Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993

Lyapunov Functions for Uncertain Systems with Applications to the Stability of Time Varying Systems

Ganapathy Chockalingam* Soura Dasgupta† Brian D. O. Anderson‡ Minyue Fu§

Abstract

This paper has three contributions. The first involves polytopes of matrices whose characteristic polynomials also lie in a polytopic set (e.g. companion matrices). We show that this set is Hurwitz or Schur invariant iff there exist multiaffinely parameterized positive definite, Lyapunov matrices which solve an augmented Lyapunov equation. The second result concerns uncertain transfer functions with denominator and numerator belonging to a polytopic set. We show all members of this set are Strictly Positive Real iff the Lyapunov matrices solving the equations featuring the Kalman-Yakubovic-Popov Lemma are multiaffinely parameterized. Moreover, under an alternative characterization of the underlying polytopic sets, the Lyapunov matrices for both of these results admit affine parameterizations. Finally, we apply the Lyapunov equation results to derive stability conditions for a class of Linear Time Varying Systems.

1 Introduction

This paper considers the existence of parameterized Lyapunov functions for the stability and passivity analysis of linear time invariant (LTI) uncertain systems and demonstrates their application to the stability analysis of a class of Linear Time Varying (LTV) systems.

The first problem considered here involves the family of matrices described below where g and $h(k)$ are $n-$vectors, F is an $n \times n$ matrix:

$$\Omega = \{A(k) = F + gh'(k) \in \Re^{n\times n} : k \in K\} \quad (1.1)$$

with

$$K = \{k = [k_1, ..., k_m]' : k_i^- \leq k_i \leq k_i^+\}. \quad (1.2)$$

and $h(k)$ affine in the elements of k. An example of such a set of matrices is a set of affinely parameterized companion matrices in the controllable form [5].

We call an $n \times n$ matrix A, σ-Hurwitz if all its eigenvalues lie in the open half plane $Re[s] < -\sigma$, for some $\sigma > 0$. Similarly, A is said to be ρ-Schur, for some $0 < \rho < 1$, if all its eigenvalues lie in the open disc $|z| < \rho$.

It is shown here that Ω is σ-Hurwitz (respectively ρ-Schur) invariant iff there exists a σ-Hurwitz (respectively ρ-Schur) matrix Δ, compatibly dimensioned vector w and a Lyapunov pair $P(k)$, $Q(k)$ depending *multiaffinely* on the elements of k, which satisfies the Lyapunov equation [1] (1.3) (respectively (1.4)) for all $k \in K$.

$$\Pi'(k)P(k) + P(k)\Pi(k) < -2\sigma P(k) - Q(k) \quad (1.3)$$

$$\Pi'(k)P(k)\Pi(k) - P(k) < -(1-\rho^2)P(k) - Q(k) \quad (1.4)$$

where

$$\Pi(k) \doteq \begin{bmatrix} \Delta & wh'(k) \\ 0 & A(k) \end{bmatrix}. \quad (1.5)$$

The P and Q appearing above will be respectively referred to as a continuous and discrete time Lyapunov pair associated with A, while the matrix P itself will be called a Lyapunov matrix associated with A.

Here a multiaffine function is one which is affine in each individual argument. We note that the fact that the parametric Lyapunov pair thus constructed displays a multiaffine dependence on k has certain appealing characteristics to be highlighted in the sequel.

Polytopic sets such as (1.1-1.2) can equivalently be described by the convex combination of their corners, i.e. for some M and suitable $h_1, ..., h_M$, one has

$$\Omega = \{\bar{A}(\lambda) = F + g(\sum_{i=1}^{M} \lambda_i h_i) : \sum_{i=1}^{M} \lambda_i = 1, \lambda_i > 0\}. \quad (1.6)$$

We will show that Lyapunov pairs under this slightly different parameterization are in fact *affine* rather than multiaffine in the λ_i.

*Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA.

†Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA-52242, USA. Supported in part by NSF grants MIP-9001170 and ECS-9211593.

