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Parametric Approach to Robust Stability with both Parametric and Nonparametric Uncertainties *

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Abstract

This paper shows that the robust stability problem for a linear system with both parametric and nonparametric uncertainties is equivalent to a robust stability problem with parametric uncertainty only. This is achieved by converting the nonparametric uncertainty into a fictitious linear parameter. When applied to systems with affine parametric uncertainty and nonparametric uncertainty, the resulting robust stability problem involves bilinear parametric uncertainty which can be simply tested. This result is also useful in computing H_∞ norm and strict positive realness for systems with parametric uncertainty.

1 Introduction

Uncertainties arising from system modeling usually come in different forms. Parametric uncertainty is often used to describe perturbations caused by unknown or uncertain physical parameters which are time-invariant or slowly drifting (and hence can be approximated by time-invariant ones). This type of uncertainty also corresponds to low-frequency variations in the system transfer function. Unparametric uncertainty, on the other hand, is popular for capturing unmodeled dynamics, fast time-varying and/or high-frequency perturbations.

This paper solves the robust stability problem of linear systems with both parametric and nonparametric uncertainties. We show that the commonly used unmodeled dynamics, either additive or multiplicative uncertainties, can be reparameterized by a single fictitious parameter which is linear, bounded and real. This reparameterization has some advantages over the standard "polar" parameterization [1, 2]. In particular, we show that the robust stability problem of a linear system with both *affine* parametric uncertainty and nonparametric uncertainty in either additive form or multiplicative form is equivalent to the robust stability of a family of polynomials with *bilinear* parametric uncertainty, i.e., linear in both the original parameters and the fictitious one. The resulting robust stability can be tested in various ways. In particular, one can use the Finite Zero Exclusion Principle in [3] which has the advantage of avoiding "frequency sweeping."

The development of our results is through an equivalence among the robust stability with nonparametric uncertainty, H_∞ performance, and strict positive realness (SPRness). For this reason, our results are applicable to the problems of testing the H_∞ performance and SPR property of transfer functions with parametric uncertainty. In other words, these problems are shown to be equivalent to the robust stability problem of a family of polynomials with parametric uncertainty with a fictitious (linear) parameter.

2 Problem Formulation and Notation

Consider a single-input-single-output unity feedback system with the open loop transfer function in s given by $G(s,q)+\Delta(s)$, where $G(s,q)\in\mathcal{G}$ is the parametric part of the transfer function which depends an parameter vector q which belongs to a bounding set $Q\in\mathbf{R}^m$, $\Delta(s)\in\mathcal{D}$ is the nonparametric uncertainty in the additive form. We restrict ourselves to additive perturbations although multiplicative ones can be treated similarly. The following assumptions are required throughout the paper:

A1 $G(s,q)$ is nth-order, strictly proper and real-valued for all $q\in Q$, i.e.,

$$G(s,q)=\frac{n(s,q)}{d(s,q)}=\frac{\sum_{i=0}^{n-1}b_i(q)s^i}{\sum_{i=0}^{n}a_i(q)s^i} \qquad (1)$$

where $a_i(q)$ and $b_i(q)$ are real-valued functions with $a_n(q)\neq 0$ for all $q\in Q$.

A2 $a_i(q)$ and $b_j(q)$ are continuous in q over Q, $0\le i\le n, 0\le j\le n-1$.

A3 $G(s,q)$ has no poles on the $j\omega$ axis for all $q\in Q$.

A4 $G(s,q)$ and $G(s,q)+\Delta(s)$ have the same number of poles in the closed right half plane, for all $\Delta(s)\in\mathcal{D}$.

A5 There exists some strictly proper, stable and minimum-phase, real-valued transfer function $\gamma(s)=\frac{\alpha(s)}{\beta(s)}$ such that

$$\mathcal{D}=\{\Delta(s)\text{ satisfying A4}: |\Delta(j\omega)|\le\gamma(j\omega)\}\ . \quad (2)$$

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The robust stability problem to be solved in this paper is to determine whether the closed-loop stability holds for all for all $q \in Q$ and $\Delta(s) \in \mathcal{D}$.

Notation: The convex set of two elements x and y is denoted by $\text{conv}[x, y]$, i.e., $\text{conv}[x, y] = \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$. Similarly, the convex family of two polynomials $a(s)$ and $b(s)$ will be denoted by $\text{conv}[a(s), b(s)]$.

3 Main Results

Several related results are to be developed. Theorem 1 shows that a rational function is SPR if and only if the convex family of two associated polynomials is robustly stable. This relationship simplifies a result by Rantzer [3] which needs two parameters to do the conversion. Theorem 2 generalizes Theorem 1 to testing the H_∞ norm of a rational function by using an equivalence between H_∞ performance and SPRness. Theorem 3, an application of Theorem 2, gives a parametric approach to the robust stability problem of linear systems with both parametric and nonparametric uncertainties.

