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Robust $\mathcal{H}_\infty$ Control for a class of Nonlinear Systems: A LMI Approach*

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ABSTRACT

This paper considers the problem of robust $\mathcal{H}_\infty$ control for a class of nonlinear systems. The nonlinear uncertainities which we allow include are those satisfying integral quadratic constraints (IQCs), and those nonlinear bounded terms satisfying matching conditions. We adopt an LMI approach to designing a state feedback controller which guarantees the so-called $\mathcal{L}_2$-gain from the exogenous input noise to the controlled output to be no larger than a prescribed value. We then extend this robust $\mathcal{H}_\infty$ result to systems for which the nonlinear uncertainty satisfies a *generalized matching conditions.*

1. INTRODUCTION

There has been some substantial interest over the past several years in extending the $\mathcal{H}_\infty$ control theory for linear systems to nonlinear systems; see [1, 2, 3, 4] for example. Like the linear case, the idea of nonlinear $\mathcal{H}_\infty$ control is to find a feedback controller, either full state or output-based, such that the mapping from an exogenous input to a controlled output is minimized in terms of an induced $\mathcal{L}_2$-gain. The common feature of the results in [1, 2, 3, 4] shows that the nonlinear H_∞ control can be solved via a particular type of Hamilton-Jacobi equations, known as Isaac equations. The solvability of an Isaac equation has been addressed in [1]. It is known [1] that if a linearized Isaac equation at an equilibrium point, which is a Riccati equation, has a solution, then the Isaac equation has a solution in the neighborhood of the equilibrium point provided that some smoothness conditions are satisfied.

The motivation of this paper stems from two facts: i) solutions to the Isaac equations are generally difficult to find, especially when the region of interest in the state space is non-local or global. Alternatively, we seek for a special class of nonlinear systems for which simple solutions exist. ii) The nonlinear H_∞ control results provided in [1, 2, 3, 4] assume a perfect model of the system, i.e, no modeling uncertainty is permitted. What we intend to do in this paper is to consider some type of uncertainty in the system model, aiming at developing a controller which is robust with respect to the modeling uncertainty.

In this paper, we consider a class of nonlinear systems which consists of a linear nominal model and nonlinear uncertainties. More precisely, the nominal model involves an exogenous input, control input and controlled output. The nonlinear uncertainties we allow include, those satisfying IQCs, and those nonlinear bounded nonlinear terms satisfying matching condition. See Section 2 for details.

For the systems discussed above, our main results in Section 4 show that the robust $\mathcal{H}_\infty$ control problems can be solved via a fixed full state feedback controller if an LMI (rather than an Isaac equation) has a solution. Furthermore, a simple procedure for constructing a desired robust nonlinear controller is also provided..

2. ROBUST $\mathcal{H}_\infty$ ANALYSIS

Consider a class of nonlinear uncertain systems described by a state-space model of the following form:

$$\begin{aligned} \dot{x}(t) &= Ax + \textstyle\sum_{i=1}^{p} D_i\xi_i(t) + B_1\omega(t) + Bf(x)x \\ y(t) &= C_1x(t) + D_oW \end{aligned} \quad (2.1)$$

where $x(t) \in \mathbb{R}^m$ is the state, $w(t) \in \mathbb{R}^q \in \mathcal{L}_2(0,\infty)$ is the bounded disturbance, $y(t) \in \mathbb{R}^j$ is the penalty variable related to some performance cost, $\xi_i(t) \in \mathbb{R}^{s_i}$, the uncertain variables satisfying the following IQCs

$$\int_0^T (\|E_ix(t)\|^2 - \|\xi_i\|^2)\, dt \geq 0,\ \forall T > 0. \quad (2.2)$$

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A, B, B_1, C_1, D_o, D_i and E_i are known constant matrices with appropriate dimensions. $f(x(t))$ is an $q \times m$ matrix function representing the nonlinear uncertainty in the autonomous part of the plant which assumes to satisfy

$$\rho(x(t)) - \|f(x(t))\| \geq 0, \ \forall x \in \mathbb{R}^m \tag{2.3}$$

for some $\rho(x(t)) \geq 0$

Remark 1. Norm bounded uncertainty has been considered in a number of papers. For instance, in [5] authors have investigated the following type of uncertainty:

$$\xi_1(t) = F(t)E_1 x(t) \tag{2.4}$$

with

$$F^t(t)F(t) \leq I, \ \forall t > 0. \tag{2.5}$$

Obviously, this type of uncertainty corresponds to our case with $p = 1$. Moreover, condition (2.5) is much more restrictive than condition (2.2).

