Lyapunov Functions for Uncertain Systems with Applications to the Stability of Time Varying Systems

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***Abstract*—This paper has three contributions. The first involves polytopes of matrices whose characteristic polynomials also lie in a polytopic set (e.g. companion matrices). We show that this set is Hurwitz or Schur invariant iff there exist multiaffinely parameterized positive definite, Lyapunov matrices that solve an augmented Lyapunov equation. The second result concerns uncertain transfer functions with denominator and numerator belonging to a polytopic set. We show all members of this set are strictly positive real iff the Lyapunov matrices solving the equations featuring in the Kalman-Yakubovic-Popov Lemma are multiaffinely parameterized. Moreover, under an alternative characterization of the underlying polytopic sets, the Lyapunov matrices for both of these results admit affine parameterizations. Finally, we apply the Lyapunov equation results to derive stability conditions for a class of linear time varying systems.**

I. Introduction

THIS PAPER CONSIDERS the existence of parameterized Lyapunov functions arising in the stability and passivity analysis of linear time invariant (LTI) uncertain systems and demonstrates their application to the stability analysis of a class of linear time varying (LTV) systems. Both discrete and continuous time settings are investigated. By way of background, we begin by citing two fundamental results from linear systems theory.

The first is relevant to the stability of LTI systems. In the sequel we call an $n \times n$ matrix A σ-Hurwitz if all its eigenvalues lie in the open half plane $Re[s] < -\sigma$, for some $\sigma > 0$. In other words, a σ-Hurwitz matrix has a continuous time stability margin of σ. Then a given matrix A is σ-Hurwitz iff there exist symmetric positive definite P, Q [1] such that

$$A'P + PA < -2\sigma P - Q. \tag{1.1}$$

Similarly, a $n \times n$ matrix A is said to be ρ-Schur, for some $0 < \rho < 1$, if all its eigenvalues lie in the open disc $|z| < \rho$. Then A is ρ-Schur iff there exist symmetric positive definite P and Q for which

$$A'PA - P < -(1-\rho^2)P - Q. \tag{1.2}$$

Thus, the stability analysis of LTI systems whose zero input dynamics is governed by A can be accomplished by using the Lyapunov function $V(x) = x'Px$; (1.1) and (1.2) are accordingly called Lyapunov equations. In the sequel, the P and Q appearing in (1.1) and (1.2) will be respectively referred to as a continuous and discrete time Lyapunov pair associated with A, while the matrix P itself will be called a Lyapunov matrix associated with A.

The second result of relevance here concerns ascertaining the strict passivity of a LTI system characterized by the transfer function $T(s) = 1 - c'(sI - A)^{-1}b$, with $[A, c]$ completely observable and $[A, b]$ completely reachable. (For ease of presentation, we have excluded the case where $T(s)$ is strictly proper. The continuous time results derived here, however, apply to the strictly proper case as well.) Recall that such a LTI continuous time system is strictly passive iff for some $\sigma > 0$, it is continuous time strictly positive real with margin σ (σ-CSPR): i.e $T(s - \sigma)$ is minimum phase, stable and obeys for all real ω

$$Re[T(j\omega - \sigma)] > 0. \tag{1.3}$$

Similarly in discrete time, strict passivity is equivalent to the existence of $0 < \rho < 1$ for which $T(\rho s)$ is minimum phase, stable and obeys for all $\omega \in [-\pi, \pi)$

$$Re[T(\rho e^{j\omega})] > 0. \tag{1.4}$$

Such a $T(s)$ will henceforth be referred to as being ρ-DSPR, (D signifies discrete time). Then the celebrated Kalman-Yakubovic-Popov (KYP) Lemma states that *T(s)* is σ-CSPR [2] iff there exist symmetric positive definite matrices P and Q and a vector q such that

$$A'P + PA + qq' < -2\sigma P - Q \tag{1.5}$$

and

$$Pb = -c - \sqrt{2}q. \tag{1.6}$$

Likewise ρ-DSPR is equivalent to the existence of P, Q and q as above and a real scalar w such that

$$A'PA - P + qq' < -(1-\rho^2)P - Q \tag{1.7}$$

$$A'Pb = -c + wq \tag{1.8}$$

$$w^2 = 2 - b'Pb. \tag{1.9}$$

Manuscript received February 9, 1993. This work was supported in part by NSF grants MIP-9001170, ECS-9211593, and ECS-9350346. This paper was recommended by Associate Editor Joseph Nossek.

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IEEE Log Number 9215702D.

As before P, Q will be called the Lyapunov pair associated with $T(s)$ and P by itself will be called a Lyapunov matrix corresponding to $T(s)$.

In this paper we consider two problems related to the existence of appropriately parameterized Lyapunov pairs for systems that admit parameterized uncertainities. The first problem concerns (1.1) and (1.2) and involves the family of matrices described below where g and $h(k)$ are $n-$vectors, F is an $n \times n$ matrix:

$$\Omega = \{A(k) = F + gh'(k) \in \Re^{n\times n} : k \in K\} \quad (1.10)$$

with

$$K = \{k = [k_1, ..., k_m]' : k_i^- \leq k_i \leq k_i^+\}. \quad (1.11)$$

and $h(k)$ affine in the elements of k. Thus, Ω is a polytope of matrices. Further, it is a trivial matter to show that the coefficients of the characteristic polynomial of these matrices also depend affinely on the k_i; i.e., the characteristic polynomials are themselves in a polytope of polynomials. An example of such a set of matrices is a set of affinely parameterized companion matrices in the controllable form [8].

Suppose now one has to determine the σ-Hurwitz (or ρ-Schur) invariance of this set of matrices via Lyapunov-based techniques. Clearly, the σ-Hurwitz or ρ-Schur nature of each individual member of Ω is equivalent to the existence of member specific Lyapunov pairs that satisfy the appropriate Lyapunov equation. The question is whether these Lyapunov pairs can be collectively described by simple functions of the uncertain parameters k_i.

It is shown here that Ω is σ-Hurwitz (respectively ρ-Schur) invariant iff there exists a σ-Hurwitz (respectively ρ-Schur) matrix Δ, compatibly dimensioned vector w and a Lyapunov pair $P(k)$, $Q(k)$ depending *multiaffinely* on the elements of k, which satisfies the Lyapunov equation (1.12) [respectively (1.13)] for all $k \in K$.

$$\Pi'(k)P(k) + P(k)\Pi(k) < -2\sigma P(k) - Q(k) \quad (1.12)$$

$$\Pi'(k)P(k)\Pi(k) - P(k) < -(1-\rho^2)P(k) - Q(k) \quad (1.13)$$

where

$$\Pi(k) = \begin{bmatrix} \Delta & wh'(k) \\ 0 & A(k) \end{bmatrix}. \quad (1.14)$$

Here a multiaffine function is one that is affine in each individual argument. We note that the fact that the parametric Lyapunov pair thus constructed displays a multiaffine dependence on k has certain appealing characteristics to be highlighted in the sequel.

Polytopic sets such as (1.10–1.11) can equivalently be described by the convex combination of their corners. In particular, for some M and suitable $h_1, ..., h_M$, one can express Ω as the set

$$\Omega = \{\bar{A}(\lambda) = F + g(\sum_{i=1}^{M} \lambda_i h_i) : \sum_{i=1}^{M} \lambda_i = 1, \lambda_i > 0\}. \quad (1.15)$$

We will show that Lyapunov pairs under this slightly different parameterization are in fact *affine* rather than multiaffine in the λ_i. This surprising difference in the underlying parameterization, despite the clear equivalence between (1.10–1.11) and (1.15), will be explained in Section 3.

The second question we address relates to the KYP lemma, specifically with respect to transfer functions whose numerator and denominator belong to two independent polytopes with defining parameter vectors $k = [k_1, ..., k_m]'$ and $l = [l_1, ..., l_r]'$, respectively. Then we show that under a suitable choice of state variable representation, the Lyapunov pairs one obtains depend multiaffinely on the elements of k and l. As with the Lyapunov equation problem, a convex combination-based representation is also considered.

We demonstrate the significance of the Lyapunov function results by extending certain linear time varying (LTV) system stability results reported in [3], [4]. We expect similar extensions of results connected with the stability of nonlinear time varying systems [4], [5], to be made possible by comparable techniques.

The first application of the KYP results is that they are used to derive the Lyapunov results that we present here. The second application is in the area of adaptive systems where the use of the KYP Lemma in proving convergence and obtaining robustness measures is all pervasive [2]. In particular, as will be argued in Section 3 the results of this paper provide a vehicle for ascertaining robustness measures for adaptive output error identification algorithms.

There are several results to be found in the literature that address the issue of Lyapunov matrices for uncertain systems. To our knowledge, the earliest such is implicit in the work of Parks [15] who shows that the Hermite matrix serves as a natural Lyapunov matrix for companion matrices, though with a rank-1, positive semidefinite Q. Given that the Hermite matrix itself is bilinear in the elements of its associated companion matrix, it is evident that companion matrices with elements that vary in independent intervals automatically admit a bilinearly parameterized Lyapunov matrix **P** with an associated rank-1 Q. This bilinear dependence is, however, destroyed when one allows, as is the case in this paper, dependent variations in the companion matrix. The next result that we are aware of is due to Thatachar and Srinath [9] who consider a family, $\Omega(\lambda)$, depending on a single parameter and claim that such a family is Hurwitz invariant iff there exists a Lyapunov matrix $P(k)$, affine in the single parameter k, such that for all $k \in K$, $A'(k)P(k) + P(k)A(k) < 0$. We believe, however, the proof given in [9] to be in error. Specifically, the errors in [9] appear as follows (the equation numbers and the variables are those of [9]): The affineness of the Lyapunov function in its equation (8) requires $\beta_k, \gamma_k, k = 1, 2, \cdots$ to be constant (independent of the parameter K). This implies that the coefficients C_k in (15) must not be constants. Therefore, the application of a result by Brockett and Willems [12], critical to the proof in [9], is not possible.