‡Department of Systems Engineering, Australian National University, ACT 2601, Australia. Also Cooperative Research Center for Robust and Adaptive Systems, supported by the Australian Commonwealth Government under the Cooperative Research Program.

§Department of Electrical and Computer Engineering, Newcastle University, NSW 2308, Australia.

The second question involves the Kalman-Yakubovic-Popov (KYP) lemma [2], for transfer functions whose numerator and denominator belong to two independent polytopes with defining parameter vectors $k = [k_1, ..., k_m]$ and $l = [l_1, ..., l_r]$ respectively. We show that under a suitable choice of state variable representation, the Lyapunov pairs one obtains depend multiaffinely on the elements of k and l. As with the Lyapunov equation problem, a convex combination based representation is also considered.

We demonstrate the significance of the Lyapunov function results by extending certain Linear Time Varying (LTV) system stability results reported in [3].

The KYP results are used to derive the Lyapunov results that we present here. Further, they can be used to determine robustness measures for adaptive output error identification algorithms.

Past results include those of [9], where the Hermite matrix was shown to be a Lyapunov matrix for a companion matrix A, albeit with a rank-1 Q. The Hermite matrix is bilinear in the elements of its associated companion matrix. This bilinear dependence is destroyed when one allows, as is the case in this paper, dependent variations. Thatachar and Srinath [6] *incorrectly* prove that the *single parameter* family, $\Omega(\lambda)$, is Hurwitz invariant iff it has an affinely parametrized Lyapunov matrix $P(k)$. The quadratic stability literature (see [7]) considers the existence of a single Lyapunov matrix P. Barring [8] these results are confined to norm bounded, as opposed to polytopic, uncertainities. The subject of [8] is the quadratic stability of the single parameter set $\{A + \lambda bc' : \lambda \in [0, \infty)\}$.

Section 2 gives preliminary results; Section 3 the KYP results; Section 4 the Lyapunov results; Section 5 LTV stability results; Section 6 the conclusion.

2 Preliminaries

This section provides certain preliminary results and assumptions.

The KYP results of this paper will be derived for sets of transfer functions

$$T = \left\{ \tau(s,k,l) = \frac{s^n - \sum_{i=1}^{n} b_i(k)s^{n-i}}{s^n - \sum_{i=1}^{n} a_i(l)s^{n-i}} \; : \; k \in K, \, l \in L \right\} \tag{2.1}$$

with K as in (1.2),

$$L = \{l = [l_1, ..., l_r]' : l_i^- \le l_i \le l_i^+\}, \tag{2.2}$$

and the $b_i(k)$ and $a_i(l)$ affine in their respective arguments. Then for suitably chosen $F \in \Re^{n\times n}$, $g \in \Re^n$, $h_1(l) \in \Re^n$ and $h_2(k) \in \Re^n$, with $[F, g]$ a completely reachable pair and $h_i(.)$ affine in their respective arguments, T can equivalently be described by

$$T = \{1 + (h_1(l) - h_2(k))'(sI - F - gh_1'(l))^{-1}g\} \tag{2.3}$$

where $k \in K$, $l \in L$. Since $h_1(l)$ and $h_2(k)$ lie in independent polytopes, it follows that T can also be expressed as

$$T = \left\{1 + (\hat{h}_1(\mu) - \hat{h}_2(\lambda))'(sI - F - g\hat{h}_1'(\mu))^{-1}g\right\} \tag{2.4}$$

where

$$\hat{h}_1(\mu) = \sum_{i=1}^{N} \mu_i \hat{h}_{1i} \, ; \; \sum_{i=1}^{N} \mu_i = 1 \, ; \; \mu_i \ge 0 \tag{2.5}$$

$$\hat{h}_2(\lambda) = \sum_{i=1}^{M} \lambda_i \hat{h}_{2i} \, ; \; \sum_{i=1}^{M} \lambda_i = 1 \, ; \; \lambda_i \ge 0 \tag{2.6}$$

Observe that $\{1 + (\hat{h}_{1i} - \hat{h}_{2j})'(sI - F - g\hat{h}_{1i}')^{-1}g\}$ represents the corners of the set T. In the sequel we will denote $\mu = [\mu_1, ..., \mu_N]'$ and $\lambda = [\lambda_1, ..., \lambda_M]'$ (note $N = 2^m$ and $M = 2^r$).