Theorem 1. *Given two n-th order real-valued polynomials $a(s)$ and $b(s)$ with the same sign for their leading coefficients, $a(s)/b(s)$ is SPR if and only if* $\text{conv}[a(s), jb(s)]$ *is robustly stable (strictly Hurwitz).*

Proof. From the definition of SPRness, we know that $a(s)/b(s)$ is SPR if and only if (i) both $a(s)$ and $b(s)$ are stable, and (ii) $\text{Re}[a(j\omega)/b(j\omega)] > 0, \ \forall -\infty < \omega < \infty$. Because the leading coefficients of $a(s)$ and $b(s)$ have the same sign, i.e., $a(j\infty)/b(j\infty) > 0$, the condition (ii) above can be replaced by

$$\text{Re}[a(j\omega)/b(j\omega)] \neq 0, \ \forall -\infty < \omega < \infty ,$$

which is actually equivalent to

$$\frac{a(j\omega)}{b(j\omega)} + j\frac{t}{1-t} \neq 0, \ \forall t \in [0, 1) . \qquad (3)$$

Using the nonzeroness of $a(j\omega)$ and $b(j\omega)$ (from the stability of $a(s)$ and $b(s)$), the inequality (3) holds if and only if

$$p(j\omega, t) = (1-t)a(j\omega) + jtb(j\omega) \neq 0 \ , \ \forall t \in [0, 1] \ . \quad (4)$$

Note that the leading coefficient of $p(s, t)$ is nonvanishing for all $t \in [0, 1]$. Finally, due to the zero exclusion principle (see [4], for example), the condition (i) and that in (4) are further equivalent to the robust stability of $\text{conv}[a(s), b(s)]$. ▽▽▽

The following lemma which establishes the relationship between SPRness and H_∞ performance is well-known; see [5], for example.

Lemma 2. *Given a strictly proper transfer function $H(s) = n(s)/d(s)$ where $n(s)$ and $d(s)$ are real-valued polynomials, $||H(s)||_\infty < 1$ if and only if the transfer function $(d(s) - n(s))/(d(s) + n(s))$ is SPR.*

Following the lemma above and Theorem 1, we have the next result which relates the H_∞ performance of a transfer function to robust stability of a family of polynomials. The proof is omitted due to its obviousness.

Theorem 3. *Given a strictly proper transfer function $H(s) = n(s)/d(s)$ where $n(s)$ and $d(s)$ are real-valued polynomials, $||H(s)||_\infty < 1$ if and only if* $\text{conv}[d(s) - n(s), j(d(s) + n(s))]$ *is robustly stable.*

Now we return to the robust stability problem formulated in Section 2. The following result is a natural combination of the small gain theorem (see, [5], for example) and Theorem 2 above.

Theorem 4. *Consider the unity feedback uncertain system described in Section 2 which satisfies A1-A5. Then, the closed-loop system is robustly stable if and only if the following family of polynomials is robustly stable:*

$$\mathcal{H}_a = \{h_a(s, q, \lambda) : q \in Q, \lambda \in [0, 1]\} \qquad (5)$$

where

$$\begin{aligned} h_a(s, q, \lambda) &= (1-\lambda)[(\beta - \alpha)d(s, q) + \beta n(s, q)] \\ &+ j\lambda[(\beta + \alpha)d(s, q) + \beta n(s, q)] \ . \end{aligned} \qquad (6)$$

Remark 1. Note that the families of polynomials $\mathcal{H}_a$ and $\mathcal{H}_m$ will involve multilinear parametric uncertainty if the open-loop transfer function $G(s, q)$ is multilinear in q, or bilinear uncertainty if $G(s, q)$ is affine in q. Effecient numerical algorithms are available for testing the robust stability with these classes of uncertainty. In particular, we draw attention to those based on the Finite Zero Principle in [3] which require testing certain zero exclusion properties at a finite number of frequencies. Since the fictitious parameter λ introduced above is a linear parameter, robust stability test can be carried out efficiently.

REFERENCES

[1] B. R. Barmish and P. P. Khargonekar, "Robust stability of feedback control systems with uncertain parameters and unmodelled dynamics," *Math. Contr. Signals Syst.*, vol. 3, pp. 197–210, 1990.

[2] H. Chapellat, M. Dahleh, and S. P. Bhattacharyya, "Robust stability under structured and unstructed perturbations," *IEEE Tran. Auto. Contr.*, vol. 35, no. 10, pp. 1100–1108, 1990.

[3] A. Rantzer, "A finite zero exclusion principle," in *Proc. MTNS*, (Amsterdam, Netherlands), pp. 239–245, 1989.

[4] B. R. Barmish, "A generalization of Kharitonov's four polynomial concept for robust stability problems with linearly dependent coefficient perturbations," *IEEE Trans. Auto. Contr.*, vol. 34, pp. 157–165, Feb. 1989.

[5] C. S. Desoer and M. Vidyasagar, *Feedback Systems-Input Output Properties.* New York, NY: Academic Press, 1975.