Subsequent work in this area has been largely confined to the quadratic stabilizability literature (see [10] and the references therein) where the issue has been one of determining a single Lyapunov matrix P that satisfies the Lyapunov equation. Most necessary and sufficient conditions for such quadratic stabilizability of uncertain matrices are confined to norm bounded uncertainties as opposed to polytopic uncertainities. The exception is a result by Boyd and Yang [11], who for a given matrix A and two vectors b and c show through a direct application of the KYP Lemma, that there exists a single Lyapunov matrix P for which both $A'P + PA < 0$ and $cb'P + Pbc' \leq 0$ iff $c'(sI - A)^{-1}b$ is positive real. This result thus provides a condition for the family of single parameter Hurwitz matrices $\{A + \lambda bc' : \lambda \in [0, \infty)\}$ to have a single Lyapunov matrix.

The rest of this paper is organized as follows. In Section 2 we present certain preliminary results that facilitate subsequent analysis. Section 3 gives a constructive proof for the result concerning uncertain passive systems. Section 4 considers the existence of multiaffine solutions to the Lyapunov equations, for the continuous and discrete time cases. Again, both the proofs are constructive and draw upon the result of Section 3 and an SPR construction result given in [5]. Sections 6 and 7, respectively, present the continuous and discrete time LTV stability results. Section 8 contains concluding remarks.

II. Preliminaries

This section provides certain preliminary results and assumptions. The first result concerns a state variable realization (SVR) of a cascade combination of biproper transfer functions. In the sequel δ refers to the degree of a polynomial.

Lemma 2.1: Consider the monic polynomials $b(s), a(s), \beta(s)$ and $\alpha(s)$ such that $\delta(b(s)) = \delta(a(s))$ and $\delta(\beta(s)) = \delta(\alpha(s))$. Suppose for matrices and vectors of suitable dimensions,

$$\frac{b(s)}{a(s)} = 1 - h'(sI - F)^{-1}g \tag{2.1}$$

and

$$\frac{\beta(s)}{\alpha(s)} = 1 + v'(sI - D)^{-1}w \tag{2.2}$$

Then with

$$\Phi = \begin{bmatrix} D & 0 \\ gv' & F \end{bmatrix} \tag{2.3}$$

$$\Gamma = [w', g']' \tag{2.4}$$

and

$$\Psi = [v', -h']', \tag{2.5}$$

$$\frac{b(s)}{a(s)}\frac{\beta(s)}{\alpha(s)} = 1 + \Psi'(sI - \Phi)^{-1}\Gamma. \tag{2.6}$$

Proof: Trivial. □

The second result is as follows.

Lemma 2.2: With $b(s)$ as in (2.1)

$$b(s) = det(sI - F - gh') \tag{2.7}$$

Proof: Results from a trivial application of the fact that $det(I + AB) = det(I + BA)$. □

The KYP results of this paper will be derived for sets of transfer functions

$$T = \left\{\tau(s, k, l) = \frac{s^n - \sum_{i=1}^n b_i(k)s^{n-i}}{s^n - \sum_{i=1}^n a_i(l)s^{n-i}} : k \in K, l \in L\right\} \tag{2.8}$$

with K as in (1.11),

$$L = \{l = [l_1, ..., l_r]' : l_i^- \leq l_i \leq l_i^+\}, \tag{2.9}$$

and the $b_i(k)$ and $a_i(l)$ affine in their respective arguments. Lemma 2.3 below shows that for suitably chosen $F \in \Re^{n\times n}$, $g \in \Re^n$, $h_1(l) \in \Re^n$ and $h_2(k) \in \Re^n$, with $[F, g]$ a completely reachable pair and $h_i(.)$ affine in their respective arguments, T can equivalently be described by

$$T=\{1+(h_1(l)-h_2(k))'(sI-F-gh_1'(l))^{-1}g : k \in K, l \in L\} \tag{2.10}$$

Since $h_1(l)$ and $h_2(k)$ lie in independent polytopes, it follows that T can also be expressed as

$$T = \left\{1 + (\hat{h}_1(\mu) - \hat{h}_2(\lambda))'(sI - F - g\hat{h}_1'(\mu))^{-1}g\right\} \tag{2.11}$$

where

$$\hat{h}_1(\mu) = \sum_{i=1}^N \mu_i\hat{h}_{1i}\,; \sum_{i=1}^N \mu_i = 1\,;\ \mu_i \geq 0 \tag{2.12}$$

$$\hat{h}_2(\lambda) = \sum_{i=1}^M \lambda_i\hat{h}_{2i}\,; \sum_{i=1}^M \lambda_i = 1\,;\ \lambda_i \geq 0 \tag{2.13}$$

Observe that $\{1 + (\hat{h}_{1i} - \hat{h}_{2j})'(sI - F - g\hat{h}_{1i}')^{-1}g\}$ represents the corners of the set T. In the sequel we will denote $\mu = [\mu_1, ..., \mu_N]'$ and $\lambda = [\lambda_1, ..., \lambda_M]'$ (note $N = 2^m$ and $M = 2^r$).

Lemma 2.3: Every set of the form in (2.8) can be expressed as in (2.10) and vice versa.

Proof:

1) Showing (2.10) ⇒ (2.8): This follows by the application of the matrix inversion lemma, which yields

$$\begin{aligned} 1 + (h_1 - h_2)'(sI - F - gh_1')^{-1}g &= 1 + (h_1 - h_2)' \\ &\quad \cdot \left[(sI - F)^{-1} + \frac{(sI - F)^{-1}gh_1'(sI - F)^{-1}}{1 - h_1'(sI - F)^{-1}g}\right]g \\ &= 1 + \frac{(h_1 - h_2)'(sI - F)^{-1}g}{1 - h_1'(sI - F)^{-1}g} \\ &= \frac{1 - h_2'(sI - F)^{-1}g}{1 - h_1'(sI - F)^{-1}g} \end{aligned} \tag{2.14}$$

2) Showing (2.8) ⇒ (2.10): For a given F define

$$f(s) = s^n + \sum_{i=1}^n f_i s^{n-i} = \det(sI - F). \tag{2.15}$$

For arbitrary completely reachable $[F, g]$, there exists T such that

$$T^{-1}FT = \bar{F} = \begin{bmatrix} 0 & I_{n-1} \\ -f_n & \dots \quad -f_1 \end{bmatrix} \tag{2.16}$$

and $\bar{g} = T^{-1}g = [0, \dots, 0, 1]'$. Then with $f = [f_n, \dots, f_1]$,

$$a(l) = [a_n(l), \dots, a_1(l)]', \tag{2.17}$$

$$b(k) = [b_n(k), \dots, b_1(k)]', \tag{2.18}$$

it is easy to verify that

$$\frac{s^n - \sum_{i=1}^n a_i(\ell)s^{n-i}}{f(s)} = 1 - \bar{h}_1'(\ell)(sI - \bar{F})^{-1}\bar{g} \tag{2.19}$$

and

$$\frac{s^n - \sum_{i=1}^n b_i(k)s^{n-i}}{f(s)} = 1 - \bar{h}_2'(k)(sI - \bar{F})^{-1}\bar{g} \tag{2.20}$$

where $\bar{h}_1(\ell) = a(\ell) - f$ and $\bar{h}_2(k) = b(k) - f$. Therefore, using (2.14), $\tau(s, k, \ell)$ is expressible as

$$\tau(s, k, l) = 1 + (\bar{h}_1(l) - \bar{h}_2(k))'(sI - \bar{F} - \bar{g}\bar{h}_1'(l))^{-1}\bar{g}. \tag{2.21}$$

Then choosing $h_i(.) = (T^{-1})'\bar{h}_i(.)$ yields the result. □

The next set of results concern certain multiaffine functions. Specifically, we consider a set of functions U:

$$U = \{u(k) \in \Re^{n\times m} : k \in K\} \tag{2.22}$$

where $u(k)$, the elements of U, are multiaffine, possibly matrix, functions of k.

In the sequel, a parameter vector k, is said to lie on an $r - face$ of K if all but r elements of k are at extreme values. Corners constitute the $0 - faces$ of K. Members of the set U will be said to lie on an $m - face$ of the respective set if the corresponding k is on an $m - face$ of K. We note the following fact concerning U.

Fact 2.3: Suppose U in (2.22) is multiaffine in k. Then, for every k on an $r - face$ of K there exist $k^{(1)}$ and $k^{(2)}$ on certain $(r-1) - faces$ of K such that: (1) $k^{(1)}$, $k^{(2)}$, k differ from each other in only one element; (2) There exists $x \in [0, 1]$, such that $k = (1-x)k^{(1)} + xk^{(2)}$ and $u(k) = (1-x)u(k^{(1)}) + xu(k^{(2)})$.

The following Lemma from [18] shows that U is uniquely defined by its corners.

Lemma 2.4: Suppose, the value of the corners of U (see (2.22)) are known. Then there exists a unique multiaffine function $u(k)$, which assumes the given values at the respective corners.

