_\infty$ filtering problem is converted into a nonlinear $\mathcal{H}_\infty$ filtering problem for a 'scaled' known system. Using Theorem 2 for a known system we obtain our result. □

Remark 3. Similar to Remark 2, under some assumptions on $\epsilon(x, \xi)$, $UHJ(x, \xi) \leq 0 \ \forall x, \xi \in \mathbb{R}^n$, is a necessary condition for the existence of a solution to the robust nonlinear $\mathcal{H}_\infty$ filtering problem.

Remark 4. For linear systems, it can be shown (Nguang and Fu, 1994) that the condition given in Theorem 3 reduces to the same conditions as given by Fu *et al.* (1992).

5. *Conclusion*

In this paper, we have established an interconnection between robust nonlinear $\mathcal{H}_\infty$ filtering and $\mathcal{H}_\infty$ nonlinear filtering. Based on this interconnection, we have provided a sufficient condition for the existence of a filter to a nonlinear system with uncertainties described by some integral functional constraints. Our condition is expressed in terms of a 'scaled' Hamilton–Jacobi inequality. This result can also be viewed as an extension of Fu *et al.* (1992) which treats linear uncertain systems. It can be shown that the result on uncertain nonlinear systems can be further extended to treat the case where there are more than one block of uncertainties. But for the sake of simple bookkeeping, the details are not presented.

During the revision of this paper, we became aware that the robust $\mathcal{H}_\infty$ filtering problem is also considered by Shergei *et al.* (1994).

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References

Ball, J. A. and J. W. Helton (1989). H_∞ control for nonlinear plants: connection with differential games. In *Proc. 28th IEEE Conf. Decision Control*, Tampa, FL, pp. 956–962.

Ball, J. A., J. W. Helton and M. L. Walker (1993). H_∞ control for nonlinear systems with output feedback. *IEEE Trans. Automat. Control*, **AC-38**, 546–559.

Basar, T. (1991). Optimum performance levels for minimax filters, predictors and smoothers. *Syst. & Control Lett.*, **16**, 309–317.

Basar, T. and G. J. Olsder (1982). *Dynamic Noncooperative Game Theory*. Academic Press, New York.

Basar, T. and P. Bernhard (1991). $\mathcal{H}_\infty$ optimal control and related minimax design problems. In *Systems and Control: Foundations and Applications*. Birkhauser, Boston.

Berman, N. and U. Shaked (1995). $\mathcal{H}_\infty$ nonlinear filtering. *Int. J. Robust and Nonlinear Control*, in press.

Fu, M., C. E. de Souza and L. Xie (1992). H_∞ estimation for uncertain systems. *Int. J. of Robust and Nonlinear Control*, **2**, 87–105.

Grimble, M. J. (1988). H_∞ design of optimal linear filters. In C. Byrnes, C. F. Martin and R. E. Saeks (Eds), *Linear Circuits, Systems and Signal Processing: Theory and Application*, pp. 533–540. North-Holland, Amsterdam.

Hill, D. J. and P. J. Moylan (1980). Dissipative dynamical systems: basic input-output and state properties. *J. Franklin Inst.*, **309**, 327–357.

Isidori, A. (1991). Feedback control of nonlinear systems. In *Proc. First European Control Conf.*, Grenoble, France, pp. 1001–1012.

Isidori, A. and A. Astolfi (1992). Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems. *IEEE Trans. Automat. Control*, **37**, 1283–1293.

Kwakernaak, H. (1986). A polynomial approach to minimax frequency domain optimization of multivariable feedback systems. *Int. J. Control*, **44**, 117–156.

Limebeer, D. J. and U. Shaked (1991). New results in H_∞ filtering. In *Proc. 9th Int. Symp. on MTNS*, Kobe, Japan, pp. 317–322.

Nagpal, K. M. and P. P. Khargonekar (1991). Filtering and smoothing in an H_∞ setting. *IEEE Trans. Automat. Control*, **AC-36**, 152–166.

Nguang, S. K. and M. Fu (1994). H_∞ filtering for known and uncertain nonlinear systems. In *Proc. of IFAC Symposium Robust Control Design*, Rio de Janeiro, Brazil, pp. 347–352.

Savkin, A. V. and I. R. Petersen (1993). Nonlinear versus linear control in the absolute stabilizability of uncertain linear systems with structured uncertainty. In *Proc. 32th IEEE Conf. Decision Control*, San Antonio, TX, pp. 172–178.

van der Schaft, A. J. (1991). A state-space approach to nonlinear H_∞ control. *Syst. & Control Lett.*, **16**, 1–8.

van der Schaft, A. J. (1992). $\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Trans. Automat. Control*, **AC-37**, 770–784.

Shergei, M., U. Shaked and C. E. de Souza (1994). Robust $\mathcal{H}_\infty$ nonlinear estimation. In *IV Workshop on Adaptive Control: Applications to Nonlinear Systems and Robotics*, Cancun, Mexico.

Willems, J. C. (1972). Dissipative dynamical systems Part I: General theory. *Arch. Rational Mech. Anal.*, **45**, 321–351.

Yakubovich, V. A. (1971). S-procedure in nonlinear control theory. Vestinink Leningrad University, Series 1, Vol. 13, no. 1, pp. 62–77.