Now, we state the robust $\mathcal{H}_\infty$ analysis problem as follows: *Given the system (2.1) and γ, determine if the system is asymptotically stable and that the following condition holds*

$$\int_0^T \|y(t)\|^2 dt < \gamma^2 \int_0^T \|w(t)\|^2 dt, \ \forall T \geq 0, \quad x(0) = 0, w \in \mathbb{R}^q \in \mathcal{L}_2(0,T) \tag{2.6}$$

for all admissible uncertainties satisfying the IQCs (2.2) and condition (2.3).

For the sake of notational convenience, we define:

$$\mathbb{E}_1^t = [E_1^t \ \cdots \ E_2^t]; \ \ \xi(t) = [\xi_1(t) \ \cdots \ \xi_p(t)] \quad \mathbb{J} = diag\{\tau_1 I_{s_1}, \cdots, \tau_p I_{j_p}\} \tag{2.7}$$

where $\tau_1, \cdots, \tau_p$ are scalars and s_i are the number of columns of D_i.

Applying the well-known $\mathcal{S}$-procedure, we have the following result:

Lemma 2.1. *The system (2.1) is exponential stable with condition (2.6) holds for all admissible uncertainties satisfying the IQCs (2.2) and condition (2.3) if there exist a K-function [6] $V(x)$, scaling parameters $\tau_1, \cdots, \tau_p > 0$ and ϵ such that the following condition holds:*

$$\begin{array}{l} L + \sum_{i=1}^p \tau_i(\|E_i x(t)\|^2 - \|\xi_i(t)\|^2) + \|C_1 x(t) \\ +D_o w\|^2 - \gamma^2\|w\|^2 + \epsilon V(x) \leq 0, \\ \forall x \in \mathbb{R}^m; \ \xi_i(t) \in \mathbb{R}^{m_i} \ and \ w \in \mathbb{R}^q \in \mathcal{L}_2(0,T), \\ \forall T \geq 0, \ with \ (x, w, \xi_i) \neq 0 \end{array} \tag{2.8}$$

where

$$L \triangleq \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x}[Ax(t) + \sum_{i=1}^p D_i \xi_i(t) + B_1\omega(t) + Bf(x(t))] \tag{2.9}$$

Proof. First, we need to show that the system (2.1) is exponential stable when $w = 0$. Obviously, the time derivative of $V(x)$ along the trajectory of the system (2.1) reads

$$\dot{V}(x) = L = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x}[Ax(t) + \sum_{i=1}^p D_i\xi_i(t) + B_1\omega(t) + Bf(x(t))] \tag{2.10}$$

Clearly, from (2.8) we have

$$\dot{V}(x) \leq -\epsilon V(x) \tag{2.11}$$

which implies that system (2.1) is exponential stabe.

To prove that condition (2.6) holds, we intergrate the left hand side of the inequality in (2.8) from 0 to T along the trajectory of the system (2.1), we have $\forall T > 0$

$$\begin{array}{l} V(x(T)) - V(x(0)) \\ + \int_0^T \sum_{i=1}^p \tau_i(\|E_i x(t)\|^2 - \|\xi_i(t)\|^2) \ dt \\ + \int_0^T (\|C_1 x(t) + D_o w\|^2 - \gamma^2\|w\|^2) \ dt < 0. \end{array} \tag{2.12}$$

Let $T \mapsto \infty$ and $x(0) = 0$, we obtain (2.6). ▽▽▽

3. ROBUST $\mathcal{H}_\infty$ SYNTHESIS

Consider a class of nonlinear systems described by a state-space model of the following form:

$$\begin{array}{ll} \dot{x}(t) & = Ax(t) + \sum_{i=1}^p D_i\xi_i(t) + B_1\omega(t) \\ & +B(I + J(x(t))u(t) + +Bf(x(t))x(t) \\ y(t) & = C_1 x(t) + D_o w \end{array} \tag{3.1}$$

where $x(t) \in \mathbb{R}^m$ is the state, $u(t) \in \mathbb{R}^q$ is the input, $w \in \mathbb{R}^q \in \mathcal{L}_2(0,T)$ is the bounded disturbance, $y(t) \in \mathbb{R}^j$ is the penalty variable related to some performance cost. The uncertain matrix function $f(x(t))$ and the uncertain variables $\xi_i(t)$ are assumed to satisfy conditions (2.2) and (2.3), respectively. The uncertain matrix $J(x(t))$ is a function satisfying:

$$\|J(x(t))\| - \eta \leq 0 \text{ for some } 0 \leq \eta \leq 1, \ \forall x(t) \in \mathbb{R}^m. \tag{3.2}$$