We next present a generalized version of the mapping theorem of Zadeh and Desoer [16], a result that we consider as being of independent interest. To this end, we introduce the notion of objects $\tilde{O}$ and convex properties $\boldsymbol{P}$, which these objects may or may not satisfy. In the sequel any quantity will be called an object if it is a member of a set over which the operations of addition (+) among its members and multiplication (.) by real scalars are well defined. Thus, rational functions and matrices are examples of objects. A property $\boldsymbol{P}$ acting on an object maps this object to either 0 or 1. For a given object O, if $\boldsymbol{P}(O) = 1$ then we say O satisfies $\boldsymbol{P}$, else it does not satisfy $\boldsymbol{P}$. A property $\boldsymbol{P}$ defined over a set of objects $\tilde{O}$ will be said to be convex if for any $O_1, O_2 \in \tilde{O}$, $\boldsymbol{P}(O_1) = 1$ and $\boldsymbol{P}(O_2) = 1$ implies that for all $x \in [0, 1]$, $\boldsymbol{P}((1-x)O_1 + xO_2) = 1$. Then the following theorem holds.

Theorem 2.1: With real scalars k_i and the set K as defined in (1.11), define the set of objects

$$\tilde{O} = \{O(k) : k \in K\} \tag{2.23}$$

where $O(k)$ is a multiaffine function of k. Suppose $\boldsymbol{P}$ is a convex property defined on $\tilde{O}$. Then every member of $\tilde{O}$ satisfies $\boldsymbol{P}$ iff each corner of $\tilde{O}$ satisfies $\boldsymbol{P}$.

Proof: We use induction. From fact 2.3 and the convexity of $\boldsymbol{P}$, if $\boldsymbol{P}$ is satisfied on each $r - face$ of $\tilde{O}$, then it is also satisfied on each $(r+1) - face$. Then provided it is satisfied at each corner it must be satisfied everywhere in $\tilde{O}$. □

By way of application for Theorem 2.1, we present the following result.

Corollory 2.1: Suppose in (2.22), $u(k)$ is a symmetric multiaffine matrix function of k. Then the maximum and minimum eigenvalues of $u(k)$ occur at the corners of U. Consequently, all members of U are positive definite iff all corners of U are positive definite.

To conclude this section on preliminaries, we impose certain restrictions on various matrices of interest.

Assumption 2.1: The pair $[F, g]$ is completely reachable. Further, for (2.10) $[F, h_1(l) - h_2(k)]$ is completely observable almost everywhere in $K \times L$, including at every corner of L and K. Likewise, for Ω, $[F, h(k)]$ is completely observable almost everywhere in K, including at every corner of K.

We note that the corner observability conditions can be assumed without loss of generality, possibly through an infinitesimal expansion of L and/or K.

Recall, that Ω will be examined for σ-Hurwitz (or ρ-Schur) invariance. Thus, to avoid trivialities we will assume that at least one member of Ω is σ-Hurwitz (or ρ-Schur). Then, through a simple affine transformation in the parameter vector k if need be, one can make the following assumption without loss of generality.

Assumption 2.2: Under continuous (respectively discrete) time settings, F is σ−Hurwitz (respectively ρ-Schur).

III. On the Kalman-Yakubovic-Popov Lemma

In this section we will address the issue of parameterized Lyapunov pairs for σ-CSPR and ρ-DSPR parameterized transfer functions as defined in (2.10) and (2.11). Before presenting the main results of this section, we wish to present the continuous and discrete time linear matrix inequality (LMI) [15]. In continuous time it is known that the satisfaction of (1.5-1.6) is equivalent to:

$$\begin{bmatrix} -(A'P + PA) - 2\sigma P - Q & (Pb + c) \\ (Pb + c)' & 2 \end{bmatrix} > 0 \tag{3.1}$$

Likewise in discrete time (1.7)–(1.9) are equivalent to

$$\begin{bmatrix} -A'PA+\rho^2P-Q & (A'Pb+c) \\ (A'Pb+c)' & 2-b'Pb \end{bmatrix} > 0 \tag{3.2}$$

Indeed, henceforth we will work exclusively with the two LMI's. The first set of results concerns the parameterization in (2.11)-(2.13).

Theorem 3.1: All members of the set (2.11)-(2.13) are σ-CSPR iff there exist symmetric $P(\mu,\lambda)$ and $Q(\mu,\lambda)$, which obey the following:

1) $\forall \mu_i$, λ_i obeying the constraints in (2.12) and (2.13), (3.3) at the bottom of this page holds.
2) For fixed μ (respectively λ), both $P(\mu,\lambda)$ and $Q(\mu,\lambda)$ are affine in the elements of λ (respectively μ).
3) $P(\mu,\lambda)>0$ and $Q(\mu,\lambda)>0$ $\forall \mu,\lambda$ obeying (2.13) and (2.12).

Proof: The "if" part of the theorem is straightforward. We focus instead on the "only if" part. Suppose all members of the set T in (2.11) to (2.13) are indeed σ-CSPR. Then since for each $i\in\{1,\dots,N\}$ and $j\in\{1,\dots,M\}$

$$T_{ij}=1-(\hat{h}_{2j}-\hat{h}_{1i})'(sI-F-g\hat{h}_{1i})^{-1}g \tag{3.4}$$

is σ-CSPR and in view of Assumption 2.1 and the fact that $\hat{h}_{1i}$ and $\hat{h}_{2j}$ represent the respective corners of the polytopes to which $h_1(l)$ and $h_2(k)$ belong, we have the existence of positive definite symmetric P_{ij}, Q_{ij} such as are displayed in (3.5) at the bottom of this page.

Defining

$$P_i(\lambda)=\sum_{j=1}^{M}\lambda_jP_{ij}\ ; \qquad Q_i(\lambda)=\sum_{j=1}^{M}\lambda_jQ_{ij} \tag{3.6}$$

one obtains from (3.5) and the convexity of the positive definiteness property, for all $i\in\{1,\dots,N\}$the inequality shown in (3.7) at the bottom of this page. Pre- and postmultiplying (3.7) by the nonsingular matrices $T_1(\lambda,i)$ and $T_1'(\lambda,i)$ with

$$T_1(\lambda)=\begin{bmatrix} I_n & \hat{h}_{1i}-\hat{h}_2(\lambda) \\ 0 & 1 \end{bmatrix} \tag{3.8}$$

one obtains (3.9), shown at the bottom of this page. Define

$$P(\mu,\lambda)=\sum_{i=1}^{N}\mu_iP_i(\lambda) \qquad \text{and} \qquad Q(\mu,\lambda)=\sum_{i=1}^{N}\mu_iQ_i(\lambda) \tag{3.10}$$

Observe from (3.6)–(3.10) that $P(\mu,\lambda)$ and $Q(\mu,\lambda)$ so obtained obeys (2). Then using yet again the underlying convexity, one has the inequality shown in (3.11) at the bottom of this page. Then pre- and postmultiplying (3.11) by $T_2(\mu,\lambda)$ and $T_2'(\mu,\lambda)$ with

$$T_2(\mu,\lambda)=\begin{bmatrix} I_n & \hat{h}_2(\lambda)-\hat{h}_1(\mu) \\ 0 & 1 \end{bmatrix} \tag{3.12}$$

yields the result. □

$$\begin{bmatrix} -(F+g\hat{h}_1'(\mu))'P(\mu,\lambda)-P(\mu,\lambda)(F+g\hat{h}_1'(\mu))-Q(\mu,\lambda)-2\sigma P(\mu,\lambda), & P(\mu,\lambda)g+\hat{h}_2(\lambda)-\hat{h}_1(\mu) \\ (P(\mu,\lambda)g+\hat{h}_2(\lambda)-\hat{h}_1(\mu))' & 2 \end{bmatrix} > 0 \tag{3.3}$$

$$\begin{bmatrix} -(F+g\hat{h}_{1i}')'P_{ij}-P_{ij}(F+g\hat{h}_{1i}')-Q_{ij}-2\sigma P_{ij}, & P_{ij}g+\hat{h}_{2j}-\hat{h}_{1i} \\ (P_{ij}g+\hat{h}_{2j}-\hat{h}_{1i})' & 2 \end{bmatrix} > 0 \tag{3.5}$$

$$\begin{bmatrix} -(F+g\hat{h}_{1i}')'P_i(\lambda)-P_i(\lambda)(F+g\hat{h}_{1i}')-Q_i(\lambda)-2\sigma P_i(\lambda), & P_i(\lambda)g+\hat{h}_2(\lambda)-\hat{h}_{1i} \\ (P_i(\lambda)g+\hat{h}_2(\lambda)-\hat{h}_{1i})' & 2 \end{bmatrix} > 0 \tag{3.7}$$

$$\begin{bmatrix} -(F+g\hat{h}_2'(\lambda))'P_i(\lambda)-P_i(\lambda)(F+g\hat{h}_2'(\lambda))-Q_i(\lambda)-2\sigma P_i(\lambda), & P_i(\lambda)g+\hat{h}_{1i}-\hat{h}_2(\lambda) \\ (P_i(\lambda)g+\hat{h}_{1i}-\hat{h}_2(\lambda))' & 2 \end{bmatrix} > 0. \tag{3.9}$$

$$\begin{bmatrix} -(F+g\hat{h}_2'(\lambda))'P(\mu,\lambda)-P(\mu,\lambda)(F+g\hat{h}_2'(\lambda))-Q(\mu,\lambda)-2\sigma P(\mu,\lambda), & P(\mu,\lambda)g+\hat{h}_1(\mu)-\hat{h}_2(\lambda) \\ (P(\mu,\lambda)g+\hat{h}_1(\mu)-\hat{h}_2(\lambda))' & 2 \end{bmatrix} > 0 \tag{3.11}$$

Several remarks are in order.