To conclude this section on preliminaries, we impose certain restrictions on various matrices of interest.

Assumption 2.1: The pair $[F, g]$ is completely reachable. Further, for (2.3) $[F, h_1(l) - h_2(k)]$ is completely observable almost everywhere in $K \times L$ including, at every corner of L and K. Similarly, for (2.4) $[F, \hat{h}_1(\mu) - \hat{h}_2(\lambda)]$ is completely observable almost everywhere, including at all corners (i.e. at all $\hat{h}_1(\mu) = \hat{h}_{1i}$ and all $\hat{h}_2(\lambda) = \hat{h}_{2j}$). Likewise, for Ω, $[F, h(k)]$ is completely observable almost everywhere in K including, at every corner of K.

We note that the corner observability conditions can be assumed without loss of generality, possibly through an infinitesimal expansion of L and/or K.

Recall, that Ω will be examined for σ-Hurwitz (or ρ-Schur) invariance. Thus, to avoid trivialities we will assume that at least one member of Ω is σ-Hurwitz (or ρ-Schur). Then, through a simple affine transformation in the parameter vector k if need be, one can make the following assumption without loss of generality:

Assumption 2.2: Under continuous (respectively discrete) time settings, F is $\sigma-$Hurwitz (respectively ρ-Schur).

3 On the KYP Lemma

A continuous time system having transfer function $T(s) = 1 - c'(sI - A)^{-1}b$, is strictly passive iff for some $\sigma > 0$, it is continuous time strictly positive real with margin σ (σ-CSPR): i.e $T(s - \sigma)$ is minimum phase, stable and obeys for all real ω

$$Re[T(j\omega - \sigma)] > 0. \tag{3.1}$$

Similarly in discrete time, strict passivity is equivalent to the existence of $0 < \rho < 1$ for which $T(\rho s)$ is minimum phase, stable and obeys for all $\omega \in [-\pi, \pi)$

$$Re[T(\rho e^{j\omega})] > 0. \tag{3.2}$$

Such a $T(s)$ will henceforth be referred to as being ρ-DSPR. In this section we will address the issue of parameterized Lyapunov pairs for σ-CSPR and ρ-DSPR parameterized transfer functions as defined in (2.3) and (2.4). The first set of results concerns the parameterization in (2.4)-(2.6).

Theorem 3.1 *All members of the set (2.4)-(2.6) are σ-CSPR iff there exist symmetric $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ which obey the following:*

(i) $\forall \mu_i, \lambda_i$ obeying the constraints in (2.4 - 2.6), dropping all arguments due to lack of column space, the following matrix is positive definite

$$\begin{bmatrix} -\hat{\Theta}'P - P\hat{\Theta} - Q - 2\sigma P, & Pg + \hat{h}_2 - \hat{h}_1 \\ (Pg + \hat{h}_2 - \hat{h}_1)' & 2 \end{bmatrix} \tag{3.3}$$

where

$$\hat{\Theta}(\mu) = F + g\hat{h}_1'(\mu) \tag{3.4}$$

(ii) For fixed μ (respectively λ), both $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ are affine in the elements of λ (respectively μ).

(iii) $P(\mu, \lambda) > 0$ and $Q(\mu, \lambda) > 0\ \forall \mu, \lambda$ obeying the constraints in (2.5) and (2.6).

In the above P, Q are called the Lyapunov pairs satisfying the KYP lemma and P by itself is called the Lyapunov Matrix. Several remarks are in order.

