Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993



# Robust Control of Systems With Both Norm Bounded and Nonlinear Uncertainties*

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## ABSTRACT

In this paper the robust stabilization problem for a class of systems with both norm-bounded and nonlinear uncertainties will be considered. The nonlinear uncertainty is assumed to satisfy a matching condition but the norm-bounded one may not. By generalizing an $H_\infty$ control result, we develop a Riccati equation technique for designing state feedback control to globally exponentially stabilize the system for all admissable uncertainties. This stabilization result is then extend to static output feedback and to systems for which the nonlinear uncertainty satisfies a *generalized matching conditions.* Furthermore, we point out that the global stability in the presence of nonlinear uncertainty may be destroyed by some arbitrarily small *mismatched uncertainty* in the input matrix, and proceed to establish the region of semi-global exponential stability of the controlled system.

## 1. INTRODUCTION

Robust stabilization of nonlinear systems has been an important research problem in recent years. Its origin can be traced back to Leitmann's paper[1] who introduced the *matching conditions* and a technique for robust stabilization of systems under these conditions. Subsequently, a great deal of work has been done to study various robust stabilization issues for *matched* nonlinearity and uncertainty ; see [2, 3, 4] for example. Most recently, the *generalized matching conditions* and also known as the *triangular structure* has been used to capture a much larger class of nonlinearity and uncertainties ( see e.g., ([5, 6, 7, 8]) along with a *back-stepping* approach for designing a guaranteed state feedback robust stabilizer.

A significant drawback of the aforementioned results is that the closed-loop system may not very robust against additional *mismatched* nonlinearity and/or uncertainty. Although, there are a number of papers appeared to deal with *mismatched uncertainties* (see, e.g., [9, 10, 11]) the results are not quite satisfactory because the additional uncertainty is not taken into account in the design of the controller. That is, the controller is designed based on the matched uncertainty only and the size of the allowable mismatched uncertainty is then calculated depending on the *robustness margin* of the resulting closed-loop system. Although, this method may work for linear systems when the mismatched uncertainty is sufficiently small. It would fail for nonlinear systems in general. As we will show that even an arbitrarily small mismatched uncertainty (in certain sense) will cause the closed-loop system to lose global stability.

In this paper, we consider the robust stabilization problem for nonlinear systems with both matched and mismatched nonlinearities and uncertainties. The matched nonlinearities are not restricted to be Lipschitz bounded. In fact, they can be bounded by almost any continuous nonlinear and time-varying functions. The mismatched part is allowed to be of large size but restricted to be Lipschitz bounded (i.e., norm-bounded) and to be in the autonomous part of the system. We show that this type of uncertain and nonlinear system can be stabilized via a fixed state feedback controller in the sense of global exponential stability if and only if the same system with the norm-bounded uncertainty alone can be robustly stabilized. The latter task can be solved by using a standard $H_\infty$ result [12]. That is, the robust stabilizability of the system with *mismatched* norm-bounded uncertainty can be determined by the solvability of an algebraic Riccati equation. If the algebraic Riccati equation has a desired solution then the robust controller can be designed by a simple procedure.

The aforementioned robust stabilization result is also extended in two cases. The first extension is to convert the state feedback controller into a static output feedback controller under some additional conditions. The second extension is to relax the *matching conditions* to the *generalized matching conditions* by restricting the norm-bounded uncertainty to a sub-system and a similar result is provided.

Another related robust stabilization problem of interest to us is when the control input matrix is subjected to some mismatched uncertainty. As mentioned earlier, we will show, via a simple example, that global exponential stability is impossible to establish even when this additional mismatched uncertainty has an arbitrarily small size. Hence, one has to settle for semi-global stability. We provide an estimate of the size of the semi-global stability region (in the state space) in terms of the size of the additional mismatched uncertainty. Furthermore, we use this estimate to show that the global stability is restored

*This work has been supported by the Australian Research Council.

when either the mismatched uncertainty in the input matrix completely vanishes, or when it is sufficiently small and the matched uncertainty or nonlinearity in the autonomous part of the system is Lipschitz bounded.