Remark 3.1: It is interesting to note that in the above, (3.7) expresses the σ-CSPR property of

$$1 - (\hat{h}_2(\lambda) - \hat{h}_{1i})'(sI - F - g\hat{h}_{1i})^{-1}g \tag{3.13}$$

while (3.9) expresses the σ-CSPR property of the *inverse* of

$$1 - (\hat{h}_2(\lambda) - \hat{h}_{1i})'(sI - F - g\hat{h}_{1i})^{-1}g \tag{3.14}$$

with clearly the same P, Q pair.

Remark 3.2: Observe that the proof given above is constructive. In particular, one must construct the Lyapunov pairs $[P_{ij}, Q_{ij}]$ (using possibly the spectral factorization method outlined in [14]), which work with the corner represented by $\hat{h}_1 = \hat{h}_{2i}$ and $\hat{h}_2 = \hat{h}_{1j}$, and then apply (3.6) and (3.10) to obtain the desired Lyapunov pairs $[P(\mu, \lambda), Q(\mu, \lambda)]$.

Remark 3.3: The special cases of (2.8) and (2.9) corresponding to the situations where the numerator is fixed and the denominator is uncertain, and where the converse holds, are of particular interest in adaptive systems and the development to be outlined in Section 4. In the case where the numerator is fixed, one can assume that

$$h_2(k) = h \ \ \forall k. \tag{3.15}$$

Likewise, the converse case of denominator fixed allows one to assume without loss of generality, that

$$h_1(\ell) = 0 \ \ \forall \ell. \tag{3.16}$$

In either case, $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ are *affine* in the underlying parameters. We next present the discrete time counterpart of theorem 3.1.

Theorem 3.2: All members of the set (1.12) are ρ-DSPR iff there exist symmetric $P(\mu, \lambda)$ and $Q(\mu, \lambda)$ which obey (ii) and (iii) of Theorem 2 and (3.17), shown at the bottom of this page.

Proof: Follows exactly as in Theorem 3.1. □

Remarks 3.2 and 3.3 apply to this situation as well. Having dispensed with the parameterization contained in (2.11), we now turn our attention to its counterpart in (2.10).

Theorem 3.3: All members of (2.10) are σ-CSPR iff there exist symmetric $P(k, l)$ and $Q(k, l)$, multiaffine in $[k', l']'$ that obey (3.18), shown at the bottom of this page.

Proof: Define h_{1i} and h_{2j} $i = 1, ..., N,\ j = 1, ..., M$ as the corners of the respective polytopes to which $h_1(l)$ and $h_2(k)$ belong. In the sequel, we will refer to the corresponding values of l and k as the ith corner of L and the jth corner of K, respectively. Define $[P_{ij}, Q_{ij}]$ as the Lyapunov pair that "works" with the corner $1 + (h_{1i} - h_{2j})'(sI - F - gh'_{1i})^{-1}g$. Define $P(k, l), Q(k, l)$ as the unique matrix function, multiaffine in the k and l, that assumes the value P_{ij} at the corresponding corners of K and L (see (2.10)).

Observe that $P(k, l)(Q(k, l))$ can equivalently be viewed in the following terms. Define for each j and with l fixed at the jth corner of L, $P_j(k)(Q_j(k))$ to be the unique multiaffine function that assumes the value $P_{ij}(Q_{ij})$ at the ith corner of K. Then for each $k \in K$, $P(k, l)(Q(k, l))$ is the unique multiaffine function that assumes the value $P_j(k)(Q_j(k))$ whenever l is fixed at the jth corner of L.

For each j, define the set

$$\tilde{O}_j = \{O_j(k) = [P_j(k), Q_j(k)] : k \in K\}. \tag{3.19}$$

$O_j(k)$ is of course multiaffine in k. Define the property $\boldsymbol{P}_j$ as being satisfied iff (3.18) holds for all $k \in K$ and with l fixed at the jth corner of L (recall, for this j, $[P(k, l), Q(k, l)] = [P_j(k), Q_j(k)]$). Then by Theorem 3.1, the property $\boldsymbol{P}_j$ is convex. Since, it is satisfied, by hypothesis, at the corners of K, it holds for all $k \in K$.

Likewise, with any fixed $k \in K$, define,

$$\tilde{O} = \{O(k, l) = [P(k, l), Q(k, l)] : k \in K, l \in L\}. \tag{3.20}$$

As before, for every k, $O(k, l)$ is multiaffine in l. Define the property $\boldsymbol{P}$ as being satisfied iff (3.18) holds for this chosen k and each $l \in L$. Again from Theorem 3.1, this property is convex, from the above it holds at the corners of L, and hence Theorem 2.1 establishes the result. □

In a similar vein we can prove the following result.

Theorem 3.4: All members of (2.10) are ρ-DSPR iff there exist symmetric $P(k, l)$ and $Q(k, l)$, multiaffine in $[k', l']'$ which obey (3.21), shown at the bottom of this page.

Remark 3.4: A self-evident modification of Remark 3.3 applies here as well.

$$\begin{bmatrix} -(F + g\hat{h}'_1(\mu))'P(\mu, \lambda)(F + g\hat{h}'_1(\mu)) - Q(\mu, \lambda) + \rho^2 P(\mu, \lambda), & (F + g\hat{h}'_1(\mu))'P(\mu, \lambda)g + \hat{h}_2(\lambda) - \hat{h}_1(\mu) \\ ((F + g\hat{h}'_1(\mu))'P(\mu, \lambda)g + \hat{h}_2(\lambda) - \hat{h}_1(\mu))' & 2 - g'P(\mu, \lambda)g \end{bmatrix} > 0 \tag{3.17}$$

$$\begin{bmatrix} -(F + gh'_1(l))'P(k, l) - P(k, l)(F + gh'_1(l)) - Q(k, l) - 2\sigma P(k, l) & P(k, l)g + h_2(k) - h_1(l) \\ (P(k, l)g + h_2(k) - h_1(l))' & 2 \end{bmatrix} > 0 \tag{3.18}$$

$$\begin{bmatrix} -(F + gh'_1(l))'P(k, l)(F + gh'_1(l)) - Q(k, l) + \rho^2 P(k, l) & (F + gh'_1(l))'P(k, l)g + h_2(k) - h_1(l) \\ ((F + gh'_1(l))'P(k, l)g + h_2(k) - h_1(l))' & 2 - g'P(k, l)g \end{bmatrix} > 0 \tag{3.21}$$

Remark 3.5: Here also the proof is constructive. As in Remark 3.2, we must now construct the Lyapunov pairs $[P_{ij}, Q_{ij}]$ (see Remark 3.2 on the construction of these pairs) that work with the transfer function that represents the combination of the i-th and j-th corners of K and L, respectively. Then the required $[P(k,\ell), Q(k,\ell)]$ is the unique multiaffine function that assumes the value $[P_{ij}, Q_{ij}]$ at the apropriate corner combination (See Lemma 2.4).

Remark 3.6: Observe, that Theorems 3.1 and 3.2 deal with parametrizations that are equivalent to those used in their respective counterparts Theorems 3.3 and 3.4. However, while for fixed λ (respectively μ) the Lyapunov pairs of Theorems 3.1 and 3.2 are collectively affine in the μ (respectively λ) parameters, even for a fixed k (respectively ℓ) the Lyapunov pairs of Theorems 3.3 and 3.4 are multiaffine in the ℓ (respectively k) parameters. This apparent paradox can be understood in terms of the following example. Consider the multiaffine function

$$p(k_1, k_2) = 1 + k_1 + k_2 + k_1 k_2 \qquad 0 \le k_1 \le 1,\ 0 \le k_2 \le 1. \tag{3.22}$$

Clearly, in the given range of $[k_1, k_2]$, $p(k_1, k_2)$ cannot be expressed as an affine function of two variables. Yet each member of this set can be expressed as a convex combination of the four corners $p(0,0)$, $p(0,1)$, $p(1,1)$ and $p(1,0)$. This latter representation, though, will be nonunique. As shown in Lemma 2.4 this example reflects a general property of all multiaffine mappings. Indeed similar considerations apply to the results of Section 4 also.

We conclude this section by explaining the utility of these results to the adaptive output error identification of an unknown system with denominator polynomial $a(s)$. To guarantee convergence, one needs to construct a single error filter $E(s)$ such that $E(s)/a(s)$ is biproper and SPR. Further the minimum and maximum eigenvalues of the P, Q pair associated with $E(s)/a(s)$, together with the adaptation gain and the degree of excitation of the input signals, constitute the major determinants of the extent to which such an algorithm is robust to modeling inadequacies.

We now argue that the results of this paper readily apply to the determination of the extreme eigenvalues of the P and Q matrices associated with $E(s)/a(s)$. To this end observe that $a(s)$ being the denominator of the system to be identified, is itself unknown. One may assume, however, that prior knowledge supplied by the physical modeling processes allows one to obtain a polytope to which the true value of $a(s)$ belongs. Denote by k the defining parameters of this polytope. Then in [6], [7] are formulated methods for constructing a single operator $E(s)$ of the required relative degree, whose product with every potential value of $1/a(s)$ is SPR. The KYP result derived here then provides multiaffine Lyapunov pairs, $P(k)$ and $Q(k)$ corresponding to all possible values of $E(s)/a(s)$. The multiaffine nature of $P(k)$ and $Q(k)$ then directly provides tight bounds on their eigenvalues. This is so (see corollory 2.1) because the extreme eigenvalues of multiaffine symmetric matrices, defined over a polytopic set in the parameters, are the eigenvalues of the matrices corresponding to the corners of this polytope. Thus the needed robustness margins can be readily obtained.

IV. Solutions To The Lyapunov Equation

In this section, we restrict our attention to the set Ω as represented in both (1.10) and (1.15) and consider suitable Lyapunov pairs for this set. The main results of this section are first formally stated.

