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Robust output feedback stabilization of a class of time-varying non-linear systems

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This paper deals with the problem of robust output feedback stabilization of a class of time-varying non-linear systems. This class of systems involves two kinds of time-varying uncertainties: those norm-bounded and those bounded by a smooth non-linear function of the output. Under the assumption that the zero dynamics of the system are uniformly asymptotically stable and some additional mild conditions, we show via a Lyapunov function approach that the uncertain system can be robustly stabilized by a time-varying non-linear output feedback controller. The order of this controller turns out to be one less than the relative degree of the uncertain system. A systematic design procedure is given for constructing the controller. Illustrative examples are given. Note that the results generalize several previous results on robust output feedback stabilization.

1. Introduction

Over the past several years, a great deal of research has been devoted to adaptive stabilization of uncertain non-linear systems with stable zero dynamics (e.g. Sastry and Isidori 1989, Taylor *et al*. 1989, Kanellakopoulos *et al*. 1991a–c, Praly *et al*. 1991, Marino and Tomei 1993a,b, Huang 1995, Khali 1996, Byrnes *et al*. 1997, Isidori 1997, Isidori *et al*. 2000). Common assumptions employed in these adaptive schemes are that the unknown parameters are time-invariant (or varying sufficiently slowly) and must enter the state equation linearly. Although these assumptions allow the conventional parameter estimation theory for linear systems to be generalized to non-linear systems, most non-linear systems do involve non-linear and/or time-varying parameters.

The assumption of linear parameters has been removed by Marino and Tomei (1993b). The key difference between that paper and the papers cited above (Sastry and Isidori 1989 and Taylor *et al*. 1989 in particular) is that the design procedure proposed in Marino and Tomei is not based on parameter estimation. That is, the controller does not attempt to achieve exact non-linearity cancelation. Rather, only stabilization of the system is of concern.

This paper is motivated by the fact that the adaptive scheme proposed in Marino and Tomei (1993b) is not suitable for systems with fast time-varying uncertainty. Since adaptive control schemes in general have difficultly with fast time-varying systems, we will be interested in designing robust stabilizers.

More precisely, this paper deals with robust output feedback stabilization of a class of time-varying non-linear systems. This class of systems is represented by what we call a *generalized output feedback form,* which involves two types of uncertainty. The first type is a set of time-varying parameters, which have uniform bounds on themselves and their derivatives. The second type includes smooth non-linear functions of the output that are uniformly bounded by a known non-linear function of the output. This generalized output feedback form is extended from Kanellakopoulos *et al*. (1991a). It turns out that many non-linear systems, including those studied in Kanellakopoulos *et al*. (1991a) and Marino and Tomei (1993a,b), can be represented by this form.

The main assumption we take is that the zero dynamics of the system are uniformly asymptotically stable. We call this condition the *minimum phase condition*. Under this assumption and a few other mild assumptions, we show that a robust output feedback

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stabilizer exists. For the case when the relative degree r of the uncertain system is one, this stabilizer is static, time-varying and non-linear. For system with a higher relative degree, $r > 1$, the stabilizer is dynamic with its order equal to $r - 1$.

Apart from the aforementioned existence result for robust stabilization, we propose a simple procedure for constructing such a stabilizer. For the case $r = 1$, this stabilizer involves three tuning parameters. For $r > 1$, a two-step design procedure is employed. The first step involves setting up an auxiliary system with $r = 1$ and robust stabilization of this system. This controller, when applied to the original system, requires the derivatives of the output. The second step constructs a dynamic filter in order for the robust stabilizer to be physically realizable. This dynamic filtering is similar to the filtered-transformation in Marino and Tomei (1993b). However, the construction of the filter is quite different because the method in Marino and Tomei relies on the time-invariant nature of the system, and hence it is inappropriate for time-varying systems. Rather, we use a method similar to the back-stepping method.

The rest of this paper is organized as follows. Section 2 defines the uncertain system to be considered. A coordinate transformation is introduced in Section 3 to isolate the zero dynamics of the system for the case of $r = 1$. We then move on presenting our main results on robust stabilization in Sections 4 and 5 for $r = 1$ and $r > 1$, respectively. Illustrative examples are given in Section 6, and some concluding remarks in Section 7.

2. System description and definitions

We consider a class of non-linear systems that can be transformed into the following *generalized output feedback form*:

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))\sigma(y,t)u(t) + f(y,\theta(t),t) \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}$ is the input; $y(t) \in \mathbb{R}$ is the output; $\theta(\cdot) \in \Omega = [\Omega_1(\cdot) \ \cdots \ \Omega_p(\cdot)]$ is a set of functionals representing time-varying unknown parameters, each element $\theta(\cdot) \in \Omega$ is a sufficiently smooth function; $f(y,\theta(t),t)$ is an $n \times 1$ vector representing the non-linear uncertainty in the autonomous part of the plant; $\sigma(y,t)$ is a known non-linear time-varying scalar function; $A(\theta(t))$ and $B(\theta(t))$ are sufficiently smooth uncertain matrix functions of $\theta(\cdot)$ with appropriate dimensions; and $f(0,\theta(t),t) = 0$, $\forall \theta(\cdot) \in \Omega$ and $\sigma(y,t) \neq 0$ for all $y \in \mathbb{R}$.

The objective of robust output feedback stabilization is to find either a static controller of the form

$$u(t) = K(y,t) \tag{2.2}$$

or a dynamic compensator of the form

$$\begin{aligned} \dot{\xi}(t) &= A_c(\xi,y,t) \\ u(t) &= C_c(\xi,y,t) \end{aligned} \tag{2.3}$$

such that the closed loop is robustly asymptotically stable.

Definition 2.1: The system (2.1) is said to have invariant relative degree r if there exists some $\epsilon > 0$ such that

$$CA^k(\theta(t))B(\theta(t)) = 0, \ \forall \ 0 \le k \le r-1, \ \theta(\cdot) \in \Omega, \ t \ge 0$$

$$|CA^r(\theta(t))B(\theta(t))| \ge \epsilon \ \ \forall \ \theta(\cdot) \in \Omega, \ t \ge 0,$$

where A^k denotes the k^{th} power of A.

Throughout, the uncertain matrix functions $A(\theta(t))$ and $B(\theta(t))$ will be assumed to have the following properties.

Assumption 2.1: *The system* (2.1) *has invariant relative degree* $r > 0$ *and*

$$\sup_{\theta(\cdot)\in\Omega, t\ge 0} \|A^{(j)}(\theta(t))\| \le M_j, \qquad j = 0,\ldots,r-1 \tag{2.4}$$

$$\sup_{\theta(\cdot)\in\Omega, t\ge 0} \|B^{(j)}(\theta(t))\| \le H_j, \qquad j = 0,\ldots,r-1, \tag{2.5}$$

where $^{(j)}$ *denotes the* jth *time derivative,* M_j *and* H_j *are constants.*

The non-linear function $f(y,\theta(t),t)$ is assumed to be smooth, satisfying the following assumption.

Assumption 2.2: *There exists a Carathéodory function* $\rho(y,t)$ *such that*

$$\left\| \frac{f(y,\theta(t),t)}{y} \right\| \le \rho(y,t); \ \forall \ (y,\theta(\cdot),t) \in \mathbb{R} \times \Omega \times \mathbb{R}_+, \tag{2.6}$$

where $\|\cdot\|$ *denotes the Euclidean norm. Also* $\lim_{t\to\infty} \rho(y,t) < \infty$ *for all* $y \in \mathbb{R}$.

A function $V : \mathbb{R} \times \mathbb{R}^p \mapsto \mathbb{R}^q$ is called Carathéodory if : (1) $V(\cdot,z)$ is Lebesque measurable for each $z \in \mathbb{R}^p$; (2) $V(t,\cdot)$ is continuous for each $t \in \mathbb{R}$; (3) for each compact set $U \subset \mathbb{R} \times \mathbb{R}^p$, there exists a Lebesque integrable function $m_u(\cdot)$ such that $\|V(t,z)\| \le m_u(t)$ for all $(t,z) \in U$. This type of function is needed primarily for ensuring the existence and continuity of the solution to a differential equation; see Corless *et al*. 1984, *passim*.

3. Coordinate transformation for $r = 1$

The notion of zero dynamics of the system plays a crucial role in deriving our main result. We consider the case $r = 1$ and define a coordinate transformation which isolates the zero dynamics. Before doing so, we assume, without loss of generality, that $B(\theta(t))$ and C are in the following form:

$$B(\theta(t)) = [b_1(\theta(t)) \ \ b_2(\theta(t)) \ \ \cdots \ \ b_n(\theta(t))]^t \tag{3.1}$$

and

$$C = [1 \ \ 0 \ \ \cdots \ \ 0]. \tag{3.2}$$

Remark 3.1: If $B(\theta(t))$ and C are in a different form, we can always transform them into (3.1) and (3.2), respectively.