## 2. SYSTEM AND PRELIMINARIES

The class of systems to be considered in this paper are described by the following state equations :

$$\begin{aligned} \Sigma: \quad \dot{x}(t) = &\ (A+\Delta A(x,t))x(t) + Bf(x,t) \\ &+ (B+\Delta B(x,t))u(t) \\ y(t) = &\ Cx(t) \end{aligned} \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $y \in \Re^p$ is the control output, $A$, $B$ and $C$ are known constant matrices with appropriate dimensions, $f(x,t)$ is an $m \times 1$ vector representing the nonlinear uncertainties in the plant, $\Delta A(x,t)$ and $\Delta B(x,t)$ are matrix functions representing uncertainties in the matrix $A$ and $B$.

The following structure for the uncertainties $\Delta A(x,t)$ and $\Delta B(x,t)$ will be assumed throughout of this paper:

*Assumption 1.*

$$\begin{aligned} \Delta A(x,t) &= DF(x,t)E_1 \\ \Delta B(x,t) &= BJ(x,t)E_2 \end{aligned} \tag{2}$$

where $F(x,t) \in \Re^{k\times j}$ and $J(x,t) \in \Re^{m\times g}$ are Carathéodory matrix functions[1] bounded by

$$F(x,t)^t F(x,t) \le \xi \text{ for some } \xi \ge 0 \tag{3}$$

and

$$\max_{J(x,t);(x,t)\in\Re^n\times\Re} \|J(x,t)E_2\| \le \gamma \tag{4}$$

for some $0 \le \gamma < 1$, and $D, E_1$ and $E_2$ are known real matrices which characterize the structure of the uncertainties. The nonlinear function $f(x,t)$ is also assumed to be a Carathéodory function and to satisfy the following assumption :

*Assumption 2.* There exists a positive scalar Carathéodory function $\rho(x,t)$ such that

$$\|f(x,t)\| \le \rho(x,t); \forall (x,t) \in \Re^n \times \Re \tag{5}$$

where $\|\cdot\|$ denotes the Euclidean norm. Also, $\lim_{t\to\infty} \rho(\cdot,t) < \infty, \forall x \in \Re^n$.

The following linear system associated with (1) will be called the *nominal* system :

$$\dot{x}(x,t) = Ax(t) + Bu(t) \tag{6}$$

[1] A function $V : \Re \times \Re^p \mapsto \Re^q$ is called Carathéodory if : i) $V(\cdot,z)$ is Lebesque measureable for each $z \in \Re^p$; ii) $V(t,\cdot)$ is continuous for each $t \in \Re$; iii) for each compact set $U \subset \Re \times \Re^p$, there exists a Lebesque integrable function $m_u(\cdot)$ such that $\|V(t,z)\| \le m_u(t)$ for all $(t,z) \in U$. This type of function is needed primarily for ensuring the existence and continuity of the solution to a differential equation; see [4] and reference thereof.

*Remark 1.* Two special cases of system (1) have been well studied. If $\Delta A(x,t) = 0$ or the matrix $D$ in (2) is equal to $B$, then we have the so-called *matching conditions.* It is well known that an uncertain system with matching conditions can be robustly stabilized via a fixed state feedback controller if and only if the nominal system (6) is stabilizable, see [2, 1] for example. Furthermore, when $f(x,t) = 0$, the robust stabilization problem can be solved by using an $H_\infty$ control method; see [12]. That is, the robust stabilizability of the system via state feedback is equivalent to the solvability of an algebraic Riccati equation. What we intend to do in this paper is to develop a unified method to treat the general case.

*Remark 2.* We emphasize that the assumption on $\Delta B(x,t)$ is not too restrictive because even an arbitrarily small mismatched (i.e., unstructured) uncertainty may cause the system to lose the global stabilizability, provided that $f(x,t)$ is not Lipschitz bounded. See Section 4 for example.

The notion of exponential stability plays an important role in this paper. The definition of it can be found in [13]. For our purpose, the following result suffices.