Theorem 4.1 Consider Ω as in (1.10), with assumptions 2.1 and 2.2 in force. Then, all members of Ω are σ-Hurwitz (respectively ρ-Schur) iff there exist σ-Hurwitz (respectively ρ-Schur) Δ, a vector w and positive definite symmetric $P(k)$ and $Q(k)$, multiaffine in k, such that for all k in K , (1.12) (respectively (1.13)) holds with $\Pi(k)$ as in (1.14)

Theorem 4.2 With Ω as in (1.15), the statement of Theorem 4.1 stands with $P(k), Q(k), \Pi(k)$ replaced by $P(\lambda), Q(\lambda), \Pi(\lambda)$, $\Pi(\lambda)$ obviously defined and $P(\lambda), Q(\lambda)$ affine in λ.

We will prove Theorem 4.1 in detail. The proof of Theorem 4.2 being similar, is omitted. The proofs to be given will be constructive, and as will become clear presently, the construction of the Lyapunov pairs can be accomplished by only considering the corners of Ω. The key results to be used fall into two categories. The first is the main result of Section 3. The second result we use is a minor variation of a construction result given in [7]. This result in [7] considers polytopes of polynomials and gives necessary and sufficient conditions under which there exists a single stable LTI operator whose product with all the members of this polytope is σ-CSPR (respectively ρ-DSPR). The variation in question is summarized in Theorem 4.3 below. In presenting this theorem, we specialize it to the needs of the present paper. Specifically, the polytope of polynomials we consider here is the set of characteristic polynomials of the members of Ω. Recall from Lemma 2.2, this is a polytope as

$$det(sI - (F + gh'(k))) = det(sI - F) - h'(k)Adj(sI - F)g \tag{4.1}$$

Theorem 4.3 Consider the set Ω as in (1.10). This set is σ-Hurwitz (respectively ρ-Schur) invariant iff there exist monic polynomials $c(s)$ and $d(s)$, with $d(s)$ σ-Hurwitz(respectively ρ-Schur) such that the transfer function

$$\frac{det(sI - (F + gh'(k)))c(s)}{d(s)} \tag{4.2}$$

is biproper and σ-CSPR (respectively ρ-DSPR) for all $k \in K$.

Proof: The proof follows from [7] and the transformation $s \to s - \sigma$ in continuous time and $z \to \rho z$ in discrete time.□

A few comments about this result are called for. Since in the continuous and discrete time settings of our problem F is respectively σ-Hurwitz and ρ-Schur with

$$f(s) = det(sI - F) \tag{4.3}$$

for sufficiently small ε, σ- Hurwitz or ρ-Schur invariance of Ω is equivalent to the existence of monic $c(s)$ and $d(s)$ as above, such that the transfer function below is σ-CSPR and

ρ-DSPR for all $k \in K$.

$$\frac{det(sI-(F+gh'(k))}{f(s)}\frac{f(s+\varepsilon)c(s)}{d(s)} \tag{4.4}$$

Further, as there are only a finite number of corners of Ω and as Lemma 2.2 and Assumption 2.1 assure that $det(sI-(F+gh'(k)))$ and $f(s)$ are coprime for all corners of K, through an arbitarily small perturbation in $c(s)$ and $d(s)$, if need be, one can ensure that the transfer function in (4.4) is free from any pole-zero cancellations at the corners of K. In the sequel we will assume

$$\delta(f(s)d(s)) = N. \tag{4.5}$$

It is clear that the choice of $c(s)$ and $d(s)$ ensures that $f(s+\varepsilon)c(s)/d(s)$ is biproper. Suppose its minimal state variable realization is $\{D, w, v, 1\}$. From Lemma 2.2, $\{F, g, -h(k), 1\}$ is a state variable realization of $[det(sI-(F+gh'(k)))]/f(s)$. Then from Lemma 1 the transfer function in (4.4) has the state variable realization $\{\Phi, \Gamma, \Psi(k), 1\}$ where Φ, Γ and $\Psi(k)$ are given by (2.3), (2.4) and (2.5) respectively.

From the foregoing discussion the following is obtained.

Lemma 4.1 The set Ω as in (1.10) is σ-Hurwitz (respectively ρ-Schur) invariant iff there exist suitably dimensioned D and v such that with Φ, Γ and $\Psi(k)$ as defined in (2.3) through (2.5)

$$1 + \Psi(k)'(sI-\Phi)^{-1}\Gamma \tag{4.6}$$

is σ-CSPR (respectively ρ-DSPR) for all $k \in K$. Further (4.6) is minimal at all corners of K.

Having proved this Lemma we now turn to proving Theorem 4.1.

Proof of Theorem 4.1: We will only show the "only if" part of the theorem, since the "if" part is trivial.

1) σ-CSPR: From Theorem 3.3 and Lemma 4.1, there exist positive definite, symmetric $P(k), Q(k)$ multiaffine in k, such that, for all $k \in K$, the matrix in (4.7) is positive definite

$$\begin{bmatrix} -\Phi'P(k)-P(k)\Phi-Q(k)-2\sigma P(k) & P(k)\Gamma-\Psi(k) \\ (P(k)\Gamma-\Psi(k))' & 2 \end{bmatrix} \tag{4.7}$$

Then pre- and postmultiplying by T and T' with

$$T = \begin{bmatrix} I & \Psi(k) \\ 0 & I \end{bmatrix} \tag{4.8}$$

one gets (4.9), shown at the bottom of this page. Thus, from the (1,1) block of (4.9),

$$(\Phi-\Gamma\Psi'(k))'P(k)+P(k)(\Phi-\Gamma\Psi'(k))<-Q(k)-2\sigma P(k). \tag{4.10}$$

Noting that, with $\Delta = D - wv'$, $(\Phi - \Gamma\Psi'(k))$ is precisely the matrix $\Pi(k)$ in (1.14), one obtains (1.12). Moreover, comparing the left- and right-hand sides of the (1,1) blocks of (1.12), proves that Δ is σ-Hurwitz.

2) ρ-DSPR: Follows exactly as above.

Observe the procedure for constructing the Lyapunov pairs is yet again constructive and, assuming that Ω is σ-Hurwitz invariant, entails the following steps:

Step I: Find the multiplier $c(s)/d(s)$ outlined in (4.2) such that the transfer function (4.2) is biproper and is σ-CSPR for all $k \in K$. This can be done using the techniques of [7] using corners of Ω alone.

Step II: Find sufficiently small ε, such that the transfer function in (4.4) is minimal at the corners of K and is σ-CSPR for all $k \in K$.

Step III: Construct $\{\Phi, \Gamma, \Psi(k), 1\}$, a state variable realization of (4.4), with properties set out in Lemma 4.1.

Step IV: As (4.4) is σ-CSPR for all $k \in K$, use the results of Section 3 to construct the multiaffinely parameterized pair $[P(k), Q(k)]$ that obeys (4.7). Again only the corners of K are needed.

Step V: The required $P(k)$ and $Q(k)$ are those obtained in Step 4.

Thus, should the set Ω be σ-Hurwitz invariant, the Lyapunov pairs can be constructed from the corners of Ω alone. This *does not* imply that σ-Hurwitz invariance of the corners of Ω, is equivalent to the σ-Hurwitz invariance of Ω. Indeed, the fact that the corner stability of a polytope of matrices does not imply the stability of the entire polytope can be gleaned from counter-examples (e.g. [21]) that show that the stability of a polytope of polynomials is not implied by the stability of its corners.

V. Stability of Continuous Time Linear Time Varying Systems

In this section, we illustrate the utility of Theorem 3.2 by employing it in the analysis of a class of linear time varying (LTV) systems. We begin with a definition.

Definition 5.1: The LTV system

$$\dot{x}(t) = A(t)x(t) \tag{5.1}$$

is exponentially asymptotically stable (EAS) with degree of stability $\gamma > 0$ if $\exists c, \alpha > 0$ such that for all $x(t_0)$ and $t \geq t_0$,

$$\|x(t)\|e^{\gamma(t-t_0)} \leq c\|x(t_0)\|e^{-\alpha(t-t_0)} \tag{5.2}$$

If $\gamma = 0$, we simply say that (5.1) is EAS.

References [3], [4] contain results that through a simple application of results in [19], [20], yield conditions for the EAS of a class of LTV systems with time variations confined to a *scalar* parameter k. In particular, suppose that,

$$A(k) = F + kgh' \tag{5.3}$$

with g, h vectors is σ-Hurwitz for all scalar fixed k lying in a given interval. Then the conditions in question, involve

$$\begin{bmatrix} -(\Phi-\Gamma\Psi'(k))'P(k)-P(k)(\Phi-\Gamma\Psi'(k))-Q(k)-2\sigma P(k) & P(k)\Gamma+\Psi(k) \\ (P(k)\Gamma+\Psi(k))' & 2 \end{bmatrix} > 0 \tag{4.9}$$

certain precise logarithmic bounds on the time variations in the parameter k, such that the EAS of

$$\dot{x}(t) = A(k(t))x(t) \tag{5.4}$$

is retained. Specifically, one obtains the theorem below.