Remark 3.1: To construct the Lyapunov pairs one must first construct $[P_{ij}, Q_{ij}]$ (using possibly the spectral factorization method outlined in [13]) which work with the corner represented by $\hat{h}_1 = \hat{h}_{2i}$ and $\hat{h}_2 = h_{1j}$. The Lyapunov pairs $[P(\mu, \lambda), Q(\mu, \lambda)]$ are constructed using (3.5) given below.

$$P(\mu, \lambda) = \sum_{j=1}^{M}\sum_{i=1}^{N} \lambda_j \mu_i P_{ij} \quad Q(\mu, \lambda) = \sum_{j=1}^{M}\sum_{i=1}^{N} \lambda_j \mu_i Q_{ij} \tag{3.5}$$

Remark 3.2: The special cases of (2.1) and (2.2) corresponding to the situations where the numerator is fixed and the denominator is uncertain, and where the converse holds, are of particular interest in adaptive systems and the development to be outlined in section 4. In the case where the numerator is fixed, one can assume that

$$h_2(k) = h \ \forall k. \tag{3.6}$$

Likewise the converse case of denominator fixed allows one to assume without loss of generality, that

$$h_1(\ell) = 0 \ \forall \ell. \tag{3.7}$$

In either case, $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ are *affine* in the underlying parameters. We next present the discrete time counterpart of Theorem 3.1.

Theorem 3.2 *All members of the set (2.4-2.6) are ρ-DSPR iff there exist symmetric $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ which obey (ii) and (iii) of Theorem 3.1 and in addition (again dropping all arguments), the following is positive definite:*

$$\begin{bmatrix} -\hat{\Theta}'P\hat{\Theta} - Q + \rho^2 P, & \hat{\Theta}'Pg + \hat{h}_2 - \hat{h}_1 \\ (\hat{\Theta}'Pg + \hat{h}_2 - \hat{h}_1)' & 2 - g'Pg \end{bmatrix}. \tag{3.8}$$

Remarks 3.1 and 3.2 apply to this situation as well. Having dispensed with the parameterization contained in (2.4), we now turn our attention to its counterpart in (2.3).

Theorem 3.3 *All members of (2.3) are σ-CSPR iff there exist symmetric $P(k, l)$ and $Q(k, l)$, multiaffine in $[k', l']'$ such that (dropping all arguments), the following is positive definite:*

$$\begin{bmatrix} -\Theta'P - P\Theta - Q - 2\sigma P, & Pg + h_2 - h_1 \\ (Pg + h_2 - h_1)' & 2 \end{bmatrix} \tag{3.9}$$

with

$$\Theta(l) = F + gh'(l). \tag{3.10}$$

We next present the discrete time counterpart of Theorem 3.3.

Theorem 3.4 *All members of (2.3) are ρ-SPR iff there exist symmetric $P(k, l)$ and $Q(k, l)$, multiaffine in $[k', l']'$ such that (dropping all arguments), the following is positive definite:*

$$\begin{bmatrix} -\Theta'P\Theta - Q + \rho^2 P, & \Theta'Pg + h_2 - h_1 \\ (\Theta'Pg + h_2 - h_1)' & 2 - g'Pg \end{bmatrix} \tag{3.11}$$

Remark 3.3: A self evident modification of Remark 3.2 applies here as well.

Remark 3.4: Here also the proof is constructive. As in Remark 3.1 we must now construct the Lyapunov pairs $[P_{ij}, Q_{ij}]$ (see Remark 3.1 on the construction of these pairs) that work with the transfer function that represents the combination of the i-th and j-th corners of K and L respectively. Then the required $[P(k, \ell), Q(k, \ell)]$ is the unique multiaffine function that assumes the value $[P_{ij}, Q_{ij}]$ at the apropriate corner combination.