The robust $\mathcal{H}_\infty$ synthesis problem associated with uncertain system (3.1) is as follows: *Given $\gamma > 0$, find a controller of the form*

$$u(t) = \phi(x) \tag{3.3}$$

such that the closed-loop system is asymptotically stable and satisfies the condition (2.6) for all admissible uncertainties satisfying conditions (2.2), (2.3) and (3.2).

Now we are ready to state our main result:

Theorem 3.1. *Given a scalar $\gamma > \sqrt{\lambda_{\max}[D_o^t D_o]}$, the uncertain system (3.1) whose uncertainties satisfying* *conditions (2.2), (2.3) and (3.2) is asymptotic stable with condition (2.6) holds if there exists* $\tau_1, \cdots, \tau_p > 0$, $\epsilon > 0$ *and* $\alpha > \frac{\lambda_{\max}[D_o^t D_o]}{\gamma^2 - \lambda_{\max}[D_o^t D_o]}$ *such that the following LMI*

$$\begin{bmatrix} \mathbb{M} & PB & PB_1 & PD \\ B^t P & -1 & 0 & 0 \\ B_1^t P & 0 & \mathbb{N} & 0 \\ D^t P & 0 & 0 & -\mathbb{J} \end{bmatrix} \leq 0 \tag{3.4}$$

has symmetric positive-definite solution P. *Where* $\mathbb{M} = A^t P + PA + (1+\alpha) C_1^t C_1 + \mathbb{E}_1^t \mathbb{J} \mathbb{E}_1 + 3\epsilon I$ *and* $\mathbb{N} = -[\gamma^2 I - \frac{(\alpha+1)}{\alpha}) D_o^t D_o]$

If this is the case, then a suitable feedback control law is given by

$$u = -\frac{B^t P x}{1-\eta}\left[\frac{1}{2} + \frac{\rho^2(x)}{\epsilon}\right] \tag{3.5}$$

Proof. The closed-loop of system (3.1) with (3.5) reads

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \textstyle\sum_{i=1}^p D_i \xi_i(t) + Bf(x)x \\ &\quad - \tfrac{B(I+J(x(t)))B^t P x}{1-\eta}\left[\tfrac{1}{2} + \tfrac{\rho^2(x)}{\epsilon}\right] + B_1 \omega(t) \\ z(t) &= C_1 x(t) + D_o w \end{aligned} \tag{3.6}$$

First, we choose our K-function $V(x) = x^t P x$. It follows from Lemma 2.1 that the robust $\mathcal{H}_\infty$ control problem is solvable using the controller in (3.5) if the following inequality holds

$$\begin{aligned} &2x^t(t) P \{Ax(t) + \textstyle\sum_{i=1}^p D_i \xi_i(t) + B_1 \omega(t) \\ &+ Bf(x(t)) - \tfrac{B(I+J(x(t)))B^t P x}{1-\eta}\left[\tfrac{1}{2} + \tfrac{\rho^2(x)}{\epsilon}\right]\} \\ &+ \textstyle\sum_{i=1}^p \tau_i (\|E_i x(t)\|^2 - \|\xi_i(t)\|^2) + \|C_1 x(t) \\ &+ D_o w\|^2 - \gamma^2 \|w\|^2 + \epsilon x^t(t) x(t) \leq 0, \forall x \in \mathbb{R}^m \text{ and} \\ &w \in \mathbb{R}^q \in \mathcal{L}_2(0,T),\ \forall T \geq 0,\ \text{with } (x, w, \xi_i) \neq 0 \end{aligned} \tag{3.7}$$

We can rewrite Eq. (3.7) as

$$\begin{aligned} &2x^t(t) P \{Ax(t) + \textstyle\sum_{i=1}^p D_i \xi_i(t) + B_1 \omega(t)\} \\ &+ \textstyle\sum_{i=1}^p \tau_i (\|E_i x(t)\|^2 - \|\xi_i(t)\|^2) \\ &+ \|C_1 x(t) + D_o w\|^2 - \gamma^2 \|w\|^2 + \epsilon x^t x \leq -2G(x), \end{aligned} \tag{3.8}$$

where

$$G(x) = x^t P B f(x) x - \frac{x^t P B (I + J(x(t)) B^t P x}{1-\eta} \times \left[\frac{1}{2} + \frac{\rho^2(x)}{\epsilon}\right]. \tag{3.9}$$