Using Assumption 2.1, we have

$$b_1(\theta(t)) \geq \underline{b}_1 = \epsilon > 0 \ \ \forall \theta(\cdot) \in \Omega, \ t \geq 0 \tag{3.3}$$

to satisfy the invariant relative degree condition. To simplify our notation, we partition $A(\theta(t))$ into

$$A(\theta(t)) = \begin{bmatrix} A_{11}(\theta(t)) & A_{12}(\theta(t)) \\ A_{21}(\theta(t)) & A_{22}(\theta(t)) \end{bmatrix}, \tag{3.4}$$

where $A_{11}(\theta(t)) \in \mathbb{R}^{1\times 1}$, $A_{12}(\theta(t)) \in \mathbb{R}^{1\times(n-1)}$, $A_{21}(\theta(t)) \in \mathbb{R}^{(n-1)\times 1}$ and $A_{22}(\theta(t)) \in \mathbb{R}^{(n-1)\times(n-1)}$. Define

$$B^*(\theta(t)) = \begin{pmatrix} \frac{b_2(\theta(t))}{b_1(\theta(t))} & \frac{b_3(\theta(t))}{b_1(\theta(t))} & \cdots & \frac{b_n(\theta(t))}{b_1(\theta(t))} \end{pmatrix}^{\mathrm{T}}. \tag{3.5}$$

Following [15, 13], we define a change of coordinates as follows:

$$\begin{pmatrix} z_1 \\ \eta_1 \\ \vdots \\ \eta_{n-1} \end{pmatrix} = Tx = \begin{pmatrix} 1 & 0 \\ -B^*(\theta(t)) & I_{n-1} \end{pmatrix} x. \tag{3.6}$$

Obviously, the transformation matrix T is non-singular. Further, Assumption 2.1 assures that both $\|T\|$ and $\|T^{-1}\|$ are uniformly bounded.

Applying the transformation in (3.6) to the system (2.1), its zero dynamics (obtained by setting $z_1 = 0$ and $\dot{z}_1 = 0$) are given by

$$\dot{\eta}(t) = \Gamma(\theta(t))\eta, \tag{3.7}$$

where

$$\Gamma(\theta(t)) = A_{22}(\theta(t)) - B^*(\theta(t))A_{12}(\theta(t)). \tag{3.8}$$

In the sequel, the following definition of minimum phase will be adopted.

Definition 3.1: The system (2.1) with invariant relative degree r is said to be *invariant minimum phase* if its zero dynamics are uniformly asymptotically stable for all $\theta(\cdot) \in \Omega$.

The following lemma establishes the equivalence between the uniformly asymptotically stability and the existence of a quadratic Lyapunov function.

Lemma 3.1: *(see Kalman and Betram 1960 for proof.) Consider the dynamic system*

$$\dot{x} = A(t)x, \tag{3.9}$$

where $x \in \mathbb{R}^n$; $A(t)$ is a sufficiently smooth and uniformly bounded matrix function. Then the following propositions are equivalent:

(P_1) The equilibrium state $x = 0$ is uniformly asymptotically stable.

(P_2) The system (3.9) admits a quadratic Lyapunov function, i.e. there exists a symmetric matrix function $P(\cdot) : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ such that:

1. *$P(\cdot)$ is continuous;*
2. *$\alpha I \leq P(t) \leq \beta I$, for all $t \in \mathbb{R}$, and some positive α and β;*
3. *The following Lyapunov derivative satisfies:*

$$L(x,t) := x^t[P(t)A(t) + A^t(t)P + \dot{P}(t)]x \leq -\lambda\|x\|^2 \tag{3.10}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and some positive constant λ.

In view of Lemma 3.1, we make the following assumption.

Assumption 3.1: *There exist a positive-definite matrix function $P(\theta(t))$ (which is not necessarily known to the designer) and known positive constants a, α and β such that $\alpha I \leq P(\theta(t)) \leq \beta I$ and the differential inequality*

$$\Gamma^t(\theta(t))P(\theta(t)) + P(\theta(t))\Gamma(\theta(t))) + \frac{d}{dt}P(\theta(t)) \leq -2aI \tag{3.11}$$

holds for all $\theta(\cdot) \in \Omega$, $t \geq 0$.

4. Robust output feedback stabilization: $r = 1$

We propose a non-linear control design procedure for robust stabilization of the system (2.1) with relative degree $r = 1$.

Define

$$F^*(y,t) = \begin{bmatrix} f_2(y,\theta(t),t) \\ f_3(y,\theta(t),t) \\ \vdots \\ f_{n-1}(y,\theta(t),t) \\ f_n(y,\theta(t),t) \end{bmatrix}. \tag{4.1}$$

Under the transformation (3.6), system (2.1) becomes

$$\begin{aligned}\dot{\eta} &= (A_{22} - B^*A_{12})\eta + (A_{22}B^* - B^*A_{11} - B^*A_{12}B^* \\ &\quad + A_{21} - \dot{B}^*)y + F^* - B^*f_1 \\ &\triangleq \Gamma(\theta(t))\eta(t) + \Xi(\theta(t))y(t) + F^*(y,\theta(t),t) \\ &\quad - B^*(\theta(t))f_1(y,\theta(t),t) \end{aligned} \tag{4.2}$$

$$\begin{aligned}\dot{y} &= A_{12}\eta + (A_{11} + A_{12}B^*)y + f_1 + b_1\sigma u \\ &\triangleq \Lambda(\theta(t))\eta(t) + \Pi(\theta(t))y(t) + f_1(y,\theta(t),t) \\ &\quad + b_1(\theta(t))\sigma(y,t)u(t), \end{aligned} \tag{4.3}$$

where all the new matrix functions are defined in an obvious way. By Assumption 2.2, the following functions are well-defined and smooth:

$$\phi(y,\theta(t),t) = \frac{F^*(y,\theta(t),t) - B^*(\theta(t))f_1(y,\theta(t),t)}{y} \tag{4.4}$$

$$\psi(y,\theta(t),t) = \frac{f_1(y,\theta(t),t)}{y}. \tag{4.5}$$

Hence, (4.2) and (4.3) can be rewritten as

$$\dot{\eta}(t) = \Gamma(\theta(t))\eta(t) + \Xi(\theta(t))y(t) + y(t)\phi(y,\theta(t),t) \tag{4.6}$$

$$\begin{aligned}\dot{y}(t) &= \Lambda(\theta(t))\eta(t) + \Pi(\theta(t))y(t) + y(t)\psi(y,\theta(t),t) \\ &\quad + b_1(\theta(t))\sigma(y,t)u(t). \end{aligned} \tag{4.7}$$

Also implied by Assumption 2.2 is that there exist two Carathéodory functions $\rho_1(y,t)$ and $\rho_2(y,t)$ such that

$$\|\phi(y,\theta(t),t)\| \le \rho_2(y,t) \tag{4.8}$$

$$|\psi(y,\theta(t),t)| \le \rho_1(y,t). \tag{4.9}$$

for all $(y,\theta(t),t) \in \mathbb{R} \times \Omega \times \mathbb{R}_+$.

Let $P(\theta(t))$ be the Lyapunov matrix in Assumption 3.1. Define

$$q(\theta(t)) = \Lambda(\theta(t)) + \Xi^t(\theta(t))P(\theta(t)). \tag{4.10}$$

We know from Assumptions 2.1 and 3.1 that $\|q(\theta(t))\| \le \bar{q}$ and $|\Pi(\theta(t))| \le \bar{\Pi}$ for all $\theta(\cdot) \in \Omega$ and $t \ge 0$, where $\bar{q}$ and $\bar{\Pi}$ are some known positive constants. Further choose constants

$$K_1 > \left(\bar{\Pi} + \frac{\bar{q}^2}{2a}\right)/\underline{b}_1;\ K_2 > \frac{1}{\underline{b}_1};\ K_3 > \frac{\beta^2}{2a\underline{b}_1}, \tag{4.11}$$

where a, β are as in Assumption 3.1 and $\underline{b}_1$ in (3.3) which follows from Assumption 2.1.

Our main result is as follows.