Remark 3.6: Observe, that Theorems 3.1 and 3.2 deal with parametrizations that are equivalent to those used in their respective counterparts Theorems 3.3 and 3.4. However, while for fixed λ (respectively μ) the Lyapunov pairs of Theorems 3.1 and 3.2 are collectively affine in the μ (respectively λ) parameters, even for a fixed k (respectively ℓ) the Lyapunov pairs of Theorems 3.3 and 3.4 are multiaffine in the ℓ (respectively k) parameters. This apparent paradox can be understood in terms of the following example. Consider the multiaffine function

$$p(k_1, k_2) = 1+k_1+k_2+k_1k_2, \quad 0 \le k_1 \le 1, 0 \le k_2 \le 1. \tag{3.12}$$

Clearly in the given range of $[k_1, k_2]$, $p(k_1, k_2)$ cannot be expressed as an affine function of two variables. Yet each member of this set can be expressed as a convex combination of the four corners $p(0,0)$, $p(0,1)$, $p(1,1)$ and $p(1,0)$. This latter representation though, will be nonunique. Indeed similar considerations apply to the results of section 4 also.

4 Solutions to the Lyapunov Equation

In this section, we restrict our attention to the set Ω as represented in both (1.1) and (1.6) and consider suitable Lyapunov pairs for this set. The main results of this section are first formally stated.

Theorem 4.1 *Consider Ω as in (1.1), with assumptions 2.1 and 2.2 in force. Then, all members of Ω are σ-Hurwitz (respectively ρ-Schur) iff there exist σ-Hurwitz (respectively ρ-Schur) Δ, a vector w and positive definite symmetric $P(k)$ and $Q(k)$, multiaffine in k, such that for all k in K , (1.3) (respectively (1.4)) holds with $\Pi(k)$ as in (1.5)*

Theorem 4.2 *With Ω as in (1.6), the statement of Theorem 4.1 stands with $P(k), Q(k), \Pi(k)$ replaced by $P(\lambda), Q(\lambda), \Pi(\lambda)$, $\Pi(\lambda)$ obviously defined and $P(\lambda)$,$Q(\lambda)$ affine in λ.*

The proofs of these theorems are constructive, and the construction of the Lyapunov pairs can be accomplished by only considering the corners of Ω. The key results used in the proof of these theorems fall into two categories. The first is the main result of section 3. The second result we use is a minor variation of a construction result given in [4]. This result in [4] considers polytopes of polynomials and gives necessary and sufficient conditions under which there exists a single stable LTI operator whose product with all the members of this polytope is σ-CSPR (respectively ρ-DSPR). The variation in question is summarized in Theorem 4.3 below.

Theorem 4.3 *Consider the set Ω as in (1.1). This set is σ-Hurwitz (respectively ρ-Schur) invariant iff there exist monic polynomials $c(s)$ and $d(s)$, with $d(s)$ σ-Hurwitz(respectively ρ-Schur) such that the transfer function*

$$\frac{det(sI-(F+gh'(k)))c(s)}{d(s)} \tag{4.1}$$

is biproper and σ-CSPR (respectively ρ-DSPR) for all $k \in K$.

A few comments about this result are called for. Since in the continuous and discrete time settings of our problem F is respectively σ-Hurwitz and ρ-Schur with

$$f(s) = det(sI-F) \tag{4.2}$$

for sufficiently small ε, σ- Hurwitz or ρ-Schur invariance of Ω is equivalent to the existence of monic $c(s)$ and $d(s)$ as above, such that the transfer function below is σ-CSPR and ρ-DSPR for all $k \in K$.

$$\frac{det(sI-(F+gh'(k)))}{f(s)} \frac{f(s+\varepsilon)c(s)}{d(s)} \tag{4.3}$$

Further, as there are only a finite number of corners of Ω, Assumption 2.1 assures that $det(sI-(F+gh'(k)))$ and $f(s)$ are coprime for all corners of K. Also through an arbitarily small perturbation in $c(s)$ and $d(s)$, if need be, one can ensure that the transfer function in (4.3) is free from any pole-zero cancellations at the corners of K. In the sequel we will assume