Theorem 5.1: Suppose $A(k)$ as in (5.3) is σ-Hurwitz for all $k \in [k^-, k^+]$. Then (5.4) is EAS. if for some $\epsilon_1, \epsilon_2, T > 0$, $\delta \in (0, \sigma)$ and all $t \geq 0$

1)

$$k(t) \in [k^- + \epsilon_1, k^+ - \epsilon_2] \tag{5.5}$$

and

2) Either

a)

$$\sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \left[\frac{d}{d\tau} ln \frac{k(\tau) - k^-}{k^+ - k(\tau)}\right]^+ d\tau < 2(\sigma - \delta) \tag{5.6}$$

where

$$[a]^+ = \begin{cases} a; & a \geq 0 \\ 0; & a < 0 \end{cases} \tag{5.7}$$

or

b)

$$\sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \left|\frac{d}{d\tau} ln \frac{k(\tau) - k^-}{k^+ - k(\tau)}\right| d\tau < 4(\sigma - \delta) \tag{5.8}$$

Several comments are in order. First, in essence each condition in Theorem 5.1 offers a trade-off between the degree of stability of the "frozen" LTI systems and the average time variation that could be withstood without losing stability. Further, as one can imagine, by choosing a larger δ (i.e., with a smaller bound on the average derivative of the logarithmic value of the time varying parameter), one can quantify the degree of EAS that the resulting time varying system is endowed with. Such a result is in [17].

Second, the results of [3], [4], [17] apply only to the *continuous time* case involving the situation where time variation is confined to a *single parameter*. No comparable result for multiparameter time varying systems is to our knowledge available; nor are we aware of similar stability results that apply to discrete time system.

Third, even for the single parameter case, the results of [3], [4], [17] are proved using a somewhat involved multiplier theory, which to our knowledge does not readily extend to multiparameter time varying system.

The principal contribution of this section is to demonstrate how the results of Section 4 can be used to readily prove a much more general set of results that: (1) Involve LTV systems with multiple time varying parameters; (2) incorporate the degree of stability considerations featuring in [17]; and (3) specialize to Theorem 5.1 in the single parameter case. Section 6 gives the corresponding discrete time result. Specifically, we prove the following.

Theorem 5.2 With $\Omega, A(k), k = [k_1, \cdots, k_m]', K$ as in (1.10), (1.11), and $h(k)$ affine in the elements of k, suppose every member of Ω is σ-Hurwitz. Then the LTV system

$$\dot{x}(t) = A(k(t))x(t) \tag{5.9}$$

is EAS with degree of stability γ, $0 < \gamma < \sigma$, if there exists $\delta \in (0, \sigma - \gamma), T > 0$ and $\epsilon_{1i}, \epsilon_{2i} > 0$, $\forall i = \{1, ..., m\}$ such that for all $t > 0$

1)

$$k_i(t) \in [k_i^- + \epsilon_{1i}, k_i^+ - \epsilon_{2i}], \qquad \forall i \in \{1, ..., m\} \tag{5.10}$$

and

2) Either

a)

$$\sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \sum_{i=1}^{n} \left[\frac{d}{d\tau} ln \frac{k_i(\tau) - k_i^-}{k_i^+ - k_i(\tau)}\right]^+ d\tau < 2(\sigma - \delta - \gamma). \tag{5.11}$$

or

b)

$$\sup_{t\geq 0} \frac{1}{T} \int_t^{t+T} \sum_{i=1}^{m} \left|\frac{d}{d\tau} ln \frac{k_i(\tau) - k_i^-}{k_i^+ - k_i(\tau)}\right| d\tau < 4(\sigma - \delta - \gamma). \tag{5.12}$$

The respective association between (5.11), (5.12) and (5.6), (5.8) is clear.

To prove this theorem we first provide a result proved in the appendix which shows that (5.12) is in fact a stronger condition than (5.11).

Lemma 5.1 With (5.10) in force, (5.12) implies (5.11).

Thus, we need only show that (5.10) and (5.11) suffice for the EAS of (5.9) with degree of stability γ.

Now Theorem 4.1 and the fact that Ω is σ-Hurwitz invariant together imply the existence of a σ-Hurwitz Δ and multiaffine symmetric positive definite matrix functions $P(k), Q(k)$ such that with $\Pi(k)$ as in (1.14), (1.12) holds for all $k \in K$. In the sequel, it will become evident that it is more convenient to work with the LTV system

$$\dot{z}(t) = \Pi(k(t))z(t) \tag{5.13}$$

rather than with (5.9). Evidently, the block upper triangular structure of $\Pi(k(t))$ and the position occupied by $A(k(t))$ in $\Pi(k(t))$ readily yield the following.

Lemma 5.2: With $\Pi(k)$ as in (1.14), if the LTV system (5.13) is EAS with degree of stability γ, then so is (5.9).

Thus, we need only show that under (5.10) and (5.11), (5.13) is EAS with degree of stability γ. To this end we first prove the following intermediate proposition.

Proposition 5.1: Suppose $\lambda = [\lambda_1, ..., \lambda_m]'$ and the square matrix function $\tilde{\Pi}(\lambda)$ is such that there exists a symmetric matrix function $\tilde{P}(\lambda)$, multiaffine in λ, such that for all $\lambda_i \in [0, \infty), i \in \{1, ..., m\}$

$$\tilde{\Pi}'(\lambda)\tilde{P}(\lambda)) + \tilde{P}(\lambda)\tilde{\Pi}(\lambda) < -2\sigma\tilde{P}(\lambda) \tag{5.14}$$

and

$$\tilde{P}(\lambda) > 0 \tag{5.15}$$

Then, the LTV system

$$\dot{\tilde{z}}(t) = \tilde{\Pi}(\lambda(t))\tilde{z}(t) \tag{5.16}$$

is EAS with degree of stability γ, $0 < \gamma < \sigma$, if there exist $M > 0, T > 0, \delta \in (0, \sigma - \gamma)$, such that $\forall t$

1)

$$\lambda_i(t) \in (0, M] \qquad \forall i \in \{1, ..., m\}. \tag{5.17}$$

and

2)

$$\sup_{t\geq 0} \frac{1}{T}\int_t^{t+T} \sum_{i=1}^{m} \left[\frac{d}{d\tau} ln\, \lambda_i(\tau)\right]^+ d\tau \; < 2(\sigma - \gamma - \delta) \tag{5.18}$$

Remark 5.1: Observe no restrictions have been placed on the nature of the dependence that $\tilde{\Pi}(\lambda)$ exhibits with respect to λ. *It need not be affine.*

Proof of Proposition 5.1: Define, $\forall i \in \{1, ..., m\}$

$$\tilde{P}_i(\lambda) = \tilde{P}(\lambda)|_{\lambda_i=0} \tag{5.19}$$

Because of (5.15),

$$\tilde{P}(0) > 0 \tag{5.20}$$

and for all $i \in \{1, ..., m\}$ and $\lambda \in [0, \infty)^m$,

$$\tilde{P}_i(\lambda) > 0. \tag{5.21}$$

The multiaffine nature of $\tilde{P}(\lambda)$ ensures that for all $i \in \{1, ..., m\}$

$$\tilde{P}(\lambda) = \tilde{P}_i(\lambda) + \lambda_i \frac{\partial \tilde{P}(\lambda)}{\partial \lambda_i}. \tag{5.22}$$

Thus for all $\lambda \in [0, \infty)^m$ and $i \in \{1, ..., m\}$

$$\frac{\partial \tilde{P}(\lambda)}{\partial \lambda_i} \geq 0. \tag{5.23}$$

Consequently, whenever $\dot{\lambda}_i(t) \geq 0$

$$\begin{aligned}
\dot{\lambda}_i(t)\frac{\partial \tilde{P}(\lambda(t))}{\partial \lambda_i(t)} &= \frac{\dot{\lambda}_i(t)}{\lambda_i(t)}\frac{\partial \tilde{P}(\lambda(t)}{\partial \lambda_i(t)}\lambda_i(t) \\
&\leq \frac{\dot{\lambda}_i(t)}{\lambda_i(t)}\left[\frac{\partial \tilde{P}(\lambda(t))}{\partial \lambda_i(t)}\lambda_i(t) + \tilde{P}_i(\lambda(t))\right] \\
&= \frac{\dot{\lambda}_i(t)}{\lambda_i(t)}\tilde{P}(\lambda(t)).
\end{aligned}$$

In other words, with $\lambda_i(t)$ obeying (5.17)

$$[\dot{\lambda}_i(t)]^+ \frac{\partial \tilde{P}(\lambda_i(t))}{\partial \lambda_i(t)} \leq \left[\frac{d}{dt} ln \lambda_i(t)\right]^+ \tilde{P}(\lambda(t)) \tag{5.24}$$

Consider now the Lyapunov function (see (5.15))

$$V(\tilde{z}(t), t) \,=\, \tilde{z}'(t)\tilde{P}(\lambda(t))\tilde{z}(t) \tag{5.25}$$

Then along (5.16), the developmeny outlined in (5.26) at the bottom of this page holds. Thus, with T as in (5.18), (5.27) at the bottom of the page holds.t Likewise observe that for each $T_1 \in (0, T)$,

$$V(\tilde{z}(t+T_1), t+T_1) \leq exp[2(\sigma-\gamma-\delta)T]V(\tilde{z}(t), t). \tag{5.28}$$

Thus, as $\lambda_i(t) \in (0, M]$, $\tilde{P}(\lambda(t))$ is bounded, a standard set of arguments provide the result.

We need one last proposition.