Theorem 4.1: *Suppose the system* (2.1) *has invariant relative degree* $r = 1$, *is invariant minimum phase and satisfies Assumptions 2.1, 2.2 and 3.1. Then there exists a static global robust output feedback stabilizer of the form* (2.2). *In particular, one can choose the following stabilizer*

$$u(y,t) = -\sigma^{-1}(y,t)(K_1 + K_2\rho_1(y,t) + K_3\rho_2^2(y,t))y(t), \tag{4.12}$$

where $\rho_1(\cdot)$ *and* $\rho_2(\cdot)$ *are given in* (4.8) *and* (4.9), *respectively;* $K_i, i = 1,2,3$ *are in* (4.11).

Proof: Let $P(\theta(t))$ be the positive definite matrix function in Assumption 3.1. A suitable choice of Lyapunov function for the system (4.6) and (4.7) with (4.12) is given by

$$V(y,\eta,t) = \frac{1}{2}y^2 + \frac{1}{2}\eta^t P(\theta(t))\eta. \tag{4.13}$$

Then the time derivative of $V(y,\eta,t)$ along the trajectories of system (4.6) and (4.7) with (4.12) reads:

$$\begin{aligned}\dot{V} &= y\{\Lambda\eta + y\psi + \Pi y - b_1(K_1 + K_2\rho_1 + K_3\rho_2^2)y\} \\ &\quad + \{\Gamma\eta + \Xi y + y\phi\}^t P\eta + \frac{1}{2}\eta^t\dot{P}\eta \\ &\le y\{\Lambda\eta + y\psi + \Pi y - b_1(K_1 + K_2\rho_1 + K_3\rho_2^2)y\} \\ &\quad + \{\Xi y + y\phi\}^t P\eta - a\eta^t\eta. \end{aligned} \tag{4.14}$$

The last step above uses (3.11). Using the triangular inequality

$$y\phi^t P\eta \le \frac{a}{2}\eta^t\eta + \frac{1}{2a}y^2\phi^t P^2\phi$$

and recalling the definition of $q(\theta(t))$ in (4.10), we further bound (4.14) and rewrite it in a more compact form:

$$\begin{aligned}\dot{V} &\le -\begin{bmatrix} y \\ \eta \end{bmatrix}^t \begin{bmatrix} b_1K_1 - \Pi & -\dfrac{q(\theta(t))}{2} \\ -\dfrac{q^t(\theta(t))}{2} & \dfrac{a}{2}I \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \\ &\quad - \left\{ b_1(K_2\rho_1 + K_3\rho_2^2) - \frac{1}{2a}\phi^t P^2\phi - \psi \right\} y^2. \end{aligned}$$

Recall the definition of $\rho_i(y,t)$ in (4.8) and (4.9) and K_i in (4.11). The choice of K_1 ensures that the first term in $\dot{V}$ above is strictly negative. Similarly, the choice of K_2 and K_3 together with ρ_1 and ρ_2 guarantee that the second term above in $\dot{V}$ is non-positive. It follows that

$$\dot{V} \le -\underline{a}(|y|^2 + \|\eta\|^2) \tag{4.15}$$

for some positive $\underline{a}$. Using (4.13) and $P(\theta(t)) \ge \alpha I$, (4.15) implies $\dot{V} \le -\hat{a}V$ for some $\hat{a} > 0$. Since $V(y,\eta)$ is radially unbounded, we have shown that the system (2.1) is globally asymptotically stable with (4.12). □

Remark 4.1: If the non-linear terms $\sigma(y,t)$ and $f(y,\theta(t),t)$ in the system (2.1) disappear, we have a linear system with time-varying uncertainty. In this case, Theorem 4.1 can be specialized to a known result in [17] as follows.

Corollary 4.1: *Consider the following uncertain linear time-varying system*

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4.16}$$

satisfying the following conditions:

L1) it has invariant relative degree $r = 1$ and satisfies Assumption 2.1; and
L2) it is invariant minimum phase and satisfies Assumption 3.1.

Then there exists a linear static output feedback controller of the form

$$u(t) = -Ky(t) \tag{4.17}$$

with some constant K which globally robustly stabilizes (4.16).

Remark 4.2: If all the non-linear functions in the system (2.1) are Lipschitz bounded (i.e, $\rho_i(y,t) = \kappa_i$ for $i = 1,2$ and $\sigma(y,t) = \kappa_3$, where $\kappa_i > 0$ for all $i = 1,2,3$), then clearly the controller given in (4.12) is a linear gain. If the uncertainty $\theta(t)$ in system (2.1) is time-invariant then Theorem 4.1 reduces to Marino and Tomei's (1993b) result.

5. Robust output feedback stabilization: $r \geq 2$

We now generalize Theorem 4.1 to systems with invariant relative degree $r > 1$. In this case, a dynamic stabilizer of the form (2.3) will be used. The design involves two steps. In the first, we consider an auxiliary system which is a cascade of (2.1) with an $(r-1)^{th}$ order differentiator at the input (figure 1). The purpose of this cascading is that the auxiliary system has invariant relative degree equal to 1. Therefore, Theorem 4.1 applies and a static output feedback stabilizer $u^*(t) = K(y,t)$, where $u^*(t)$ is the input to the auxiliary system.

However, this controller, if viewed as a mapping from the output $y(t)$ to $u(t)$, is non-casual because the differentiator (5.2) is not physically realizable. In the second step, we show that the differentiator can be replaced by a suitable filtered approximation without destroying the stability properties of the corresponding closed-loop (see figure 2). The filtered mapping from $y(t)$ to $u(t)$ will be casual and forms a suitable dynamic stabilizer. The details are given below.

Step 1: We consider the following auxiliary system

$$\begin{aligned} \dot{x}(t) &= A(\theta(t)))x(t) + B(\theta(t))\mathcal{P}(t) + f(y,\theta(t),t) \\ y(t) &= Cx(t), \end{aligned} \tag{5.1}$$

where

$$\mathcal{P}(t) = p_0 u^*(t) + p_1 u^{*(1)}(t) + p_2 u^{*(2)}(t) + \cdots + p_{r-2} u^{*(r-2)} + u^{*(r-1)}(t) \tag{5.2}$$

$u^*(t)$ is the control input to (5.1), and the scalars $p_0, p_1, \ldots, p_{r-2}$ are arbitrarily chosen as long as the polynomial

$$\mathcal{P}(s) = s^{r-1} + \cdots + p_1 s + p_0 \tag{5.3}$$

is Hurwitz. Now we show that the auxiliary system (5.1) is causal and has invariant relative degree equal to 1. This is revealed by a sequence of coordinate transformations so that the transformed system does not contain the derivatives of $u^*(t)$. To this end, let

$$\bar{x}(t) = x(t) - B(\theta(t))\{p_1 u^*(t) + p_2 u^{*(1)}(t) + \cdots + p_{r-2} u^{*(r-3)}(t) + u^{*(r-2)}(t)\}, \tag{5.4}$$

then

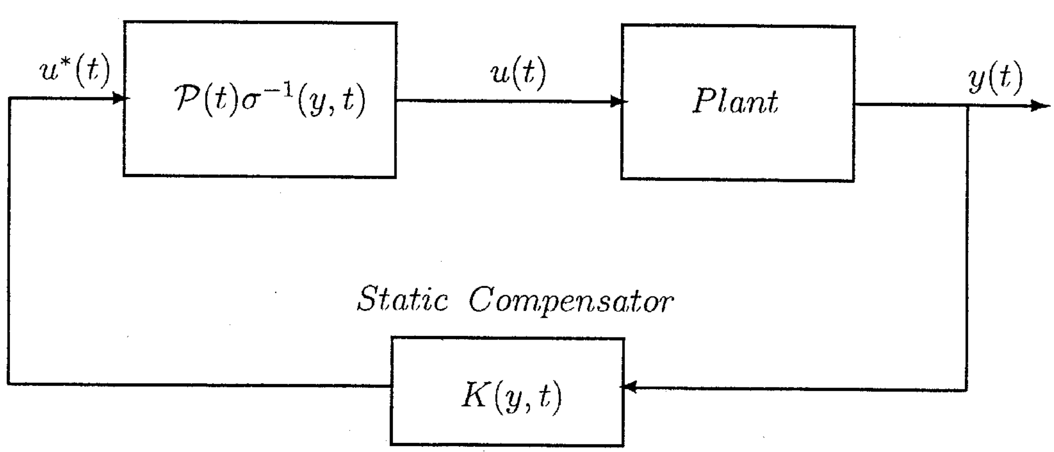

Figure 1. Linear differentiator cascading with the plant.