Appendix A. *Proof of Theorem* 2

Rewrite the augmented system (1) with (5) as follows:

$$\begin{aligned}
\dot{x}(t) &= A(x) + H_1(x)v(t) + B(x)\omega(t), \quad x(0) = 0,\\
\dot{\xi}(t) &= a(\xi) + b(\xi)C(x) + b(\xi)H_2(x)v + b(\xi)D(x)w(t),\\
\xi(0) &= 0,\\
y(t) &= C(x) + H_2(x)v + D(x)w(t),\\
z(t) &= L(x),\\
\hat{z}(t) &= l(\xi),
\end{aligned} \tag{A.1}$$

where $w(\cdot), v(\cdot) \in \mathcal{L}_2(0,\infty)$ and

$$f_1(x(\cdot), \xi(\cdot), v(\cdot)) = \int_0^\infty (\|E(x)\|^2 - \|v(t)\|^2)\mathrm{d}t \geq 0. \tag{A.2}$$

The $\mathcal{H}_\infty$ filtering requirement becomes

$$\begin{aligned}
&f_0(x(\cdot), \xi(\cdot), w(\cdot), v(\cdot))\\
&\quad = \int_0^\infty (\|w(t)\|^2 - \|z(t) - \hat{z}(t)\|^2)\mathrm{d}t \geq 0.
\end{aligned} \tag{A.3}$$

Using Lemma 1, (A.3) holds for all $x(\cdot), w(\cdot)$ and $v(\cdot)$ satisfying (A.2) if and only if there exist $\tau_0 \geq 0$, $\tau_1 \geq 0$ with $\tau_0 + \tau_1 > 0$, such that

$$\tau_0 f_0(\cdot) - \tau_1 f_1(\cdot) \geq 0 \tag{A.4}$$

for all $w(\cdot), v(\cdot) \in \mathcal{L}_2(0,\infty)$ and $x(\cdot), \xi(\cdot)$ satisfying (A.1). We show that both τ_0 and τ_1 must be positive by excluding the following two cases:

Case 1: $\tau_1 = 0$. In this case, $\tau_0 > 0$ and $f_0(\cdot) \geq 0$. Setting $w = 0$ and using (A.3), we have $z(t) - \hat{z}(t) = 0$ for all $t \geq 0$ and $v(\cdot)$. But this is impossible due to Assumption 2. So $\tau_1 > 0$.

Case 2: $\tau_0 = 0$. Setting $v = 0$ and using (A.2), we have $E(x(t)) = 0$ for all $t \geq 0$ and $w(\cdot)$. Again this possibility is excluded by Assumption 2. So $\tau_0 > 0$.

Consequently, there exists $\tau > 0$ such that

$$\int_0^\infty \left(\left\| \begin{bmatrix} w(t) \\ \tau v(t) \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} L(x) \\ \tau E(x) \end{bmatrix} - \begin{bmatrix} l(\xi) \\ 0 \end{bmatrix} \right\|^2 \right) dt \geq 0. \tag{A.5}$$

Necessity: Now suppose (6) holds for the augmented system (1) with (5), i.e.

$$\int_0^T \|z(t) - \hat{z}(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt \quad \forall w(\cdot) \in \mathcal{L}_2(0, T), \quad T > 0. \tag{A.6}$$

We need to show that the following inequality holds for the system (19) with the same filter:

$$\int_0^T \|\bar{z}(t) - \hat{\bar{z}}(t)\|^2 dt \leq \int_0^T \|\bar{w}(t)\|^2 dt \quad \forall \bar{w}(\cdot) \in \mathcal{L}_2(0, T), \quad T > 0. \tag{A.7}$$

Without loss of generality, we assume

$$\bar{w}(t) = 0 \quad \forall t > T. \tag{A.8}$$

Choosing

$$\begin{bmatrix} w(\cdot) \\ \tau v(\cdot) \end{bmatrix} = \bar{w}(\cdot) \tag{A.9}$$

we force the state trajectory of the system (A.1) to be identical to that of (19) with (5). From the analysis above, we have

$$\int_0^\infty (\|\bar{z}(t) - \hat{\bar{z}}(t)\|^2 - \|\bar{w}(t)\|^2) dt \leq 0, \tag{A.10}$$

which implies (A.7) because of (A.8).

Sufficiency: Conversely, suppose (A.7) holds for the augmented system (19) with (5). Then, for any $T > 0$ and $w(\cdot) \in \mathcal{L}_2(0, T)$, we need to show that (A.6) holds for the augmented system (1) with the same filter. Indeed, for any $T > 0$, $w(\cdot) \in \mathcal{L}_2(0, T)$ and $v(\cdot) \in \mathcal{L}_2(0, \infty)$ satisfying (A.2), we can assume that $w(t) = 0 \ \forall t > T$ without loss of generality. Choose

$$\bar{w} = \begin{bmatrix} w(t) \\ \tau v(t) \end{bmatrix} \quad \forall 0 \leq t \leq T \tag{A.11}$$

and $\bar{w}(t) = 0 \ \ \forall t > T$. Then, (A.7) implies that (A.5), and then (A.3), i.e.

$$\int_0^\infty (\|z(t) - \hat{z}(t)\|^2 - \|w(t)\|^2) dt \leq 0. \tag{A.12}$$

Since $w(\cdot)$ is truncated, we obtain (A.6).