Lemma 1. *[13] Given an n-dimensional continuous-time system*

$$\dot{x}(t) = f(x(t),t), \tag{7}$$

*where $f(x(t),t)$ is a Carathéodory vector function, suppose there exists a Lyapunov function $V(x,t)$ with the following properties:*

$$\lambda_1\|x(t)\|^2 \le V(x,t) \le \lambda_2\|x(t)\|^2; \forall (x,t) \in \Re^n \times \Re, \tag{8}$$

$$\dot{V}(x,t) \le -\lambda_3\|x(t)\|^2 + \epsilon e^{-\beta t}; \forall (x,t) \in \Re^n \times \Re, \tag{9}$$

*where $\lambda_1, \lambda_2, \lambda_3, \epsilon$ and $\beta$ are some positive scalar constants. Then, the system (7) is globally exponentially stable. Furthermore, suppose (9) is replaced by*

$$\dot{V}(x,t) \le -\lambda_3(1-\alpha(x))\|x(t)\|^2 + \epsilon e^{-\beta t} \tag{10}$$

*for some continuous function $\alpha(x)$ with $|\alpha(x)| < 1, \forall \|x\| \le \xi, \xi > 0$. Then, system (7) is semi-globally exponentially stable with the stability region given by $M = \{x(t) : \|x\| \le \xi, x \in \Re^n\}$.*

*Remark 3.* [13] The convergence rate of the system (7) with (8)-(9) is given as follows:

$$\|x(t)\| \le \begin{cases} \left[\frac{\lambda_2}{\lambda_1}\|x(0)\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1} t e^{-\lambda t}\right]^{\frac{1}{2}} \\ if \beta = \lambda, \\ \left[\frac{\lambda_2}{\lambda_1}\|x(0)\|^2 e^{-\lambda t} + \frac{\epsilon}{\lambda_1(\lambda-\beta)} e^{-\beta t} \right. \\ \left. \quad - \frac{\epsilon}{\lambda_1(\lambda-\beta)} e^{-\lambda t}\right]^{\frac{1}{2}} \\ if \beta \ne \lambda, \end{cases} \tag{11}$$

where $\lambda = \frac{\lambda_3}{\lambda_2}$.

## 3. MAIN RESULTS

In this section, we present a state feedback stabilization result for the system (1) under Assumptions 2.1-2.2. This result will then be extended to static output feedback under additional assumptions, and to systems with genernalized matching conditions.

### 3.1. State feedback

Given the system $(\Sigma)$ in (1), we are searching for a state feedback stabilizer of the following form:

$$u(t) = \phi_c(x,t), \tag{12}$$

where $\phi_c(x,t)$ is a Carathéodory function. We now state our main result:

**Theorem 2.** *The system $(\Sigma)$ satisfying Assumptions 2.1-2.2 is globally exponentially stabilizable via a nonlinear state feedback controller (12) if and only if there exists $\epsilon > 0$ and a positive definite symmetric matrix $Q \in \Re^{n\times n}$, such that the following algebraic Riccati equation*

$$\begin{aligned}\tfrac{1}{2}\{A^tP + PA + \epsilon\xi PDD^tP - 2PBB^tP \\ + \tfrac{1}{\epsilon}E_1^tE_1\} + Q = 0\end{aligned} \tag{13}$$

*has a positive definite symmetric solution $P$. If this is the case, then a suitable stabilizing control law is given by*

$$u(t) = -Kx(t) - \frac{1}{(1-\gamma)}\phi_c(x,t), \tag{14}$$

*where*

$$K = B^tP \tag{15}$$

$$\phi_c(x,t) = \frac{B^tPx(\rho(x,t) + \gamma\|B^tPx\|)^2}{\|B^tPx\|(\rho(x,t) + \gamma\|B^tPx\|) + \epsilon^* e^{-\beta t}} \tag{16}$$