$$\delta(f(s)d(s)) = N. \tag{4.4}$$

It is clear that the choice of $c(s)$ and $d(s)$ ensures that $f(s+\varepsilon)c(s)/d(s)$ is biproper. Suppose its minimal state variable realization (SVR) is $\{D, w, v, 1\}$. Then it is easy to show that $\{\Phi, \Gamma, \Psi(k), 1\}$ is a SVR of (4.3) where Φ, Γ and $\Psi(k)$ are given by: $\Gamma = [w', g']'$, $\Psi = [v', -h']'$ and

$$\Phi = \begin{bmatrix} D & 0 \\ gv' & F \end{bmatrix}. \tag{4.5}$$

Since this represents a σ-CSPR or a ρ-DSPR transfer function, the results of Section 3, provide appropriately parameterized Lyapunov pairs for this transfer function. Then one can show that with $\Delta = D - wv'$, these Lyapunov pairs are precisely the ones we seek.

5 Stability of Linear Time varying Systems

Definition 5.1: The LTV system

$$\dot{x}(t) = A(t)x(t) \tag{5.1}$$

is exponentially asymptotically stable (EAS) with degree of stability $\gamma > 0$ if $\exists c, \alpha > 0$ such that for all $x(t_0)$ and $t \ge t_0$,

$$||x(t)||e^{\gamma(t-t_0)} \le c||x(t_0)||e^{-\alpha(t-t_0)} \tag{5.2}$$

If $\gamma = 0$, we simply say that (5.1) is EAS.

Reference [3] contains results that through a simple application of results in [11,12], yield conditions for the EAS of a class of LTV systems with time variations confined to a *scalar* parameter k. In particular, suppose that,

$$A(k) = F + kgh' \tag{5.3}$$

with g, h vectors is σ-Hurwitz for all scalar fixed k lying in a given interval. Then the conditions in question, involve certain precise logarithmic bounds on the time variations in the parameter k, such that the EAS of

$$\dot{x}(t) = A(k(t))x(t) \qquad (5.4)$$

is retained. Specifically one obtains the Theorem below.

Theorem 5.1 : *Suppose $A(k)$ as in (5.3) is σ-Hurwitz for all $k \in [k^-, k^+]$. Then (5.4) is EAS. if for some $\epsilon_1, \epsilon_2, T > 0$, $\delta \in (0, \sigma)$ and all $t \geq 0$*

$$(a)\, k(t) \in [k^- + \epsilon_1, k^+ - \epsilon_2] \qquad (5.5)$$

and
(b) either

$$(i) \sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \left[\frac{d}{d\tau} ln \frac{k(\tau) - k^-}{k^+ - k(\tau)}\right]^+ d\tau < 2(\sigma - \delta) \qquad (5.6)$$

where

$$[a]^+ = \begin{cases} a; & a \geq 0 \\ 0; & a < 0 \end{cases} \qquad (5.7)$$

or

$$(ii) \sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \left|\frac{d}{d\tau} ln \frac{k(\tau) - k^-}{k^+ - k(\tau)}\right| d\tau < 4(\sigma - \delta) \qquad (5.8)$$

Several comments are in order. First in essence each condition in Theorem 5.1 offers a trade-off between the degree of stability of the "frozen" LTI systems and the average time variation that could be withstood without losing stability. Further as one can imagine, by choosing a larger δ, i.e. with a smaller bound on the average derivative of the logarithmic value of the time varying parameter, one can quantify the degree of EAS that the resulting time varying system is endowed with. Such a result is in [9].

Second, the results of [3,9] apply only to the *continuous time* case involving the situation where time variation is confined to a *single parameter*. No comparable result for multiparameter time varying systems is to our knowledge available; nor are we aware of similar stability results that apply to discrete time system.

Third, even for the single parameter case, the results of [3,9] are proved using a somewhat involved multiplier theory, which to our knowledge does not readily extend to multiparameter time varying system.