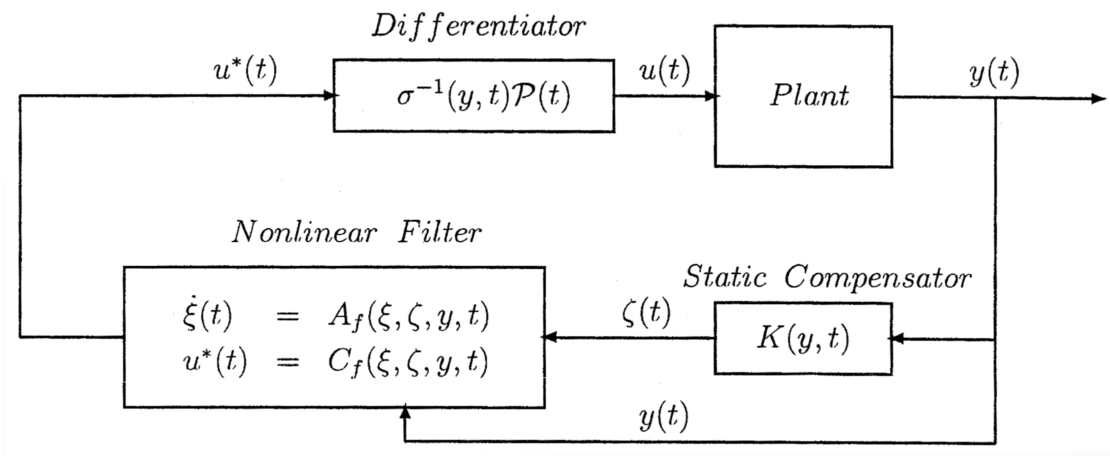

Figure 2. Plant with a dynamic compensator.

$$\begin{aligned}\dot{\bar{x}}(t) &= A(\theta(t))x(t) + B(\theta(t))\sigma(y,t)u(t) + f(y,\theta(t),t)\\ &\quad - \dot{B}(\theta(t))\{p_1u^*(t) + p_2u^{*(1)}(t) + \cdots\\ &\quad + p_{r-2}u^{*(r-3)}(t) + u^{*(r-2)}(t)\}\\ &\quad + B(\theta(t))\{p_1u^{*(1)}(t) + p_2u^{*(2)}(t) + \cdots\\ &\quad + p_{r-2}u^{*(r-2)}(t) + u^{*(r-1)}(t)\}\\ &= A(\theta(t))\bar{x}(t) + B(\theta(t))p_0u^*(t) + [A(\theta(t))B(\theta(t))\\ &\quad - \dot{B}(\theta(t)) + B(\theta(t))]\\ &\quad \times \{p_1u^*(t) + p_2u^{*(1)}(t) + \cdots + p_{r-2}u^{*(r-1)}(t)\\ &\quad + u^{*(r-2)}(t)\} + f(y,\theta(t),t) \qquad (5.5)\\ y(t) &= C\bar{x}(t). \qquad (5.6)\end{aligned}$$

As we can see in (5.5) the highest order of derivative of $u^*(t)$ is $(r-2)$. Now redefine the state variables

$$\begin{aligned}\bar{\bar{x}}(t) &= \bar{x}(t) - [A(\theta(t))B(\theta(t)) - \dot{B}(\theta(t))]\{p_2u^*(t)\\ &\quad + p_3u^{*(1)}(t) + \cdots + p_{r-2}u^{*(r-4)}(t) + u^{*(r-3)}(t)\}.\end{aligned} \qquad (5.7)$$

Repeat the procedure as described above until all the derivatives of $u^*(t)$ disappear in the state equation. Eventually, we can write the system (5.1) as

$$\begin{aligned}\dot{x}^*(t) &= A(\theta(t))x^*(t) + D(\theta(t))u^*(t) + f(y,\theta(t),t)\\ y(t) &= Cx^*(t),\end{aligned} \qquad (5.8)$$

where $D(\theta(t))$ is determined by the recursive procedure described above. It can shown that $D(\theta(t))$ contains the terms $A^iB, i = 0,1,\ldots,r-1$ and their derivatives as well as A^rB. Therefore the system (5.8) has invariant relative degree 1 following from Assumption 2.1. Further, (5.8) satisfies Assumptions 2.2 and 3.1. In particular, the zero dynamics of (5.8) consist of the inverse dynamics of the differentiator and the zero dynamics of the original system. Since both of them are uniformly asymptotically stable, the overall zero dynamics are also uniformly asymptotically stable. Employing Theorem 4.1, a static controller of the form $u^*(t) = K(y,t)$ as in (5.8) can be constructed which globally robustly stabilizes the system (5.8).

Step 2: Suppose we have found a smooth static stabilizer $u^* = K(y,t)$ as in (4.12). Our job here is to find a nonlinear dynamic filter depicted in figure 2 such that the filtered compensator is also a stabilizer and has relative degree equal to $(r-1)$. Inspired by Marino and Tomei (1993b), we claim that the required filter can be of the following form:

$$\begin{aligned}u^*(t) &= -\xi_1\\ \dot\xi_1 &= -\xi_1 + \xi_2\\ \dot\xi_2 &= -\xi_2 + \xi_3\\ &\vdots\\ \dot\xi_{r-2} &= -\xi_{r-2} + \xi_{r-1}\\ \dot\xi_{r-1} &= -\xi_{r-1} + \xi_r^*(y,\xi_1,\ldots,\xi_{r-1},t),\end{aligned} \qquad (5.9)$$

where $\xi_r^*(\cdot)$ is a smooth non-linear function to be chosen which depends on $K(y,t)$. Our design method is a recursive procedure similar to the well-known back-stepping procedure. To show that the filtered controller will not destroy the stability properties of the corresponding closed loop, we need to introduce a lemma.

Consider the system:

$$\begin{aligned}\dot\eta(t) &= \Upsilon(\theta(t))\eta(t) + y\mathcal{F}_1(y,\theta(t),t)\\ \dot y(t) &= \Lambda(\theta(t))\eta(t) + y\mathcal{F}_2(y,\theta(t),t) + b_1(\theta(t))u(t),\end{aligned} \qquad (5.10)$$

which satisfies the following conditions:

C1 : all the matrix functions defined in (5.10) are sufficiently smooth;
C2: $\|\Upsilon(\theta(t))\| \le M_\Gamma$, $\|\Lambda(\theta(t))\| \le M_\Lambda$ and $\bar{b}_1 \ge b_1(\theta(t)) \ge \underline{b}_1 > 0$ for all $\theta(\cdot) \in \Omega, t \ge 0$, where M_Γ, $M_\Lambda, \bar{b}_1$ and $\underline{b}_1$ are known constants; and
C3: $\|\mathcal{F}_1(y, \theta(t), t)\| \le \varrho_1(y, t)$ and $\|\mathcal{F}_2(y, \theta(t), t)\| \le \varrho_2(y, t)$ for all $\theta(\cdot) \in \Omega, t \ge 0$, where $\varrho_1(y, t)$ and $\varrho_2(y, t)$ are known Carathéodory functions.

Lemma 5.1: *Consider the system (5.10) with the controller*

$$\begin{aligned} u(t) &= -\xi_1 \\ \dot{\xi}_1 &= -\xi_1 + \xi_2 \\ \dot{\xi}_2 &= -\xi_2 + \xi_3 \\ \vdots &= \quad \vdots \\ \dot{\xi}_k &= -\xi_k + \xi_{k+1}{}^*(y, \xi_1, \ldots, \xi_k, t). \end{aligned} \tag{5.11}$$

Suppose there exist smooth functions $\xi^*{}_1(y, t)$,

$$\xi^*{}_2(y, \xi_1, t), \ldots, \xi_{k+1}{}^*(y, \xi_1, \ldots, \xi_k, t)$$

satisfying $\xi^*{}_1(0, t) = 0$ *and a Lyapunov function*

$$V_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k) = \frac{1}{2}y^2 + \frac{1}{2}\eta^t P(\theta(t))\eta + \sum_{i=1}^{k} \tilde{\xi}_i^2, \tag{5.12}$$

where

$$\begin{aligned} \tilde{\xi}_1 &= \xi_1 - \xi_1^*(y, t) \\ \tilde{\xi}_2 &= \xi_2 - \xi_2^*(y, \xi_1, t) \\ \vdots &= \ \vdots \\ \tilde{\xi}_{k-1} &= \xi_{k-1} - \xi_{k-1}^*(y, \xi_1, \ldots, \xi_{k-2}, t) \\ \tilde{\xi}_k &= \xi_k - \xi_k^*(y, \xi_1, \ldots, \xi_{k-1}, t). \end{aligned} \tag{5.13}$$