Proposition 5.2: Suppose the matrices $\Pi(k)$ and symmetric $P(k)$ are such that for all $k \in K$ (see (1.11)): (1) (1.12) holds; and (2) $P(k) > 0$. Define for each $i \in \{1, ..., m\}$, $\lambda = [\lambda_1, ..., \lambda_m]'$

$$\lambda_i \,=\, \frac{k_i - k_i^-}{k_i^+ - k_i} \tag{5.29}$$

$$\tilde{P}(\lambda) \,=\, P(k)|_{k_i = \frac{k_i^+\lambda_i + k_i^-}{\lambda_i + 1}} \tag{5.30}$$

$$\begin{aligned}
\dot{V}(\tilde{z}(t), t) &= \tilde{z}'(t)[\tilde{\Pi}(\lambda(t))\tilde{P}(\lambda(t)) + \tilde{P}(\lambda(t))\tilde{\Pi}(\lambda(t))]\tilde{z}(t) + \tilde{z}'(t)\dot{\tilde{P}}(\lambda(t))\tilde{z}(t) \\
&< -2\sigma V(\tilde{z}(t), t) + \tilde{z}'(t)\left[\sum_{i=1}^{m}\dot{\lambda}_i(t)\frac{\partial \tilde{P}(\lambda(t))}{\partial \lambda_i(t)}\right]\tilde{z}(t) \\
&\leq -2\sigma V(\tilde{z}(t), t) + \tilde{z}'(t)\left[\sum_{i=1}^{m}[\dot{\lambda}_i(t)]^+\frac{\partial \tilde{P}(\lambda(t))}{\partial \lambda_i(t)}\right]\tilde{z}(t) \\
&\leq -2\sigma V(\tilde{z}(t), t) + \tilde{z}'(t)\left[\sum_{i=1}^{m}\left[\frac{d}{dt} ln \lambda_i(t)\right]^+ \tilde{P}(\lambda(t))\right]\tilde{z}(t) \\
&= (-2\sigma + \sum_{i=1}^{m}\left[\frac{d}{dt} ln \lambda_i(t)\right]^+)V(\tilde{z}(t), t).
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
V(\tilde{z}(t+T), t+T) &\leq exp\left[-2\sigma T + \int_t^{t+T}\sum_{i=1}^{m}[\frac{d}{dt} ln \lambda_i(\tau)]^+ d\tau\right]V(\tilde{z}(t), t) \\
&\leq e^{-2\gamma T}e^{-2\delta T}V(\tilde{z}(t), t).
\end{aligned} \tag{5.27}$$

$$\hat{P}(\lambda) = [\prod_{i=1}^{m}(\lambda_i + 1)]\tilde{P}(\lambda) \tag{5.31}$$

$$\tilde{\Pi}(\lambda) = \Pi(k)|_{k_i = \frac{k_i^+ \lambda_i + k_i^-}{\lambda_i + 1}} \tag{5.32}$$

Then:

1) (5.13) is EAS with degree of stability γ, iff (5.16) has this property.
2) Whenever $k(t)$ obeys (5.10, 5.11), $\lambda(t)$ obeys (5.17, 5.18) for some finite M.
3) (5.14) and (5.15) hold.

Proof: Parts (1) and (2) follow by noting that (5.29) is equivalent to

$$k_i = \frac{k_i^+ \lambda_i + k_i^-}{\lambda_i + 1} \tag{5.33}$$

and (5.32). To prove (3) observe, that because of (1.12), positive definiteness of $Q(k)$ and (5.33)

$$\tilde{\Pi}(\lambda)\hat{P}(\lambda) + \hat{P}(\lambda)\tilde{\Pi}(\lambda) < -2\sigma\hat{P}(\lambda) \tag{5.34}$$

Thus (5.31) and the positivity of λ_i, yield the result. □

We can now prove Theorem 5.2.

Proof of Theorem 5.2: Suppose (5.10) and (5.11) hold. Then, from (2) of Proposition 5.2, under (5.29), (5.17), and (5.18) hold. Likewise, from (3) of Proposition 5.2, (5.14) and (5.15) hold. Thus, Proposition 5.1 shows that (5.16) is EAS. Hence, (1) of Proposition 5.2 shows that (5.13) is EAS. Then Lemma 5.2 completes the proof.

VI. Stability of Discrete Time Linear Time Varying Systems

This section extends the results of Section 5 to the discrete time case. We note that similar results have been hitherto unknown even for the single parameter case. We begin with the analogy of Definition 5.1.

Definition 6.1: The LTV system

$$x(t+1) = A(t)x(t) \tag{6.1}$$

is EAS with degree of stability $(1-\rho)$, (i.e. it is ρ-EAS) , $0 < \rho < 1$, if $\exists c > 0$, $0 < \delta < 1$ such that $\forall t_0$ and $t \geq t_0$, t and t_0 integers,

$$\frac{\|x(t)\|}{\rho^{t-t_0}} \leq c\|x(t_0)\|\delta^{t-t_0} \tag{6.2}$$

The main result we will derive is as follows.

Theorem 6.1: Suppose $A(k)$ as in Theorem 5.2 is such that $\forall k \in K$, $A(k)$ is ρ- Schur. Then

$$x(t+1) = A(k(t))x(t) \tag{6.3}$$

is γ-EAS, $0 < \rho < \gamma < 1$, if there exist integer $T > 0$, a δ obeying $0 < \rho < \delta < 1$ and ϵ_{1i} and ϵ_{2i} as in Theorem 5.2, such that for all integer $t > 0$, (5.10) holds together with either (6.4) or (6.5) below:

$$\sup_{t\geq 0} \frac{1}{T}\sum_{j=t}^{t+T-1}\sum_{i=1}^{m}\left[\ln\frac{\lambda_i(j+1)}{\lambda_i(j)}\right]^+ \leq 2\ln(\frac{\gamma\delta}{\rho}). \tag{6.4}$$

$$\sup_{t\geq 0} \frac{1}{T}\sum_{j=t}^{t+T-1}\sum_{i=1}^{m}\left|\ln\frac{\lambda_i(j+1)}{\lambda_i(j)}\right| \leq 4\ln(\frac{\gamma\delta}{\rho}). \tag{6.5}$$

where,

$$\lambda_i(t) = \frac{k_i(t) - k_i^-}{k_i^+ - k_i(t)} \tag{6.6}$$

By using techniques very similar to that in Lemma (5.1) one readily obtains the following anologous lemma.

Lemma 6.1: Suppose the λ_i are as in (6.6), and that (5.10) holds. Then (6.5) implies (6.4).

As before, the hypotheses of Theorem 6.1 ensure the existence of $\Pi(k)$ as in (1.12) for which certain multiaffine, symmetric positive definite $P(k), Q(k)$ form the discrete time Lyapunov pairs. Further the γ-EAS of

$$z(t+1) = \Pi(k(t))z(t) \tag{6.7}$$

suffices for the γ-EAS of (6.3). Thus, it is the γ-EAS nature of (6.7), that we will seek to establish. To this end, the following analogy of proposition 5.1 will provide the key.

Proposition 6.1: Suppose $\tilde{\Pi}(\lambda), \lambda = [\lambda_1, ..., \lambda_m]'$ is such that there exists $\tilde{P}(\lambda)$ multiaffine in λ for which for all $\lambda \in [0,\infty)^m$ and some $0 < \rho < 1$,

$$\tilde{\Pi}'(\lambda)\tilde{P}(\lambda)\tilde{\Pi}(\lambda) - \rho^2\tilde{P}(\lambda) < 0. \tag{6.8}$$

$$\tilde{P}(\lambda) > 0. \tag{6.9}$$

Suppose also that for some M and all integer $t \geq 0$

$$\lambda_i(t) \in (0, M) \qquad \forall i \in \{1, ..., m\} \tag{6.10}$$

and moreover that (6.4) holds. Then

$$\tilde{z}(t+1) = \tilde{\Pi}(\lambda(t))\tilde{z}(t) \tag{6.11}$$

is γ-EAS.

$$\begin{aligned} V(\tilde{z}(t+1), t+1) &= \tilde{z}'(t+1)\tilde{P}(\lambda(t+1))\tilde{z}(t+1) \\ &= \tilde{z}'(t)\tilde{\Pi}'(\lambda(t))\tilde{P}(\lambda(t+1))\tilde{\Pi}(\lambda(t))\tilde{z}(t) \\ &\leq \tilde{z}'(t)\tilde{\Pi}'(\lambda(t))\tilde{P}(\lambda(t))\tilde{\Pi}(\lambda(t))\tilde{z}(t)\nu(t) \end{aligned} \tag{6.14}$$

Proof: Lemma A.2 in the appendix proves that under the hypotheses of the Theorem,

$$\tilde{P}(\lambda(t+1)) \leq \prod_{i=1}^{m}\{1+\left[\frac{\lambda_i(t+1)-\lambda_i(t)}{\lambda_i(t)}\right]^{+}\}\tilde{P}(\lambda(t)). \tag{6.12}$$

Consider the Lyapunov function

$$V(\tilde{z}(t),t) = \tilde{z}'(t)\tilde{P}(\lambda(t))\tilde{z}(t). \tag{6.13}$$

Equation (6.14), at the bottom of the preceding page, holds where

$$\nu(t)=\prod_{i=1}^{m}\{1+\left[\frac{\lambda_i(t+1)-\lambda_i(t)}{\lambda_i(t)}\right]^{+}\}. \tag{6.15}$$

Thus from (6.8) and (6.15)

$$V(\tilde{z}(t+1),t+1) \leq \rho^2\nu(t)V(\tilde{z}(t),t), \tag{6.16}$$

whence

$$V(\tilde{z}(t+T),t+T) \leq \rho^{2T}\left(\prod_{j=1}^{t+T-1}\nu(j)\right)V(\tilde{z}(t),t). \tag{6.17}$$

Therefore, from (6.4), (6.15), and (6.17)

$$\begin{aligned} ln\frac{V(\tilde{z}(t+T),t+T)}{V(\tilde{z}(t),t)} &\leq 2Tln\;\rho+\sum_{j=t}^{t+T-1} ln\;\nu(j) \\ &< 2Tln\;\rho+2Tln\left(\frac{\gamma\delta}{\rho}\right) \\ &= 2Tln\;(\gamma\delta). \end{aligned} \tag{6.18}$$

In the above, we have utilized the following equality:

$$\left[\ln\frac{\lambda_i(j+1)}{\lambda_i(j)}\right]^{+} = \ln\{1+\left[\frac{\lambda_i(j+1)-\lambda_i(j)}{\lambda_i(j)}\right]^{+}\}. \tag{6.19}$$

The result follows from considerations similar to those employed in the proof of Proposition 5.1. □

Then a proposition anologous to Proposition 5.2 proves the result.