*$\beta$ and $\epsilon^*$ are any positive scalars.*

**Proof.** The necessity follows from the fact that robust stabilization via nonlinear state feedback controller implies robust stabilization via a linear state feedback controller [14]. To prove the sufficiency, we let $V(x) = \frac{1}{2}x^tPx$ be a Lyapunov candidate for system $(\Sigma)$ with (14). The upper and lower bounds in (8) on $V(x)$ are given by $\lambda_1 = \frac{1}{2}\lambda_{\min}[P]$ and $\lambda_2 = \frac{1}{2}\lambda_{\max}[P]$. The time derivative of $V(x(t))$ along $(\Sigma)$ is given by

$$\begin{aligned}\dot V(x(t)) \;=\; & x^tP(A + DF(t)E_1 - BK)x \\ & - \tfrac{1}{(1-\gamma)}x^tPB\phi_c(x,t) \\ & - \tfrac{1}{(1-\gamma)}x^tPBJ(x,t)tE_2\phi_c(x,t) \\ & x^tPB[f(t,x) + J(x,t)tE_2Kx]\end{aligned} \tag{17}$$

Using the triangular inequality

$$x^tPDF(x,t)E_1x \le \frac{1}{2}x^t\{\epsilon\xi PDD^tP + \frac{1}{\epsilon}E_1^tE_1\}x \tag{18}$$

for any $x \in \Re$ and $\epsilon > 0$. Then, we obtain

$$\begin{aligned}\dot V(x(t)) \;\le\; & \tfrac{1}{2}x^t[A^tP + PA]x + \tfrac{1}{2}\epsilon\xi PDD^tPx \\ & + \tfrac{1}{2\epsilon}x^tE_1^tE_1x - x^tPBB^tPx \\ & + x^tPB[f(x,t) + J(x,t)E_2Kx] \\ & - \tfrac{1}{(1-\gamma)}x^tPBFJ(x,t)E_2\phi_c(x,t) \\ & - \tfrac{1}{(1-\gamma)}x^tPB\phi_c(x,t)\end{aligned} \tag{19}$$

Then by using (13) , we have

$$\begin{aligned}\dot V(x(t)) \;\le\; & x^tPB[f(x,t) + J(x,t)E_2K] \\ & - \tfrac{1}{(1-\gamma)}x^tPBJ(x,t)E_2\phi_c(x,t) \\ & - \tfrac{1}{(1-\gamma)}x^tPB\phi_c(x,t) - x^tQx(t).\end{aligned} \tag{20}$$

Consider the last two terms of (20). Using the bound on $J(x,t)E_2$ given in (4), we have the following inequality

$$\begin{aligned}& \tfrac{1}{(1-\gamma)}x^tPB\phi_c(x,t) + \tfrac{1}{(1-\gamma)}x^tPBJ(x,t)E_2\phi_c(x,t) \\ & = \tfrac{1}{(1-\gamma)}x^tPB(I + J(x,t)E_2)\phi_c(x,t) \\ & \ge x^tPB\phi_c(x,t)\end{aligned} \tag{21}$$

This leads to the following result:

$$\begin{aligned}\dot V(x(t)) \;\le\; & +x^tPB[f(x,t) + J(x,t)E_2K - \phi_c(x,t)] \\ & - x^tQx(t).\end{aligned} \tag{22}$$

Then, utilizing (16), we have

$$\begin{aligned}\dot V(x(t)) \;\le\; & -x^t(t)Qx(t)\|B^tPx\|(\rho(x,t) + \gamma\|Kx\|) \\ & - \frac{\{\|B^tPx\|(\rho(x,t) + \gamma\|Kx\|)\}^2}{\|B^tPx\|(\rho(x,t) + \gamma\|Kx\|) + \epsilon^*e^{-\beta t}} \\ = \; & \frac{\|B^tPx\|(\rho(x,t) + \gamma\|Kx\|)\epsilon^*e^{-\beta t}}{\|B^tPx\|(\rho(x,t) + \gamma\|Kx\|) + \epsilon^*e^{-\beta t}} \\ & - x^t(t)Qx(t) \qquad (23) \\ \le \; & -x^tQx + \epsilon^*e^{-\beta t}. \qquad (24)\end{aligned}$$

Therefore, $(\Sigma)$ is globally exponentially stabilized, according to Lemma 2.1.

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*Remark 4.* We note that Theorem 3.1 is a generalization of some known results. More precisely, when $J(x,t)$ and $f(x,t)$ in $(\Sigma)$ are set to zero, our result will reduce to a result by Petersen [15]. Also Dawson, Qu and Carroll's result [16] will follow by setting $F(x,t)$ and $J(x,t)$ in $(\Sigma)$ to be zero.