In addition, assume that the time derivative of (5.12) *along system* (5.10) *with* (5.11) *is*

$$\dot{V}_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k) \le -a_k V_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k), \tag{5.14}$$

where a_k *is a positive constant. Then there exists a smooth function* $\xi^*{}_{k+2}(y, \xi_1, \ldots, \xi_{k+1}, t)$ *for the system* (5.10), (5.11) (*with* ξ^*_{k+1} *replaced by* ξ_{k+1}) *and*

$$\dot{\xi}_{k+1} = -\xi_{k+1} + \xi_{k+2}{}^*(y, \xi_1, \ldots, \xi_{k+1}, t) \tag{5.15}$$

together with a Lyapunov function

$$V_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}) = \frac{1}{2}y^2 + \frac{1}{2}\eta^t P(\theta(t))\eta + \sum_{i=1}^{k+1} \tilde{\xi}_i^2, \tag{5.16}$$

where

$$\tilde{\xi}_{k+1} = \xi_{k+1} - \xi_{k+1}{}^*(y, \xi_1, \ldots, \xi_k, t) \tag{5.17}$$

such that the time-derivative of V_{k+1} *satisfies*

$$\begin{aligned} &\dot{V}_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}) \\ &\quad \le -a_{k+1} V_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}), \end{aligned} \tag{5.18}$$

where a_{k+1} *is a positive constant. Furthermore, we can choose*

$$\begin{aligned} \xi^*_{k+2} = &-(\Gamma(y, \xi, t) + a_{k+1})\tilde{\xi}_{k+1} - \tilde{\xi}_k + \xi_{k+1} \\ &+ \mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t), \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} \mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t) = &\frac{\partial \xi^*{}_{k+1}}{\partial \xi_1}(-\xi_1 + \xi_2) + \cdots \\ &+ \frac{\partial \xi^*{}_{k+1}}{\partial \xi_k}(-\xi_k + \xi_{k+1}) + \frac{\partial \xi^*{}_{k+1}}{\partial t} \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \Gamma(y, \xi, t) = &\left(\frac{\partial \xi^*{}_{k+1}}{\partial y}\right)^2 \frac{1}{\alpha_1} M_\Lambda^2 + \frac{1}{\alpha_2}\varrho_2^2(y, t) \\ &+ \frac{1}{\alpha_3}\bar{b}_1^2 + \frac{1}{\alpha_4}\left(\bar{b}_1 \frac{\xi^*{}_1(y, t)}{y}\right)^2 \end{aligned} \tag{5.21}$$

for some (small) positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. *In the above,* $M_\Lambda, \bar{b}_1$ *and* ϱ_2 *are given in Conditions C2 and C3.*

Proof: see appendix A. □

Applying the lemma above to the system (5.10), we start with $\xi_1{}^*(y, t) = K(y, t)$, and then obtain $\xi^*{}_i(\cdot)$, $i = 2, \ldots, r$, recursively. Combining the non-linear filter (5.9) with the differentiator (5.2) and the static compensator (4.12) yields the overall dynamics compensator as follows:

$$\begin{aligned} u(t) = &-\sigma^{-1}(y, t)[p_0\xi_1(t) + p_1\xi_1^{(1)}(t) + p_2\xi_1^{(2)}(t) + \cdots \\ &+ p_{r-2}\xi_1^{(r-2)}(t) + \xi_1^{(r-1)}(t)] \\ \dot{\xi}_1 &= -\xi_1 + \xi_2 \\ \dot{\xi}_2 &= -\xi_2 + \xi_3 \\ &\vdots \\ \dot{\xi}_{r-2} &= -\xi_{r-2} + \xi_{r-1} \\ \dot{\xi}_{r-1} &= -\xi_{r-1} + \xi_r{}^*(y, \xi_1, \ldots, \xi_{r-1}, t). \end{aligned} \tag{5.22}$$

Equivalently, the stabilizing controller is given by

$$\begin{aligned}u(t) &= -\sigma^{-1}(y,t)[p_0\xi_1 - p_1(\xi_1-\xi_2) - \cdots \\ &\quad - p_{r-2}(\xi_{r-2}-\xi_{r-1}) - \xi_{r-1} \\ &\quad + \xi_r{*}(y,\xi_1,\ldots,\xi_{r-1},t)] \\ \dot\xi_1 &= -\xi_1+\xi_2 \\ \dot\xi_2 &= -\xi_2+\xi_3 \\ &\vdots \\ \dot\xi_{r-2} &= -\xi_{r-2}+\xi_{r-1} \\ \dot\xi_{r-1} &= -\xi_{r-1}+\xi_r{*}(y,\xi_1,\ldots,\xi_{r-1},t),\end{aligned} \tag{5.23}$$

where $\xi_r{*}(y,\xi_1,\ldots,\xi_{r-1},t)$ given in (5.19) with $(k+1)$ replaced by r. Clearly (5.23) is physically realizable.

Now we state our result as follows.

Theorem 5.1: *Suppose the system* (2.1) *has an invariant relative degree* $r>1$, *invariant minimum phase and satisfying Assumptions 2.1 and 2.2. Take any Hurwitz polynomial* $\mathcal{P}(s)$ *as in* (5.3) *and suppose the auxiliary system (5.8) satisfies Assumption 3.1. Then there exists an* $(r-1)^{th}$ *order dynamic output feedback stabilizing controller of the form* (2.3).

Proof: Consequences of *Steps* 1 and 2. □

If the system under consideration is linear with time-varying uncertainty, then we have a result in [17] as follows.

Corollary 5.1: *Consider the following uncertain linear time-invariant system*

$$\begin{aligned}\dot x(t) &= A(\theta(t))x(t)+B(\theta(t))u(t) \\ y(t) &= Cx(t),\end{aligned} \tag{5.24}$$

satisfying the following conditions:

H1) it has invariant relative degree $r>1$ *and satisfies Assumption 2.1;*
H2) it is invariant minimum phase and its zero dynamics satisfies Assumption 3.1.

Then there exists a linear dynamic compensator output feedback controller of the form

$$\begin{aligned}\dot\xi(t) &= A_c\xi(t)+B_cy(t) \\ u(t) &= C_c\xi(t)+D_cy(t).\end{aligned} \tag{5.25}$$

Remark 5.1: Theorem 5.1, can be viewed as a generalization of the results in [18, 19, 13, 17]. If all the non-linear functions of the system (2.1) are Lipschitz bounded, (i.e, $\rho_i(y,t)=\kappa$ for $i=1,2$ and $\rho_3=\kappa_3$, where $\kappa_i>0$ for all $i=1,2,3$) then, clearly the controller is a linear time-invariant dynamics compensator as in (5.25). If the uncertainty $\theta(t)$ in system (2.1) is time-invariant, then Marino and Tomei's (1993b) results follow immediately from Theorem 5.1.

6. Numerical examples

6.1. *Example 6.1*

Consider the following non-linear uncertain system:

$$\begin{aligned}\dot x_1 &= x_2+ye^{\theta(t)y} \\ \dot x_2 &= x_3 \\ \dot x_3 &= u \\ y &= x_1+2x_2+x_3,\end{aligned} \tag{6.1}$$

where $|\theta(t)|\le 1$. Define a transformation

$$\mathcal{T}:\begin{cases}\eta_1 = x_1 \\ \eta_2 = x_2 \\ y = x_1+2x_2+x_3,\end{cases} \tag{6.2}$$

which is globally non-singular. Apply ($\mathcal{T}$) on system (6.1) we have

$$\begin{aligned}\dot\eta_1 &= \eta_2+ye^{\theta(t)y} \\ \dot\eta_2 &= -\eta_1-2\eta_2+y \\ \dot y &= -3\eta_2-2\eta_1+2y+ye^{\theta(t)y}+u,\end{aligned} \tag{6.3}$$

which has invariant relative degree equal to 1. Also, the zero dynamics are given by

$$\begin{aligned}\dot\eta_1 &= \eta_2 \\ \dot\eta_2 &= -\eta_1-2\eta_2,\end{aligned} \tag{6.4}$$

which are uniformly asymptotically stable. With

$$P=\begin{bmatrix}1.5 & 0.5\\ 0.5 & 0.5\end{bmatrix},\quad A_z=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},$$

we have

$$A_z^tP+PA_z=-I. \tag{6.5}$$

Compared with (4.6) and (4.7), we have

$$\begin{aligned}\psi(y,\theta(t),t) = \phi(y,\theta(t),t) &= e^{\theta(t)y} \\ \rho_1(y,t)=\rho_2(y,t) &= e^y.\end{aligned}$$

Clearly the system (6.3) satisfies all the assumptions in Theorem 4.1. Subsequently, a stabilizing controller is given by

$$u=-(285y+10ye^y+25ye^{2y}). \tag{6.6}$$

The closed-loop response of the system (6.1) with (6.6) and the control input are given in figure 3 and 4, respectively. We take the initial condition $x(0)=(-2,5,-9)$ and the uncertainty $\theta(t)=\sin(t)$.