VII. Conclusion

We have derived parameterized Lyapunov functions for both uncertain passive and stable systems. We also have demonstrated their utility to the stability analysis of specific classes of LTV systems. Several future lines of research emerge from this paper. First, can the parameterization of Theorem 4.1 be further simplified to render $P(k)$ and $Q(k)$ affine in k? Second, can one force $P(k), Q(k)$ to have the same dimension as $A(k)$? Third, just as the results presented here have been beneficial in deriving stability theorems for a class of LTV systems, we expect they would provide similar benefit to the analysis of a wide class of nonlinear time varying systems.

Appendix

A. Proof of Lemma 5.1

Define

$$l_i(t) = ln\left[\frac{k_i(t)-k_i^-}{k_i^+-k_i(t)}\right] \tag{8.1}$$

Then the following Lemma trivially proves the result. □

Lemma A.1: Suppose there exist $M_1, M_2 > 0$ such that the real functions $l_i(t)$, $i = 1,...,m$, obey

1)
$$-M_1 \leq l_i(t) \leq M_2 \quad \forall t \geq 0 \; and \tag{8.2}$$

2) There exists T, η and $\gamma > 0$ such that
$$\sup_{t\geq 0}\frac{1}{T}\int_t^{t+T}\sum_{i=1}^{m}\left|\frac{d}{d\tau}l_i(\tau)\right|d\tau < 4\eta. \tag{8.3}$$

Then $\forall\mu > 0$, there exists $T_1(\mu)$ such that

$$\sup_{t\geq 0}\frac{1}{T_1}\int_t^{t+T_1}\sum_{i=1}^{m}\left[\frac{d}{d\tau}l_i(\tau)\right]^{+}d\tau \leq 2(\eta+\mu) \tag{8.4}$$

Proof: Suppose (8.4) fails for some $\mu = \mu_2 > 0$. Then for this μ_2 and every T_1 there exists $t_1(\mu_2, T_1)$ for which, dropping the arguments of t_1

$$\frac{1}{T_1}\int_{t_1}^{t_1+T_1}\sum_{i=1}^{m}\left[\frac{d}{d\tau}l_i(\tau)\right]^{+}d\tau > 2(\eta+\mu_2) \tag{8.5}$$

Choose $T_1 = NT$ where N is to be specified later. Then

$$\int_{t_1}^{t_1+T_1}\sum_{i=1}^{m}\left[\frac{d}{d\tau}l_i(\tau)\right]^{+}d\tau > 2(\eta+\mu_2)\,NT \tag{8.6}$$

Defining $[f(t)]^- = min[0, f(t)]$, note that for any $f(t)$

$$|f(t)| = [f(t))]^+ - [f(t)]^- \tag{8.7}$$

Then (8.3), (8.6) and (8.7) imply

$$\int_{t_1}^{t_1+NT}\sum_{i=1}^{m}\left\{[\frac{d}{d\tau}l_i(\tau)]^{-}\right\}d\tau > -2(\eta-\mu_2)\,NT \tag{8.8}$$

$$\begin{aligned} \sum_{i=1}^{m}l_i(t_1+T_1) &= \sum_{i=1}^{m}l_i(t_1)+\int_{t_1}^{t_1+T_1}\sum_{i=1}^{m}[\frac{d}{d\tau}l_i(\tau)]d\tau \\ &\geq -mM_1+\int_{t}^{t_1+T_1}\sum_{i=1}^{m}[\frac{d}{d\tau}l_i(\tau)]^{+}d\tau+\int_{t}^{t_1+T_1}\sum_{i=1}^{m}[\frac{d}{d\tau}l_i(\tau)]^{-}d\tau \\ &\geq -mM_1+2(\eta+\mu_2)NT-2(\eta-\mu_2)NT \\ &= -mM_1+4\mu_2NT \end{aligned} \tag{8.10}$$

From (8.2),

$$-mM_1 \le \sum_{i=1}^{m} l_i(t) \le mM_2 \quad \forall t \ge 0 \tag{8.9}$$

Thus from (8.6), (8.7), (8.8), and (8.9) it follows that (8.10) holds. (See bottom of preceding page.) Choosing

$$N > \frac{m(M_2 + M_1)}{4\mu_2 T} \tag{8.11}$$

one finds that the upper bound of (8.9) is violated at $t = t_1 + T_1$. The contradiction proves the result. □

B. A Lemma Required to Prove Theorem 6.1

Lemma A.2: Suppose the symmetric matrix function $\tilde{P}(\lambda)$ is multiaffine in the elements of $\lambda = [\lambda_1, ..., \lambda_m]'$, and is moreover, positive semidefinite for all $\lambda \in [0, \infty)^m$. Then, if for all t, $\lambda(t) \in [0, \infty)^m$, one has

$$\tilde{P}(\lambda(t+1)) \le \prod_{i=1}^{m}\{1 + \left[\frac{\lambda_i(t+1) - \lambda_i(t)}{\lambda_i(t)}\right]^+\}\tilde{P}(\lambda(t)) \tag{8.12}$$

Proof: Since $\tilde{P}(\lambda)$ is multiaffine in the elements of λ, one can always express it as follows: with $S = \{1, ..., m\}$

$$\tilde{P}(\lambda) = \sum_{r \subset S}(\prod_{i \in r}\lambda_i)\tilde{P}_r \ge 0 \tag{8.13}$$

Since $\tilde{P}(\lambda) \ge 0, \forall \lambda \in [0, \infty)^m$, one can readily establish the fact that

$$\tilde{P}_r \ge 0 \quad \forall r \subset S \tag{8.14}$$

Use induction on m. When $m = 1$

$$\begin{aligned}\tilde{P}(\lambda(t+1)) - \tilde{P}(\lambda(t)) &= [\lambda_1(t+1) - \lambda_1(t)]\tilde{P}_{\{1\}} \\ &\le \left[\frac{\lambda_1(t+1) - \lambda_1(t)}{\lambda_1(t)}\right]^+ \lambda_1(t)\tilde{P}_{\{1\}} \\ &\le \left[\frac{\lambda_1(t+1) - \lambda_1(t)}{\lambda_1(t)}\right]^+ [\tilde{P}_\phi + \lambda_1(t)\tilde{P}_{\{1\}}]\end{aligned}$$

whence

$$\tilde{P}(\lambda(t+1)) \le \left\{1 + \left[\frac{\lambda_1(t+1) - \lambda_1(t)}{\lambda_1(t)}\right]^+\right\}\tilde{P}(\lambda(t)) \tag{8.15}$$

Suppose the result holds for $m = l$. Define $S_l = \{1, ..., l\}$. Then, for $m = l + 1$, (8.16) at the bottom of the page holds.

$$\begin{aligned}\tilde{P}(\lambda(t+1)) &= \tilde{P}_\phi + \sum_{r \subset S_l}(\prod_{i \in r}\lambda_i(t+1))\tilde{P}_r \\ &\quad + \lambda_{l+1}(t+1)\sum_{r \subset S_l}(\prod_{i \in r}\lambda_i(t+1))\ \tilde{P}_{r \cup \{l+1\}} \\ &\le \prod_{i=1}^{l}\{1 + \left[\frac{\lambda_i(t+1) - \lambda_i(t)}{\lambda_i(t)}\right]^+\}[\tilde{P}_\phi + \sum_{r \subset S_l}(\prod_{i \in r}\lambda_i(t))\tilde{P}_r] \\ &\quad + \lambda_{l+1}(t+1)\prod_{i=1}^{l}\{1 + \left[\frac{\lambda_i(t+1) - \lambda_i(t)}{\lambda_i(t)}\right]^+\}\left[\sum_{r \subset S_l}(\prod_{i \in r}\lambda_i(t))\tilde{P}_{r \cup \{l+1\}}\right] \\ &= \prod_{i=1}^{l}\{1 + \left[\frac{\lambda_i(t+1) - \lambda_i(t)}{\lambda_i(t)}\right]^+\}\left[\tilde{P}(\lambda(t)) + [\lambda_{l+1}(t+1) - \lambda_{l+1}(t)]\sum_{r \subset S_l}(\prod_{i \in r}\lambda_i(t))\tilde{P}_{r \cup \{l+1\}}\right]s \\ &\le \prod_{i=1}^{l}\{1 + \left[\frac{\lambda_i(t+1) - \lambda_i(t)}{\lambda_i(t)}\right]^+\}\left[\tilde{P}(\lambda(t)) + \left[\frac{\lambda_{l+1}(t+1) - \lambda_{l+1}(t)}{\lambda_{l+1}(t)}\right]^+ \lambda_{l+1}(t)\tilde{P}(\lambda(t))\right]\end{aligned} \tag{8.16}$$

□

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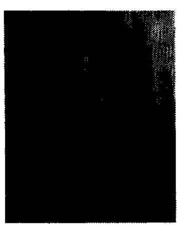

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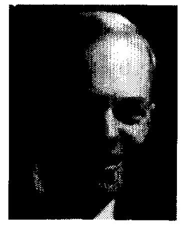

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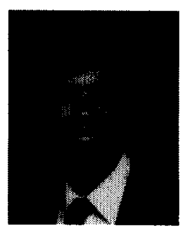

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