Let us derive an upper bounds for term $G(x)$ apply triangular inequality and condition (3.2) on it will lead to

$$\begin{aligned} G(x) &\leq \frac{1}{\epsilon} x^t P B B^t P x \rho^2(x) + \epsilon x^t x \\ &\quad - \frac{1}{\epsilon} x^t P B B^t P x \rho^2(x) - \frac{1}{2} x^t P B B^t P x \\ &= \epsilon x^t x - \frac{1}{2} x^t P B B^t P x. \end{aligned} \tag{3.10}$$

Also by applying triangular inequality on the term $\|C_1 x + D_o w\|^2$, we have

$$\begin{aligned} \|C_1 x + D_o w\|^2 &\leq (1+\alpha) \|C_1 x(t)\|^2 \\ &\quad + \frac{(\alpha+1)}{\alpha} D_o^t D_o \|w\|^2, \end{aligned} \tag{3.11}$$

where $\alpha > \frac{\lambda_{\max}[D_o^t D_o]}{\gamma^2 I - \lambda_{\max}[D_o^t D_o]}$. Substitute these two upper bounds into (3.8), we have

$$\begin{aligned} &2x^t(t) P \{Ax(t) + \textstyle\sum_{i=1}^p D_i \xi_i(t) + B_1 \omega(t)\} \\ &+ \textstyle\sum_{i=1}^p \tau_i (\|E_i x(t)\|^2 - \|\xi_i(t)\|^2) \\ &+ (1+\alpha) \|C_1 x(t)\|^2 - [\gamma^2 I - \tfrac{(\alpha+1)}{\alpha}) D_o^t D_o \|w\|^2 \\ &+ \epsilon x^t(t) x(t) \leq -2x^t(\epsilon I - \tfrac{1}{2} P B B^t P) x, \end{aligned} \tag{3.12}$$

or we can rewrite it as follows:

$$\mathbb{X}^t \begin{bmatrix} \mathbb{M} - PBB^t P & PB_1 & PD \\ B_1^t P & \mathbb{N} & 0 \\ D^t P & 0 & -\mathbb{J} \end{bmatrix} \mathbb{X} \leq 0. \tag{3.13}$$

where $\mathbb{X} = [x\ w\ \xi]^t$. Clearly, to have (3.13) holds $\forall x \in \mathbb{R}^m$ and $w \in \mathbb{R}^q \in \mathcal{L}_2(0,T)$, $\forall T \geq 0$, with $\mathbb{X} \neq 0$ is equivalent to having

$$\begin{bmatrix} \mathbb{M} - PBB^t P & PB_1 & PD \\ B_1^t P & \mathbb{N} & 0 \\ D^t P & 0 & -\mathbb{J} \end{bmatrix} \leq 0. \tag{3.14}$$

Apply well-known Schur complements on (3.14) will lead (3.4), hence we conclude our proof. $\nabla\nabla\nabla$

4. EXTENSION TO GENERALIZED MATCHING CONDITIONS

This section extends the main result in previous section to a more general class of nonlinear uncertain sysems, namely nonlinear systems satisfying the so-called generalized matching conditions. The reader is referred to [7] and [8] for the definition of generalized matching conditions and the *back stepping* robust control design procedure.

In this section, we allow the following type of systems:

$$\dot{x}_1 = Ax + \sum_{i=1}^p D_i \xi_i + B(I + J(x_1)) x_2 \tag{4.1}$$

$$+ B f_1(x_1) x_1 + B_1 w(t) \tag{4.2}$$

$$\dot{x}_2 = f_2(\mathbb{X}_2) \mathbb{X}_2 + x_3 + b_2 w \tag{4.3}$$

$$\cdots$$

$$\dot{x}_i = f_i(\mathbb{X}_i) \mathbb{X}_i + x_{i+1} + b_i w \tag{4.4}$$

$$\cdots$$

$$\dot{x}_n = f_n(\mathbb{X}_n) \mathbb{X}_n + b_n w + u \tag{4.5}$$

$$y(t) = c(\mathbb{X}_n) + d_o(\mathbb{X}_n) w \tag{4.6}$$

where $x_1(t) \in \mathbb{R}^m$, $\mathbb{X}_i \triangleq [x_1\ \cdots\ x_i]$, and $f_i(\mathbb{X}_i)$ are uncertain nonlinear matrix functions satisfying the same condition as in (3.2). The functions $c(\mathbb{X}_q)$ and $d_o(\mathbb{X}_n)$ are assumed to satisfy the following inequalities:

$$c(T(\mathbb{Z}_n))c^t(T(\mathbb{Z}_n)) \leq \sum_{k=1}^{n} z_k^t C_k^t C_k z_k \tag{4.7}$$

$$\|D_o(T(\mathbb{Z}_n))\| \leq D_o \tag{4.8}$$

where $\mathbb{Z}_n \triangleq [z_1 \; z_2 \; \cdots, z_n]$ and $T(\cdot)$ is any known nonlinear matrix mapping.

Corollary 4.1. *The system (4.2)-(4.6) satisfying conditions (2.2), (2.3), (3.2), (4.7) and (4.8) is asymptotically stabilizable with condition (2.6) if there exists* $\tau_1, \cdots, \tau_p > 0$, $\epsilon > 0$ *and* $\alpha > \frac{2\lambda_{\max}[D_o^t D_o]}{\gamma^2 - \lambda_{\max}[D_o^t D_o]}$ *such that the following LMI*

$$\begin{bmatrix} \mathbb{M} - \frac{\epsilon}{2} I & PB & PB_1 & PD \\ B^t P & -1 & 0 & 0 \\ B_1^t P & 0 & \mathbb{N} & 0 \\ D^t P & 0 & 0 & -\frac{\mathbb{J}}{2} \end{bmatrix} \leq 0 \tag{4.9}$$

has symmetric positive-definite solution P.

For proof refer to Appendix A.

5. CONCLUSION

In this paper, we have solved the problem of robust $\mathcal{H}_\infty$ control problem for a class of nonlinear systems. The nonlinear uncertainities which we allow include, those satisfying integral quadratic constraints (IQCs), and those general bounded nonlinear terms satisfying matching conditions. We have presented a state feedback control design technique by using an LMI. This result extends the standard H_∞ state feedback results for linear uncertain systems [5] and some robust stabilization results for uncertain system with matching conditions [9, 10].

Appendix A : Proof of Corollary 4.1

Proof. Step 1. Define $z_1 = x_1$ and $z_2 = x_2 - \phi_1(z_1, t)$, where $\phi_1(z_1, t)$ is a smooth function yet to be determined. The first equation of the system (4.2) can be written as

$$\dot{z}_1 = Az_1 + \sum_{k=1}^{p} D_k \xi_k + B_1 w + Bf(z_1)z_1 + B(I + J(z_1))[z_2 + \phi_1(z_1)] \tag{A.1}$$

First ignore the term $B(I + J(z_1))z_2$. If we choose $V_1(z_1) = z_1^t P z_1$ and

$$\phi_1(z_1) = -\frac{B^t P z_1}{1-\eta}\left[\frac{1}{2} + \frac{\rho^2(z_1)}{\epsilon}\right], \tag{A.2}$$

we have

$$\begin{aligned} L_1 &\triangleq \frac{\partial V_1(z_1)}{\partial z_1}\dot{z}_1 \\ &= 2z_1^t(t)P\left\{Az_1(t) + \sum_{k=1}^{p} D_k\xi_k(t) + B_1\omega(t) + Bf(z_1(t)) + \frac{B(I + J(z_1(t)))B^t P z_1}{1-\eta} \times \left[\frac{1}{2} + \frac{\rho^2(x)}{\epsilon}\right]\right\} && \text{(A.3)} \\ &\leq 2z_1^t P\left\{Az_1 + \sum_{k=1}^{p} D_k\xi_k + B_1\omega(t) - \frac{1}{2}BB^t P z_1\right\} + 2\epsilon z_1^t z_1. && \text{(A.4)} \end{aligned}$$

Note that $\phi_1(z_1)$ given in (A.2) is a smooth function, hence its partial derivative exists.