### 3.2. Static output feedback

In certain applications it is more desirable to use output feedback control rather than state feedback. The output feedback control problem for nonlinear uncertain systems is very difficult to solve in general, because observers are hard to construct. It is, however, simple to extend the state feedback stabilization result in Theorem 3.1 to the static output feedback under some additional conditions.

*Assumption 3.* There exists a positive scalar Carathéodory function

$$\rho(y,t) \geq \|f(x,t)\|, \ \forall (x,t) \in \Re^n \times \Re, \tag{25}$$

where $y = Cx$ as in (1).

Now, we state our static output feedback stabilization result :

**Corollary 1.** *The system ($\Sigma$) satisfying Assumptions 2.1 and 3.1 is globally exponentially stabilizable via a nonlinear and time-invariant static output feedback controller, if and only if there exists $\epsilon > 0$, positive-definite symmetric matrix $Q \in \Re^{n\times n}$ and a constant matrix $H \in \Re^{m\times p}$ such that the following algebraic Riccati equation*

$$\begin{aligned}\tfrac{1}{2}\{A^tP + PA + \epsilon\xi PDD^tP - 2PBB^tP \\ + \tfrac{1}{2}\epsilon E_1^tE_1\} + Q = 0\end{aligned} \tag{26}$$

*has a positive definite symmetric solution $P$ which satisfies the following constraint:*

$$B^tP = HC. \tag{27}$$

*In this case, a suitable stabilizing control law is given by*

$$u(t) = -Hy(t) - \frac{1}{(1-\gamma)}\phi_c(y,t), \tag{28}$$

*where*

$$\phi_c(y,t) = \frac{Hy(\rho(y,t)+\gamma\|Hy\|)^2}{\|Hy\|(\rho(y,t)+\gamma\|Hy\|)+\epsilon^* e^{-\beta t}}, \tag{29}$$

*$\beta$ and $\epsilon^*$ are any positive scalars.*

### 3.3. Extension to Generalized Matching Conditions

This section extends the main result in Section 3.1 to a more general class of nonlinear uncertain systems, namely nonlinear systems satisfying the so-called generalized matching conditions [5]. These systems are of the following form:

$$\dot{x}_1 = f_1(x_1,t) + g_1(x_1,x_2,t) \tag{30}$$

$$\dot{x}_2 = f_2(x_1,x_2,t) + g_2(x_1,x_2,x_3,t) \tag{31}$$

$$\cdots$$

$$\dot{x}_m = f_m(x_1,\ldots,x_m,t) + g_m(x_1,\ldots,x_m,u,t) \tag{32}$$

where, for simplicity, $x_i(t) \in \Re, g_i(\cdot) : \Re^{i+2} \mapsto \Re$ are continuous functions satisfying

$$x_{i+1}g_i(x_1,\ldots,x_{i+1},t) \geq \gamma_i x_{i+1}^2, \forall x_1,\ldots,x_{i+1},t \in \Re,$$

for some constants $\gamma_i > 0$, $f_i(x_1,\ldots,x_i,t)$ are Carathéodory functions with similar boundes as $f(x,t)$ in Assumption 2.2. This system can be stabilized by using a recursive design procedure called *backstepping* method [5, 7]. The first step of this method involves the stabilization of (30) by using $x_2$ as a fictitious control input. Denoting the control law by $x_2 = \phi_1(x_1,t)$, a coordinate transformation is then carried out : $z_1 = x_1$, $z_2 = x_2 - \phi_1(x_1,t)$, and equation (31) is rewritten in terms of

$$\dot{z}_2 = \tilde{f}_2(z_1,z_2,t) + \tilde{g}_2(z_1,z_2,x_3,t). \tag{33}$$

Now the same procedure above is applied to (33). That is, $x_3$ is used as a ficitious control input and a control law is designed to globally exponentially stabilizes (33). The design will be completed when the above recursive procedure reaches $u(t)$ and a stabilizing control law is found. It is known [5, 7] that the system (30)-(32) is globally exponentially stabilizable under some very mild assumptions.