The principal contribution of this section is to demonstrate how the results of section 4 can be used to readily prove a much more general set of results that: (a) involve LTV systems with multiple time varying parameters; (b) incorporate the degree of stability considerations featuring in [9]; and (c) specialize to Theorem 5.1 in the single parameter case. We also give the corresponding discrete time result. Specifically, we prove the following.

Theorem 5.2 *With $\Omega, A(k), k = [k_1, \cdots, k_m]'$, K as in (1.10), (1.2), and $h(k)$ affine in the elements of k, suppose every member of Ω is σ-Hurwitz. Then the LTV system*

$$\dot{x}(t) = A(k(t))x(t) \qquad (5.9)$$

is EAS with degree of stability γ, $0 < \gamma < \sigma$, if there exists $\delta \in (0, \sigma - \gamma), T > 0$ and $\epsilon_{1i}, \epsilon_{2i} > 0$, $\forall i = \{1, ..., m\}$ such that for all $t > 0$

$$(a)\, k_i(t) \in [k_i^- + \epsilon_{1i}, k_i^+ - \epsilon_{2i}], \quad \forall i \in \{1, ..., m\} \qquad (5.10)$$

and (b) either

$$(i) \sup_{t\geq 0} \tfrac{1}{T} \int_t^{t+T} \textstyle\sum_{i=1}^{n} \left[\frac{d}{d\tau} ln \frac{k_i(\tau) - k_i^-}{k_i^+ - k_i(\tau)}\right]^+ d\tau < 2(\sigma - \delta - \gamma). \qquad (5.11)$$

or

$$(ii) \sup_{t\geq 0} \tfrac{1}{T} \int_t^{t+T} \textstyle\sum_{i=1}^{m} \left|\frac{d}{d\tau} ln \frac{k_i(\tau) - k_i^-}{k_i^+ - k_i(\tau)}\right| d\tau < 4(\sigma - \delta - \gamma). \qquad (5.12)$$

The respective association between (5.11,5.12) and (5.6, 5.8) is clear.

The following lemma shows that (5.12) is in fact a stronger condition than (5.11).

Lemma 5.1 *With (5.10) in force, (5.12) implies (5.11).*

Thus we need only show that (5.10,5.11) suffice for the EAS of (5.9) with degree of stability γ.

Now Theorem 4.1 and the fact that Ω is σ-Hurwitz invariant together imply the existence of a σ-Hurwitz Δ and multiaffine symmetric positive definite matrix functions $P(k), Q(k)$ such that with $\Pi(k)$ as in (1.5),(1.3) holds for all $k \in K$. In the sequel, it will become evident that it is more convenient to work with the LTV system

$$\dot{z}(t) = \Pi(k(t))z(t) \qquad (5.13)$$

rather than with (5.9). Evidently, the block upper triangular structure of $\Pi(k(t))$ and the position occupied by $A(k(t))$ in $\Pi(k(t))$ readily yield the following.

Lemma 5.2 *With $\Pi(k)$ as in (1.5), if the LTV system (5.13) is EAS with degree of stability γ, then so is (5.9).*

Thus, we need only show that under (5.10) and (5.11), (5.13) is EAS with degree of stability γ. Then the result we seek follows from the following Proposition and the results of the previous Section.

Proposition 5.1: *Suppose, (1.3) holds with $P(k)$ multiaffine in the elements of k. Suppose also that there exist γ, $0 < \gamma < \sigma$, $\delta \in (0, \sigma - \gamma)$, $T > 0$ and $\epsilon_{1i}, \epsilon_{2i} > 0$, $\forall i = \{1, ..., m\}$ such that for all $t > 0$, (5.10, 5.11) hold. Then (5.13) is EAS with degree of stability γ.*

We next extend Theorem 5.2 to the discrete time case. We note that similar results have been hitherto unknown even for the single parameter case. We begin with the analogy of Definition 5.1.