Step i ($2 \leq i \leq n-1$). We repeat the above procedure, defining $z_{i+1} = x_{i+1} - \phi_i(z_1, \cdots, z_i)$, then

$$\begin{aligned} \dot{z}_i = \sum_{k=1}^{i} f_{ik}(z_1, \cdots, z_i)z_k + [z_{i+1} + \phi_i(z_1, \cdots, z_i)] + b_i w + \frac{\partial\phi_{i-1}(z_1, \cdots, z_{i-1})}{\partial z_1}Bw \\ + \sum_{k=2}^{i-1}\frac{\partial\phi_{i-1}(z_1, \cdots, z_{i-1})}{\partial z_k}b_k w + \frac{\partial\phi_{i-1}(z_1, \cdots, z_{i-1})}{\partial z_1}\sum_{k=1}^{p} D_k\xi_k. \end{aligned} \tag{A.5}$$

The functions $f_{ik}(z_1, \cdots, z_i)$ are defined as follows:

$$\begin{aligned} \sum_{k=1}^{i} f_{ik}(z_1, \cdots, z_i)z_k = f_i(z_1, z_2 + \phi_1(z_1), \cdots, z_i \\ + \phi_{i-1}(z_1, \cdots, z_{i-1}))\begin{bmatrix} z_1 \\ z_2 + \phi_1(z_1) \\ \cdots \\ z_i + \phi_{i-1}(z_1, \cdots, z_{i-1}) \end{bmatrix} + \psi_{i-1}(z_1, \cdots, z_i) \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} \psi_{i-1}(z_1, \cdots, z_i) = \sum_{k=1}^{i-1}\frac{\partial\phi_{i-1}(z_1, \cdots, z_{i-1})}{\partial z_k} \times \\ \{\sum_{l=1}^{k} f_{kl}(z_1, \cdots, z_k)z_k + [z_{k+1} + \phi_k(z_1, \cdots, z_k)] + \psi_{k-1}(z_1, \cdots, z_k)\}. \end{aligned} \tag{A.7}$$

Choose our new function $V_i(z_1, \cdots, z_i) = z_1^t P z_1 + \sum_{k=2}^{i} z_k^2$, we have

$$\begin{aligned} L_i &\triangleq \sum_{k=1}^{i}\frac{\partial V_i(z_1, \cdots, z_i)}{\partial z_k}\dot{z}_k \\ &= L_{i-1} + 2z_{i-1}z_i + 2z_i\left\{\sum_{l=1}^{i} f_{il}(z_1, \cdots, z_i)z_l + [z_{i+1} + \phi_i(z_1, \cdots, z_i)] + \frac{\partial\phi_{i-1}(z_1, \cdots, z_{i-1})}{\partial z_1}Bw \right. \end{aligned}$$

$$+\sum_{k=2}^{i-1}\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_k}b_k w$$
$$+\ \frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}\sum_{k=1}^{p}D_k\xi_k\Bigg\} \quad \text{(A.8)}$$

Applying triangular inequality, we have

$$2z_i\sum_{k=1}^{i-1}f_{ik}(z_1,\cdots,z_k)z_k \le$$
$$\frac{2n}{\epsilon}z_i^2\sum_{k=1}^{i-1}f_{ik}^2(z_1,\cdots,z_i)+\sum_{k=1}^{i-1}\frac{\epsilon}{2n}z_k^t z_k$$
$$2z_i\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}\sum_{k=1}^{p}D_k\xi_k \le$$
$$z_i^2\sum_{k=1}^{p}\frac{1}{\sigma_{ik}}\left(\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}D_k\right)^2+\sum_{k=1}^{p}\sigma_{ik}\|\xi_k\|^2$$
$$2z_i\left[\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}B+\right.$$
$$\left.\sum_{k=2}^{i-1}\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_k}b_k+b_i\right]w \le$$
$$z_i^2\frac{1}{\beta_i}\left(\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}B\right.$$
$$\left.+\sum_{k=2}^{i-1}\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_k}b_k+b_i\right)^2+\beta_i\|w\|^2 \quad \text{(A.9)}$$

Substituting these upper bounds into (A.8) yields

$$L_i \le L_{i-1}+2z_i[\psi(z_1,\cdots,z_i)+z_{i-1}+\phi_i(z_1,\cdots,z_i)]$$
$$+\sum_{k=1}^{i-1}\frac{\epsilon}{2n}z_k^t z_k+\sum_{k=1}^{p}\sigma_{ik}\|\xi_k\|^2+\beta_i\|w\|^2$$