In this sub-section, we further generalize the generalized matching conditions to allow the following type of systems:

$$\dot{x}_1 = (A + \Delta A(x_1,t))x_1 + Bf(x_1,t) + (B + \Delta B(x_1,t))x_2 \tag{34}$$

$$\dot{x}_2 = f_2(x_1,x_2,t) + g_2(x_1,x_2,x_3,t) \tag{35}$$

$$\cdots$$

$$\dot{x}_m = f_m(x_1,\ldots,x_m,t) + g_m(x_1,\ldots,x_m,u,t) \tag{36}$$

where $x_1(t)$ is allowed to be a vector and $\Delta A(x_1,t)$, $\Delta B(x_1,t)$ and $f(x_1,t)$ are as in section 3.1, and the rest of the system is same as in (30)-(32).

Now, the design procedure is very similar to the one for the system (30)-(32), except that in the first step of design Theorem 3.1 is applied. It is not difficult to see that as long as the algberaic Riccati equation (13) has a solution, then the generalized system (34)-(36) can be globally exponentially stabilized in the same way as for (30)-(32).

We finally point out that the conditions for the system (30)-(32) can be relaxed [5]. For example, $x_1$ can be allowed to be multi-dimensional to some extend, and weaker conditions on $g_i(\cdot)$ are also allowed. In these cases, our generalization still applies.

## 4. ROBUSTNESS ANALYSIS OF THE CLOSED-LOOP SYSTEM

The purpose of this section is to analyze the robustness of the closed loop system ($\Sigma$) with (14) or (28). We first show via an example that the global stability of the closed-loop system is very fragile in the sense that it may be destroyed with a slight additional perturbation in the input matrix. This nonrobustness property is due to the presence of nonlinear $f(x,t)$, i.e is not only for the controllers given in Section 3, but for a large class of stabilizing controllers. Based on this observation, we derive a robustness analysis result which gives a relationship between the size of additional uncertainty(in certain sense) and size of a guaranteed semi-global stability region of the perturbed system.

## 4.1. Nonrobustness of Global Exponential Stability

Consider the following simple example :

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - \delta u(t) \\ \dot{x}_2(t) &= x_1^2(t) + x_2^2(t) + u(t) \end{aligned} \quad (37)$$

where $\delta$ is a constant to be specified. When $\delta = 0$, the system (37) satisfies the matching conditions (i.e., Assumption 2.1), and therefore is globally stabilizable. We claim that the system (37) is not globally stabilizable when $\delta \neq 0$. Indeed, define

$$z(t) = x_1(t) + \delta x_2(t) \quad (38)$$

then we have

$$\dot{z}(t) = x_2(t) + \delta x_1^2(t) + \delta x_2^2(t). \quad (39)$$

Without loss of generality, we assume $\delta > 0$ (otherwise, we can change the sign of $u$). Choose the initial condition $x_1(0)$ and $x_2(0)$ such that

$$z(0) = \delta^{-1}, x_2(0) + \delta x_1^2(0) + \delta x_2^2(0) > 0. \quad (40)$$

Then, we argue that $\dot{z}(t) > 0 \ \forall t \geq 0$. Indeed, from (39) and (40) it is obvious that $\dot{z}(0) > 0$. Let, on the contrary, $t_o > 0$ be the least time at which $\dot{z}(t_o) = 0$. Then, (39) implies that

$$-\delta^{-1} < x_2(t_o) < 0$$

and in this case,

$$\dot{z}(t_o) \geq -\frac{\delta^{-1}}{4} + \delta x_1^2(t_o).$$

Note that

$$\begin{aligned} x_1(t_o) &= z(t_o) - \delta x_2(t_o) \\ &\geq z(t_o) \geq z(0) = \delta^{-1}. \end{aligned}$$

Hence,

$$\dot{z}(t_o) \geq -\frac{\delta^{-1}}{4} + \delta^{-1} = \frac{3\delta^{-1}}{4} > 0$$

contradicting the assumption $\dot{z}(t_o) = 0$. That is, $\dot{z}(t) > 0$ $\forall t \geq 0$ and the system(37) is not globally stabilizable.