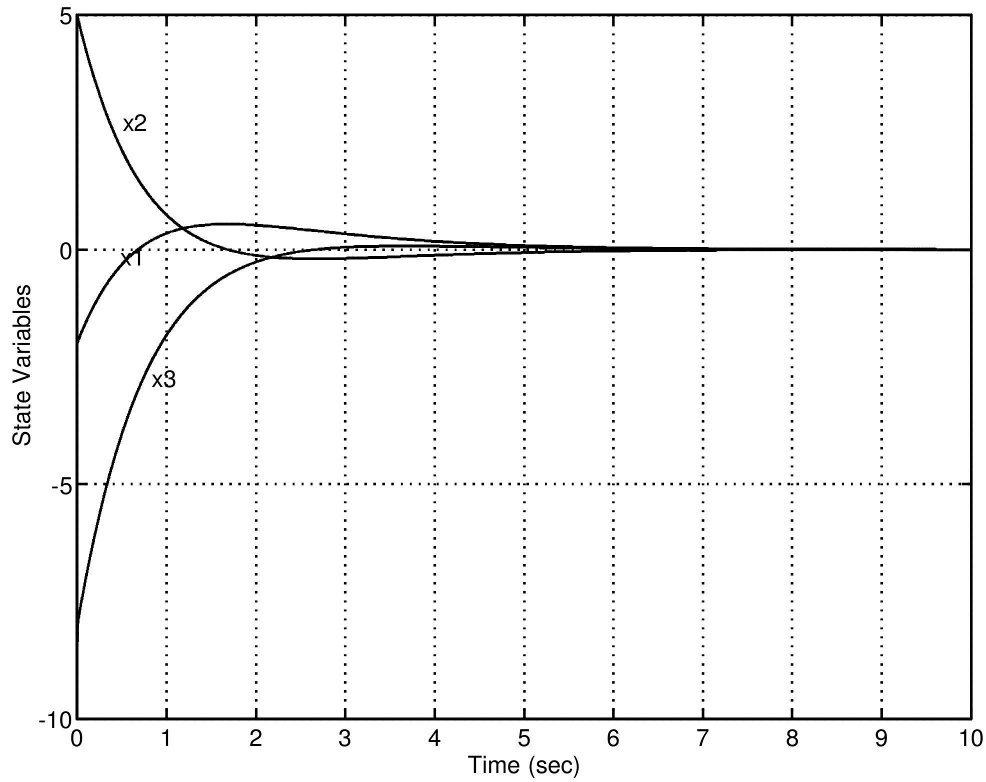

Figure 3. Closed-loop responses.

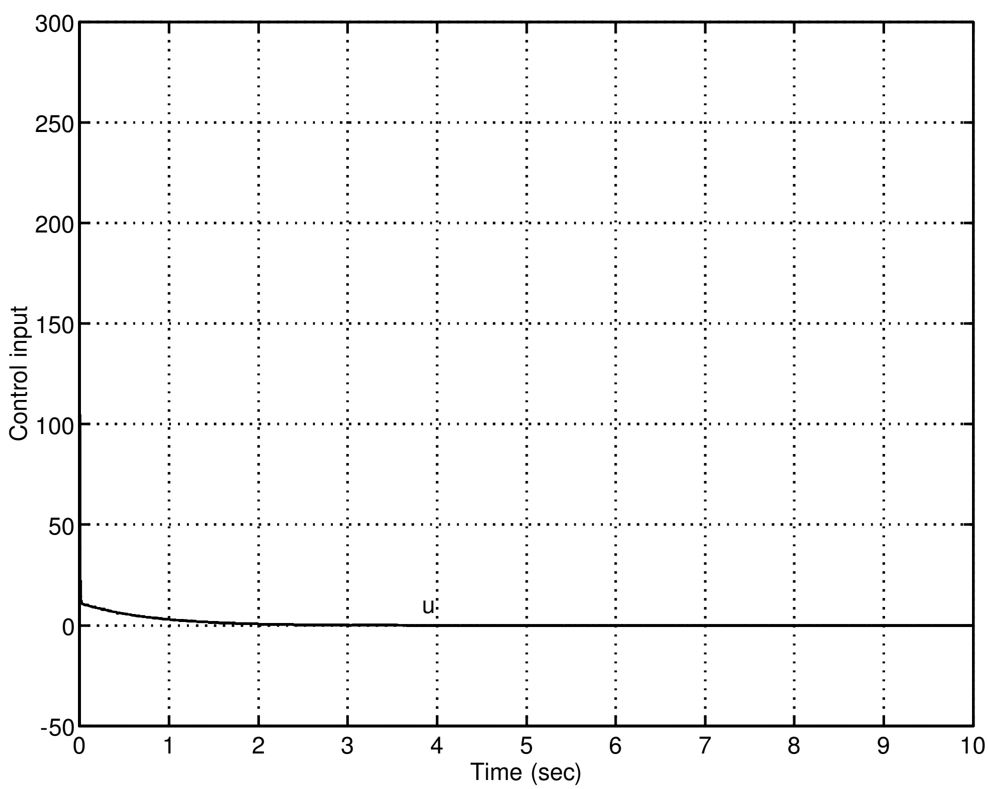

Figure 4. Control input.

6.2. *Example 6.2*

Given the following non-linear system

$$\dot{x}_1 = (2+\theta_1(t))x_2 + x_1^{(1+\theta_2(t))}$$
$$\dot{x}_2 = u \qquad (6.7)$$
$$y = x_1,$$

with $|\theta_1(t)| \le 1$, $|\dot{\theta}(t)| \le 1$ and $0 \le \theta_2(t) \le 1$. This system satisfies the assumptions in Theorem 5.1 with invariant relative degrees $r = 2$.

Step 1: Reduce the relative degree of the system (6.7) to 1 by cascading the following differentiator:

$$u(t) = u^*(t) + \dot{u}^*(t) \qquad (6.8)$$

at the input. Clearly the system (6.7) with the differentiator (6.8) is

$$\dot{x}_1 = (2+\theta_1(t))x_2 + x_1^{(1+\theta_2(t))}$$
$$\dot{x}_2 = u^*(t) + \dot{u}^*(t) \qquad (6.9)$$
$$y = x_1.$$

Following the procedure described in *Step* 1 (in section 5), the system (6.9) can be written as

$$\dot{\bar{x}}_1 = (2+\theta_1(t))\bar{x}_2 + (2+\theta_1(t))u^*(t) + \bar{x}_1^{(1+\theta_2(t))}$$
$$\dot{\bar{x}}_2 = u^*(t) \qquad (6.10)$$
$$y = \bar{x}_1,$$

where $\bar{x}_1 = x_1$ and $\bar{x}_2 = x_2 - u^*(t)$. It easy to verify that system (6.10) has relative degree 1 and the zero dynamics are the same as the zero of the differentiator (6.8) which are stable. Applying Theorem 4.1, the static compensator is found to be

$$K(y,t) = (1 + (2+y^2) + (2+y^2)^2)y. \qquad (6.11)$$

Step 2: Using Theorem 5.1, the non-linear filter is found to be

$$u^*(t) = -\xi_1$$
$$\dot{\xi}_1(t) = -\xi_1 + \xi^*{}_2(\xi_1, y, t), \qquad (6.12)$$

where

$$\xi_2^*(\xi_1, y, t) = -(\Gamma(\xi_1, y, t) + 10)(\xi_1 - K(y,t)) + \xi_1, \qquad (6.13)$$

with

$$\Gamma(\xi_1, y, t) = (7 + 15y^2 + 5y^4)^2(36 + (2+y^2)^2 + 9\{3 + y^2 + (2+y^2)^2]\}^2). \qquad (6.14)$$

The overall dynamics compensator is given by

$$u(t) = -\xi_2^*(\xi, y, t)$$
$$\dot{\xi}_1(t) = -\xi_1 + \xi_2^*(\xi_1, y, t). \qquad (6.15)$$

The closed-loop response of (6.7) with (6.15) and the control input are given in figures 5 and 6, respectively. In the simulation, the initial condition $x(0) = [1, -1]^{\mathrm{T}}$ and the uncertainties $\theta_1(t) = \sin(t)$ and $\theta_2(t) = \frac{1}{2}(1 + \sin(t))$.