where

$$\psi(z_1,\cdots,z_i)\ge\Bigg\|\frac{2n}{\epsilon}\Big[(f_{i(i-1)}(z_1,\cdots,z_i)+1)^2$$
$$+\sum_{k=2}^{i-1}f_{ik}^2(z_1,\cdots,z_i)\Big]+f_{ii}(z_1,\cdots,z_i)$$
$$+\sum_{k=1}^{p}\frac{1}{\sigma_{ik}}\left(\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}D_k\right)^2$$
$$+\frac{1}{\beta_i}\left(\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_1}B\right.$$
$$\left.+\sum_{k=2}^{i-1}\frac{\partial\phi_{i-1}(z_1,\cdots,z_{i-1})}{\partial z_k}b_k+b_i\right)^2\Bigg\| \quad \text{(A.10)}$$

is a known smooth function. Following the design given in Step 1, for $i=2,3,\cdots,n-1$ we choose we choose for $i=2,3,..n-1$

$$\phi_i(z_1,\cdots,z_i)=-z_i[\varphi_{i-1}(z_1,\cdots,z_i)+z_{i-1}+\frac{\epsilon}{2}]. \quad \text{(A.11)}$$

and for $i=n$ we choose

$$u=-z_n\left[\varphi_{n-1}(z_1,\cdots,z_n)+z_{n-1}+\frac{\epsilon}{2}\right]. \quad \text{(A.12)}$$

Then

$$L_i \le L_{i-1}+\textstyle\sum_{k=2}^{i-1}\frac{\epsilon}{2n}z_k^t z_k+\sum_{k=1}^{p}\sigma_{ik}\|\xi_i\|^2$$
$$+\textstyle\sum_{k=1}^{i}\beta_k\|w\|^2 \quad \text{(A.13)}$$

Note that $\phi_2(z_1,\cdots,z_i)$ is also a smooth function, that is, it does not contain first order Euclidean norms.

Now we need to show that with the choice of u given in (A.12), the following condition holds:

$$\int_0^T\|c(\mathbb{X}_n)+d_o(\mathbb{X}_n)\|^2dt<\gamma^2\int_0^T\|w(t)\|^2dt \quad \text{(A.14)}$$

$\forall T\ge 0,\ x(0)=0, w\in\mathbb{R}^q\in\mathcal{L}_2(0,T)$.

In terms of z coordinates, we need to show that $\forall T\ge 0,\ z(0)=0, w\in\mathbb{R}^q\in\mathcal{L}_2(0,T)$

$$\int_0^T\|c(T(\mathbb{Z}_n))+d_o(T(\mathbb{Z}_n))\|^2dt<\gamma^2\int_0^T\|w(t)\|^2dt, \quad \text{(A.15)}$$

where the transformation mapping T is given as follows:

$$T:\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}=\begin{bmatrix}z_1\\z_2+\phi_1(z_1)\\\cdots\\z_n+\phi_n(z_1,\cdots,z_n)\end{bmatrix} \quad \text{(A.16)}$$

Following Lemma 2.1 we know that if we can show that $\forall\mathbb{Z}_n\in\mathbb{R}^{m+n}$; $\xi_k(t)\in\mathbb{R}^{s_k}$ and $w\in\mathbb{R}^q\in\mathcal{L}_2(0,T)$, $\forall T\ge 0$, with $(\mathbb{Z}_n,w,\xi_k)\ne 0$

$$L_n+\textstyle\sum_{k=1}^{p}\tau_k(\|E_k z_1(t)\|^2-\|\xi_k(t)\|^2)$$
$$+\|h(T(\mathbb{Z}_n))+D_o(T(\mathbb{Z}_n))w\|^2-\gamma^2\|w\|^2$$
$$+\epsilon z_1^t(t)z_1(t)\le 0 \quad \text{(A.17)}$$

which also implies that condition (A.15) holds. First we look the term L_n, by using L_i derived in (A.13), it is easy to show that

$$L_n\le-\tfrac{\epsilon}{2}\textstyle\sum_{k=2}z_k^t z_k+\frac{\epsilon}{2}z_1^t z_1$$
$$+\textstyle\sum_{k=1}^{p}\sum_{l=1}^{n}\sigma_{lk}\|\xi_k\|^2+\sum_{k=1}^{n}\beta_k\|w\|^2. \quad \text{(A.18)}$$