We emphasize that the loss of global stabilizability above holds for arbitrarily sufficiently small $|\delta|$ and this phenomenon actually exists for a large class of systems.

## 4.2. Estimate of Semi-global Exponential stability Bound

Consider the robust controller (12) for the system $(\Sigma)$ with an additional uncertainty in the input matrix satisfying the following assumption:

*Assumption 4.*
$$\Delta B_u(x,t) = D_u F_u(x,t) E_u \text{ and } \|F_u(x,t)\| \leq \eta, \quad (41)$$

where, $D_u$ and $E_u$ are matrices representing the structure of the additional uncertainty.

The time derivative of the Lyapunov function $V(x) = x^t P x$ along the trajectory or trajectories of the system $(\Sigma)$ will be given by (see (24))

$$\begin{aligned} \dot{V}(x(t)) &= -x^t P \Delta B_u(x,t) \frac{1}{(1-\gamma)} \phi_c(x(t)) + \epsilon^* e^{-\beta t} \\ &\quad - x^t P \Delta B_u(x,t) K x - \lambda_3 \|x\|^2 \end{aligned} \quad (42)$$

Due to Assumption 4, Eq.(42) can be rewritten as

$$\dot{V}(x(t)) \leq -\lambda_3 \|x\|^2 + \epsilon^* e^{-\beta t} + \eta r(x,t), \quad (43)$$

where,

$$r(x,t) = \|x^t P D_u\| \|E_u\| (|\frac{\phi_c(x,t)}{(1-\gamma)}| + |Kx|). \quad (44)$$

Rewriting it in a more compact form, we get

$$\dot{V}(x(t)) \leq -\lambda_3 (1 - \eta \lambda_u(x,t)) \|x(t)\|^2 + \epsilon^* e^{-\beta t}, \quad (45)$$

where

$$\lambda_u(x,t) = \frac{r(x,t)}{\lambda_3 \|x\|^2}. \quad (46)$$

Note that the term $\epsilon^* e^{-\beta t}$ can be chosen to be uniformly arbitrarily small. According to Lemma 2.1, the exponential stability region of the system can be determined by the function $\lambda_u(x,t)$ in (45). This is summarized in the following result:

**Theorem 3.** *Suppose the system (1) satisfying Assumptions 2.1-2.2 is globally exponentially stabilized via the stabilizing control law (13)-(15) in Theorem 3.1. Also suppose the system's input matrix is subject to an additional uncertainty given by (41). Then for any $\xi > 0$, $M = \{x : \|x\| \leq \xi : x \in \Re^n\}$ is a region of semi-global exponential stability of the closed-loop system if*

$$\eta < \lambda_u^{-1}(x,t), \forall x \in M, \forall t \geq 0 \quad (47)$$

*provided that $\epsilon^*$ and $\beta$ are chosen to be sufficiently small.*

*Remark 5.* The function $\lambda_u(x,t)$ in (46) is unbounded in general, hence, no global exponential stability is guaranteed by (45) for the system $(\Sigma)$, except for two special cases. The first case is obvious: $\eta = 0$; i.e, the unmodelled uncertainty $\Delta B_u(x,t)$ disappears completely. In this case, (45) recovers (24). The second case is if $\rho(x,t)$ is Lipschitz bounded then it is straightforward to see from (44) and (46) that $\|\lambda_u(x,t)\|$ is uniformly bounded. Thus, the global exponential stability of the system $(\Sigma)$ is established by (45) as long as

$$0 < \eta < \min\{\lambda_u^{-1}(x,t) : (x,t) \in \Re^n \times \Re\}. \quad (48)$$

## 5. CONCLUSION

In this paper, we have solved a robust stabilization problem for systems with both norm bounded and nonlinear uncertainties. We have presented a state feedback controller design technique to globally exponentially stabilize the system. In a special case, the state feedback controller can be implemented via static output feedback. We have further analyzed the robustness of the controller in the presence of some unmodelled uncertainty which causes a possible loss of global exponential stability. Consequently, some estimate of the region of semi-global exponential stability in the state space is provided.

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