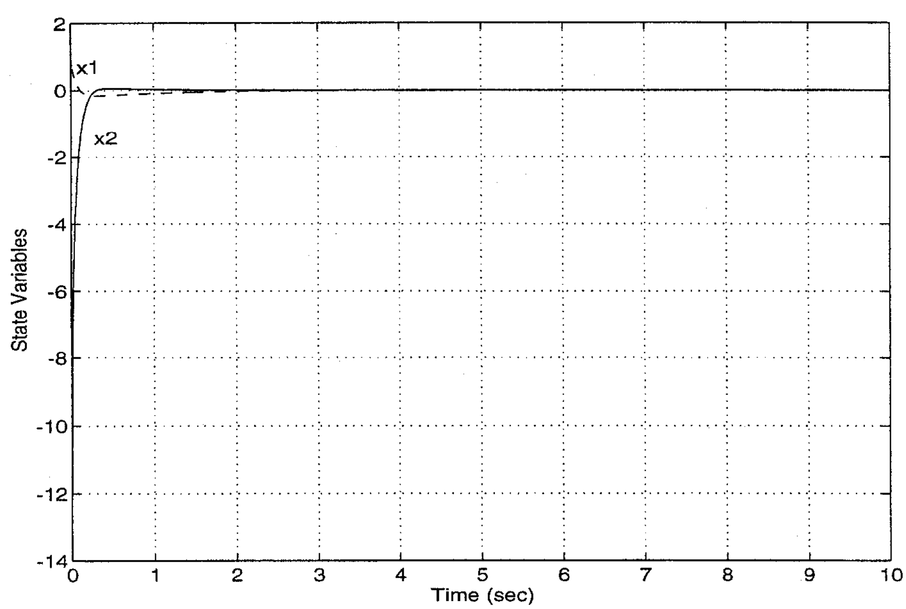

Figure 5. Closed-loop responses.

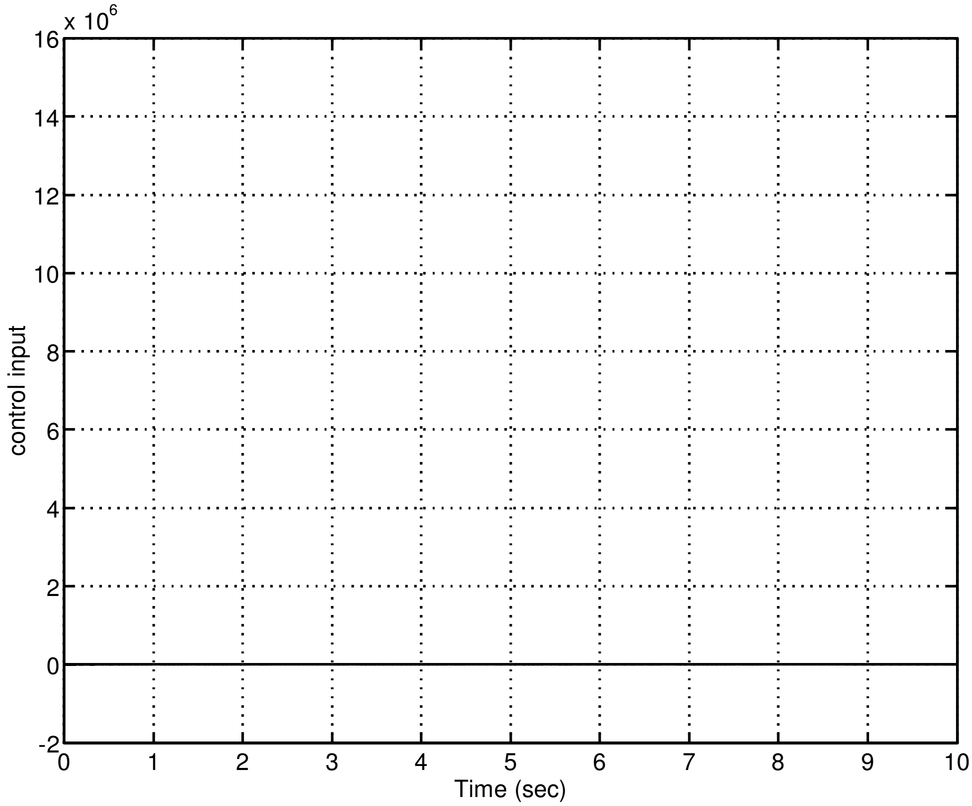

Figure 6. Control input.

7. Conclusion

We solved the robust stabilization problem for systems that are in the *generalized output feedback form*. Under the assumption that the zero dynamics are uniformly asymptotically stable, together with some other mild assumptions, we showed that the uncertain system can be robustly stabilized by a time-varying non-linear output feedback controller. We presented a new systematic procedure for constructing the time-varying non-linear controller. Illustrative examples were given to illustrate the the design procedure. Finally, the results given in Kwakernaak (1982), Wei and Barmish (1988), Marino and Tomei (1993b) and Fu and Li (1992) can be derived as special cases of our results.

Acknowledgment

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Appendix A. Proof of Lemma 5.1

Consider system (5.10) with the controller (5.11) and the Lyapunov function $V_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k)$ in (5.12). We have

$$
\begin{aligned}
\dot{V}_k &= y\dot{y} + \tfrac{1}{2}\eta^t \dot{P}(\theta(t))\eta + \eta^t P(\theta(t))\dot{\eta} + 2\sum_{i=1}^{k} \tilde{\xi}_i \dot{\tilde{\xi}}_i \\
&= y\Lambda(\theta(t))\eta(t) + y^2 \mathcal{F}_2(y, \theta(t), t) - y b_1(\theta(t))\xi_1 \\
&\quad + \tfrac{1}{2}\eta^t \dot{P}(\theta(t))\eta \\
&\quad + \eta^t P(\theta(t))[\Upsilon(\theta(t))\eta(t) + y\mathcal{F}_1(y, \theta(t), t)] \\
&\quad + 2\sum_{i=1}^{k-1} \tilde{\xi}_i \Big\{ -\xi_i + \xi_{i+1} - \mathcal{D}_i(y, \xi_1, \ldots, \xi_i, t) \, . \\
&\quad - \frac{\partial \xi^*{}_i}{\partial y}[\Lambda(\theta(t))\eta(t) + y\mathcal{F}_2(y, \theta(t), t) - y b_1(\theta(t))\xi_1] \Big\} \\
&\quad + 2\tilde{\xi}_k \Big\{ -\xi_k + \xi^*{}_{k+1} - \mathcal{D}_k(y, \xi_1, \ldots, \xi_k, t) \\
&\quad - \frac{\partial \xi_k^*}{\partial y}[\Lambda(\theta(t))\eta(t) + y\mathcal{F}_2(y, \theta(t), t) - y b_1(\theta(t))\xi_1] \Big\} \\
&\triangleq L_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k), \qquad \text{(A.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{D}_i(y, \xi_1, \ldots, \xi_i, t) &= \frac{\partial \xi_i^*}{\partial \xi_1}(-\xi_1 + \xi_2) + \cdots \\
&\quad + \frac{\partial \xi_i^*}{\partial \xi_{i-1}}(-\xi_{i-1} + \xi_i) + \frac{\partial \xi_i^*}{\partial t}. \qquad \text{(A.2)}
\end{aligned}
$$

Note that $\mathcal{D}_i(y, \xi_1, \ldots, \xi_i, t)$ is a known function.

The time-derivative of (5.16) along system (5.10) with controller (5.11) and (5.15) is

$$\begin{aligned}\dot{V}_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}) &= y\dot{y} + \tfrac{1}{2}\eta^t \dot{P}(\theta(t))\eta + \dot{\eta}^t P(\theta(t))\eta + 2\sum_{i=1}^{k+1} \tilde{\xi}_i \dot{\tilde{\xi}}_i \\ &= y\Lambda(\theta(t))\eta(t) + y^2\mathcal{F}_2(y, \theta(t), t) - yb_1(\theta(t))\xi_1 \\ &\quad + \tfrac{1}{2}\eta^t \dot{P}(\theta(t))\eta + \dot{\eta}^t P(\theta(t))\eta + 2\sum_{i=1}^{k-1} \tilde{\xi}_i \dot{\tilde{\xi}}_i \\ &\quad + 2\tilde{\xi}_k \dot{\tilde{\xi}}_k + 2\tilde{\xi}_{k+1}\dot{\tilde{\xi}}_{k+1}. \end{aligned} \tag{A.3}$$