Substituting (A.18) into (A.17), we have

$$2z_1^tP\left\{Az_1+\sum_{k=1}^{p}D_k\xi_k+B_1\omega(t)-\frac{1}{2}BB^tPz_1\right\}$$
$$+\frac{4\epsilon}{2}z_1^t z_1-\frac{\epsilon}{2}\sum_{k=2}z_k^t z_k+\sum_{k=1}^{n}\beta_k\|w\|^2$$
$$+\sum_{k=1}^{p}\left\{\sum_{l=1}^{n}\sigma_{lk}-\tau_k\right\}\|\xi_k\|^2+\sum_{i=1}^{p}\tau_i\|E_i z_1(t)\|^2$$
$$+\|c(T(\mathbb{Z}_n))+D_o(T(\mathbb{Z}_n))w\|^2-\gamma^2\|w\|^2\le 0 \quad \text{(A.19)}$$

Applying triangular inequality and the conditions (4.7) and (4.8) on $\|c(T(\mathbb{Z}_n)) + D_o(T(\mathbb{Z}_n))w\|^2$, we have

$$\|c(T(\mathbb{Z}_n)) + D_o(T(\mathbb{Z}_n))w\|^2 \leq$$
$$(1+\alpha)\|c(T(\mathbb{Z}_n))\|^2 + \frac{(\alpha+1)}{\alpha}\|D_o(T(\mathbb{Z}_n))w\|^2$$
$$\leq (1+\alpha)\sum_{k=1}^{n} z_k^t C_k^t C_k z_k + \frac{(\alpha+1)}{\alpha}\|D_o w\|^2$$

Substituting this upper bound into (A.19) yields

$$2z_1^t P\left\{Az_1 + \sum_{k=1}^{p} D_k\xi_\kappa + B_1\omega(t) - \frac{1}{2}BB^tPz_1\right\}$$
$$+\frac{4\epsilon}{2}z_1^t z_1 - \sum_{k=2}^{n}[\frac{\epsilon}{2} - C_k^2]z_k^2 + \sum_{i=1}^{p}\tau_i\|E_i z_1(t)\|^2$$
$$-\sum_{k=1}^{p}\left\{\tau_k - \sum_{l=1}^{n}\sigma_{lk}\right\}\|\xi_k\|^2 + (1+\alpha)z_1^t C_1^t C_1 z_1$$
$$-\left[\gamma^2 - \sum_{k=1}^{n}\beta_k - \frac{(\alpha+1)}{\alpha}D_o^t D_o\right]\|w\|^2 \leq 0 \quad (A.20)$$

To simplify our argument, we make the following choices:

$$\sum_{l=1}^{n}\sigma_{lk} = \frac{\tau_k}{2} \quad (A.21)$$
$$\epsilon > 2\max\{C_2^2, C_3^2, \cdots, C_n^2\} \quad (A.22)$$
$$\sum_{k=1}^{n}\beta_k = \frac{1}{\alpha} \quad (A.23)$$

Using the above choices, we have

$$2z_1^t P\left\{Az_1 + \sum_{k=1}^{p} D_k\xi_k + B_1\omega(t) - \frac{1}{2}BB^tPz_1\right\}$$
$$+\frac{5\epsilon}{2}z_1^t z_1 + \sum_{i=1}^{p}\tau_i\|E_i z_1(t)\|^2 - \sum_{k=1}^{p}\frac{\tau_k}{2}\|\xi_k\|^2$$
$$+(1+\alpha)z_1^t C_1^t C_1 z_1 - \left[\gamma^2 - \frac{(\alpha+2)}{\alpha}D_o^t D_o\right]\|w\|^2 \leq 0$$

or we can rewrite it as follows:

$$\mathbb{X}^t\begin{bmatrix} \mathbb{M} - \frac{1\epsilon}{2}I - PBB^tP & PB_1 & PD \\ B_1^tP & \mathbb{N} & 0 \\ D^tP & 0 & -\frac{\mathbb{J}}{2}\end{bmatrix}\mathbb{X} \leq 0. \quad (A.24)$$

where $\mathbb{X} = [z_1\ w\ \xi]^t$. Clearly, to have (A.24) holds $\forall z_1 \in \mathbb{R}^m$ and $w \in \mathbb{R}^q \in \mathcal{L}_2(0,T)$, $\forall T \geq 0$, with $\mathbb{X} \neq 0$ is equivalent to having

$$\begin{bmatrix} \mathbb{M} - \frac{1\epsilon}{2}I - PBB^tP & PB_1 & PD \\ B_1^tP & \mathbb{N} & 0 \\ D^tP & 0 & -\frac{\mathbb{J}}{2}\end{bmatrix} \leq 0. \quad (A.25)$$

Apply well-known Schur complements on (A.25) will lead (4.9), hence we conclude our proof. ▽▽▽

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