Definition 5.2: The discrete time LTV system

$$x(t+1) = A(t)x(t) \tag{5.14}$$

is EAS with degree of stability $(1-\rho)$, (i.e. it is ρ-EAS) , $0 < \rho < 1$, if $\exists c > 0$, $0 < \delta < 1$ such that $\forall t_0$ and $t \geq t_0$, t and t_0 integers,

$$\frac{\|x(t)\|}{\rho^{t-t_0}} \leq c\|x(t_0)\|\delta^{t-t_0}. \tag{5.15}$$

The required result is as follows.

Theorem 5.3 *Suppose $A(k)$ as in Theorem 5.2 is such that $\forall k \in K$, $A(k)$ is ρ- Schur. Then*

$$x(t+1) = A(k(t))x(t) \tag{5.16}$$

is γ-EAS, $0 < \rho < \gamma < 1$, if there exist integer $T > 0$, a δ obeying $0 < \rho < \delta < 1$ and ϵ_{1i} and ϵ_{2i} as in Theorem 5.2, such that for all integer $t > 0$, (5.10) holds together with either (5.17) or (5.18) below:

$$\sup_{t \geq 0} \frac{1}{T} \sum_{j=t}^{t+T-1} \sum_{i=1}^{m} \left[\ln \frac{\lambda_i(j+1)}{\lambda_i(j)} \right]^+ \leq 2 \ln \left(\frac{\gamma\delta}{\rho}\right). \tag{5.17}$$

$$\sup_{t \geq 0} \frac{1}{T} \sum_{j=t}^{t+T-1} \sum_{i=1}^{m} \left| \ln \frac{\lambda_i(j+1)}{\lambda_i(j)} \right| \leq 4 \ln \left(\frac{\gamma\delta}{\rho}\right). \tag{5.18}$$

where,

$$\lambda_i(t) = \frac{k_i(t) - k_i^-}{k_i^+ - k_i(t)} \tag{5.19}$$

References

[1] J.C.Willems, *Stability Theory of Dyanamical Systems,* Nelson, 1970.

[2] B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly and B. Riedle, *Stability of Adaptive systems,* MIT press, 1986.

[3] M. Freedman and G. Zames, "Logarithmic variation criteria for the stability of systems with time varying gains", *J. SIAM Control,* vol. 6(3), pp. 487-507, 1968.

[4] B.D.O. Anderson, S. Dasgupta, P.P. Khargonekar, F.J. Kraus and M. Mansour, "Robust strict positive realness: characterization and construction", *IEEE Transac. on Circuits and Systems,* vol. 37, pp. 869-876, 1990.

[5] K.J.Astrom and B.Wittenmark, *Computer controlled systems: Theory and Design,* 2nd Ed., Prentice Hall, 1990.

[6] M.A.L.Thatachar and M.D. Srinath, "Stability of linear time invariant systems", *IEEE Transac. on Automatic control,* vol. 12, 1966.

[7] P.P.Khargonekar, K. Zhou and I.R. Petersen, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and H^∞ control theory", *IEEE Transac. on Automatic control,* vol.35, pp. 356-361, 1990.

[8] S.Boyd and Q.Yang, "Structured and simultaneous Lyapunov functions for system stability problems", *International Journal of Control,* vol. 49, pp. 2215-2240, 1989.

[9] P.C. Parks, "A new proof of the Routh-Hurwitz stability criterion using the second method of Lyapunov", *Proceedings of the Cambridge Phil. Soc.,* vol 58, pp 669-672, 1962.

[10] J.B. Moore and B.D.O. Anderson, "Trade-off between loop gain time-variations and relative stability of linear systems", *Proceedings of the 3rd Hawaii International Conference on Systems Sciences,* 1970, pp 1052 - 1055.

[11] L.M. Silverman and B.D.O. Anderson, "Controllability, observability and stability of linear systems", *SIAM Journal of Control,* vol. 6, 1968, pp 121-130.

[12] B.D.O. Anderson and J.B. Moore, " New results in linear systems stability", *SIAM Journal of Control,* vol. 7, 1969.7, 1969.

[13] S.Lefschetz, *Stability of nonlinear control systems,* Academic Press, 1965.