The second last term of (A.3) can be re-expressed as

$$\begin{aligned}2\tilde{\xi}_k \dot{\tilde{\xi}}_k &= 2\tilde{\xi}_k\left(-\xi_k + \xi_{k+1} - \frac{\partial \xi_k^*}{\partial y}\dot{y} - \mathcal{D}_k(y, \xi_1, \ldots, \xi_k, t)\right) \\ &= 2\tilde{\xi}_k\Bigg(- \xi_k + \xi_{k+1}^*(y, \xi_1, \ldots, \xi_k, t) \\ &\quad - \frac{\partial \xi_k^*}{\partial y}[\Lambda(\theta(t))\eta(t) \\ &\quad + \mathcal{F}_2(y, \theta(t), t) - yb_1(\theta(t)\Big)\xi_1] \\ &\quad - \mathcal{D}_k(y, \xi_1, \ldots, \xi_k, t)) + 2\tilde{\xi}_k\tilde{\xi}_{k+1}. \end{aligned} \tag{A.4}$$

Using (A.1)) and (A.4) we can re-express (A.3) into

$$\begin{aligned}\dot{V}_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}) &= L_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k) \\ &\quad + 2\tilde{\xi}_k\tilde{\xi}_{k+1} + 2\tilde{\xi}_{k+1}\dot{\tilde{\xi}}_{k+1}. \end{aligned} \tag{A.5}$$

Knowing that

$$\begin{aligned}\dot{\tilde{\xi}}_{k+1} &= -\xi_{k+1} + \xi_{k+2}^* - \frac{\partial \xi^*{}_{k+1}}{\partial y}\dot{y} \\ &\quad - \mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t), \end{aligned} \tag{A.6}$$

the last term in (A.5) can be written as

$$\begin{aligned}2\tilde{\xi}_{k+1}\dot{\tilde{\xi}}_{k+1} &= 2\tilde{\xi}_{k+1}\Bigg(-\xi_{k+1} + \xi_{k+2}^* - \frac{\partial \xi_{k+1}^*}{\partial y}\dot{y} \\ &\quad - \mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t)\Bigg). \end{aligned} \tag{A.7}$$

Now look at the second last term of (A.7)

$$SLT(y, \eta, \xi, \theta(t), t) \triangleq -2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}\dot{y}. \tag{A.8}$$

Using the system equations (5.10), we have

$$\begin{aligned}SLT(y, \eta, \xi, \theta(t), t) &= 2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}[\Lambda(\theta(t))\eta(t) \\ &\quad + y\mathcal{F}_2(y, \theta(t), t) - b_1(\theta(t))\xi_1]. \end{aligned} \tag{A.9}$$

We know that $\xi_1 = \tilde{\xi}_1 + \xi_1^*(y, t)$. Using the fact that $\xi_1^*(y, t)$ is a smooth function with $\xi_1^*(0, t) = 0$, we have $\xi_1 = \tilde{\xi}_1 + y\xi_1^{**}(y, t)$, where $\xi_1^{**}(y, t) = \xi_1^*(y, t)/y$ is also a smooth function. Rewrite (A.9) as follows:

$$\begin{aligned}SLT(y, \eta, \xi, \theta(t), t) &= -2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}[\Lambda(\theta(t))\eta(t) \\ &\quad + y\mathcal{F}_2(y, \theta(t), t) \\ &\quad - b_1(\theta(t))(\tilde{\xi}_1 + y\xi_1^{**}(y, t))]. \end{aligned} \tag{A.10}$$

Now applying the triangular inequality

$$ab \leq \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2 \tag{A.11}$$

for any given $\alpha > 0$, we get

$$\begin{aligned}&- 2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}\Lambda(\theta(t))\eta(t) \\ &\quad \leq \frac{1}{\alpha_1}\left(\frac{\partial \xi_{k+1}^*}{\partial y}\right)^2 ||\Lambda(\theta(t))||^2 \tilde{\xi}_{k+1}^2 + \alpha_1\eta^t\eta \\ &- 2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}y\mathcal{F}_2(y, \theta(t), t) \\ &\quad \leq \frac{1}{\alpha_2}\left(\frac{\partial \xi_{k+1}^*}{\partial y}\right)^2 \mathcal{F}_2^2(y, \theta(t), t)\tilde{\xi}_{k+1}^2 + \alpha_2 y^2 \\ &- 2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}b_1(\theta(t))\tilde{\xi}_1 \\ &\quad \leq \frac{1}{\alpha_3}\left(\frac{\partial \xi_{k+1}^*}{\partial y}\right)^2 b_1^2(\theta(t))\tilde{\xi}_{k+1}^2 + \alpha_3\tilde{\xi}_1^2 \\ &- 2\tilde{\xi}_{k+1}\frac{\partial \xi_{k+1}^*}{\partial y}b_1(\theta(t))y\xi_1^{**}(y, t) \\ &\quad \leq \frac{1}{\alpha_4}\left(\frac{\partial \xi^*{}_{k+1}}{\partial y}\right)^2 [b_1(\theta(t))\xi_1^{**}(y, t)]^2 \xi_{k+1}^2 + \alpha_4 y^2. \end{aligned}$$

It follows that

$$\begin{aligned}&SLT(y, \eta, \xi, \theta(t), t) \\ &\quad \leq \left(\frac{\partial \xi_{k+1}^*}{\partial y}\right)^2 \Bigg(\frac{1}{\alpha_1}||\Lambda(\theta(t))||^2 + \frac{1}{\alpha_2}\mathcal{F}_2^2(y, \theta(t), t) \\ &\quad + \frac{1}{\alpha_3}b_1^2(\theta(t)) + \frac{1}{\alpha_4}[b_1(\theta(t))\xi_1^{**}(y, t)]^2\Bigg)\tilde{\xi}_{k+1}^2 \\ &\quad + \alpha_1\eta^t\eta + \alpha_2 y^2 + \alpha_3\tilde{\xi}_1^2 + \alpha_4 y^2, \end{aligned} \tag{A.12}$$

for any positive constants $\alpha_i, i = 1$–4. Using Conditions C2 and C3, the function $SLT(y, \eta, \xi, \theta(t), t)$ is bounded as follows:

$$\begin{aligned} SLT(y, \eta, \xi, \theta(t), t) & \\ \le \left(\frac{\partial \xi^*_{k+1}}{\partial y}\right)^2 & \left[\frac{1}{\alpha_1} M^2_\Lambda + \frac{1}{\alpha_2}\varrho^2_2(y,t) + \frac{1}{\alpha_3}\bar{b}^2_1 + \frac{1}{\alpha_4}(\bar{b}_1 \xi^{**}_1(y,t)^2\right]\tilde{\xi}^2_{k+1} \\ & + \alpha_1 \eta^t \eta + \alpha_2 y^2 + \alpha_3 \tilde{\xi}^2_1 + \alpha_4 y^2 \\ = \Gamma(y, \xi, t)\tilde{\xi}^2_{k+1} & + \alpha_1 \eta^t \eta + \alpha_2 y^2 + \alpha_3 \tilde{\xi}^2_1 + \alpha_4 y^2, \end{aligned} \tag{A.13}$$

where $\Gamma(y, \xi, t)$ in defined in (5.21). Substituting (A.12) and (A.7) into (A.3), we have

$$\begin{aligned} \dot{V}_{k+1}(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_{k+1}) & \\ \le L_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k) & + \alpha_1 \eta^t \eta + (\alpha_2 + \alpha_4) y^2 \\ & + \alpha_3 \tilde{\xi}^2_1 + 2\tilde{\xi}_{k+1}[\tilde{\xi}_k - \xi_{k+1} + \xi^*_{k+2} \\ & - \mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t) + \Gamma(y, \xi, t)\tilde{\xi}_{k+1}]. \end{aligned} \tag{A.14}$$

Note that from (5.14) we know that

$$L_k(y, \eta, \tilde{\xi}_1, \ldots, \tilde{\xi}_k) \le -a_k\left(\frac{1}{2}y^2 + \frac{1}{2}\eta^t P(\theta(t))\eta + \sum_{i=1}^{k}\tilde{\xi}^2_i\right). \tag{A.15}$$

Using (A.14), (A.15) and ξ^*_{k+2} as in (5.19), we obtain

$$\dot{V}_{k+1} \le -a_k\left(\frac{1}{2}y^2 + \frac{1}{2}\eta^t P(\theta(t))\eta + \sum_{i=1}^{k}\tilde{\xi}^2_i\right) + \alpha_1 \eta^t \eta + (\alpha_2 + \alpha_4) y^2 + \alpha_3 \tilde{\xi}^2_1 - 2a_{k+1}\tilde{\xi}^2_{k+1}.$$

Take α_i to be sufficiently small. We can find a_{k+1} such that (5.18) holds.

Finally, we note that $\mathcal{D}_{k+1}(y, \xi_1, \ldots, \xi_{k+1}, t)$ and $\Gamma(y, \xi, t)$ are smooth functions. This implies that ξ^*_{k+2} is also smooth. □

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