$$\sup_{0\le k<\infty}\|v(k)-\hat{\bar{v}}_k(0)\|$$
$$\le \|\bar{C}_2\|\cdot \underbrace{\sup_{0\le k<\infty}\sum_{j=0}^{k-1}\|[(TA_2+I)^{k-j-1}-(e^{\bar{A}_2T})^{k-j-1}]\|}_{\to 0\ (T\to 0)} T\|B_2\|\cdot r_1$$
$$+\|\bar{C}_2\|\cdot \sup_{0\le k<\infty}\sum_{j=0}^{k-1}\|(e^{\bar{A}_2T})^{k-j-1}\|T\cdot \underbrace{\left\|-\frac{1}{T}\int_{\xi=0}^{T}e^{\bar{A}_2(T-\xi)}\bar{B}_2\,d\xi+B_2\right\|}_{\to 0\ (T\to 0)}\cdot r_1$$
$$+\|\bar{C}_2\|\cdot \sup_{0\le k<\infty}\sum_{j=0}^{k-1}\|(e^{\bar{A}_2T})^{k-j-1}\|T\cdot\frac{1}{T}\int_{\xi=0}^{T}\left\|e^{\bar{A}_2(T-\xi)}\bar{B}_2\right\|d\xi\cdot r_2$$
$$+\sum_{j=0}^{\infty}\|(TA_2+I)^j\|T\cdot\|B_2\|\cdot\left(\|\bar{C}_2+\Delta C_2\|\cdot\underbrace{\Delta\rho}_{\to 0\ (T\to 0)}+\underbrace{\|\Delta C_2\|}_{\to 0\ (T\to 0)}\cdot r_1\right) \quad (20)$$

inputs to the systems. Then, (17) is satisfied. Limits in (18) come from part 1) of the Proof for Theorem 1. □

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Pole Placement via Static Output Feedback is NP-Hard

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***Abstract*—This note proves that the problem of pole placement via static output feedback for linear time-invariant systems is NP-hard.**

***Index Terms*—Computational complexity, pole placement, static output feedback.**

I. Introduction

This note is motivated by the following long-standing problem of static output feedback stabilization (SOFS): *Given a linear time-invariant system, determine if it is stabilizable via static output feedback*. This is arguably one of the most fundamental yet unsolved control problems; see [1]. There have been a number of attempts recently to analyze the computational complexity of this problem. In [2], it is shown that the problem of finding a static output feedback stabilizer from a given bounded set (a hypercube) is NP-complete. In [3], it is shown that a matrix inequality problem closely related to the SOFS problem is NP-hard. This matrix inequality problem, involving two linear matrix inequalities and a nonconvex coupling condition, is related to the SOFS problem in the sense that the latter can be transformed into the former.

In this note, we consider the problem of static output feedback pole placement (SOFPP): *Given a linear time-invariant system and a set of desired poles, determine if there exists a static output feedback controller such that the closed-loop system contains poles at these desired locations*. For some special cases where the numbers of inputs and outputs are very small, constructive methods are available for SOFPP; see [4]. It is also known that generic pole placement using static output feedback is not feasible; see, e.g., [5]–[7]. The difficulty, however, is that it is not clear how difficult it is to determine the solvability of the

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SOFPP problem for a given system and a given set of desired poles. Our result shows that the SOFPP problem is, unfortunately, NP-hard.

II. Main Result

The SOFPP problem can be formally stated as follows: Given an nth-order linear time-invariant system

$$\delta x = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{1}$$

where $\delta x = \dot{x}$ for the continuous-time case, or $\delta x = x(t+1)$ for the discrete-time case, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{r\times n}$, and a set of desired eigenvalues λ_i, $i = 1, 2, \ldots, q \leq n$, determine if there exists a static output feedback controller

$$u(t) = -Ky(t), \qquad K \in \mathbf{R}^{m\times r} \tag{2}$$

such that λ_i, $i = 1, 2, \ldots q$, are eigenvalues of $A - BKC$.Without loss of generality, (A, B) and (A, C) are assumed to be controllable and observable pairs, respectively.

Remark 1: Note that we have set $q \leq n$. This is because it is generally impossible to assign all n poles arbitrarily. Obviously, this happens when $mr < n$. Even when $mr \geq n$, arbitrary pole placement may not be possible. This is because the closed-loop characteristic polynomial, $p(s)$, is multiaffine in K, implying that the domain of the mapping from K to $p(s)$ may not cover all the nth order polynomials. □

Denote by $G(s) = C(sI - A)^{-1}B$ the open-loop transfer function, it is clear that the SOFPP problem above is equivalent to finding K such that

$$\det(I + KG(\lambda_i)) = 0, \qquad i = 1, 2, \ldots, q \tag{3}$$

In the sequel, we assume that A, B, and C are rational matrices.

Theorem 1: The SOFPP problem is NP-hard.

The proof of the result above is done by transforming a known NP-complete problem, the so-called *partition* problem, into the SOFPP problem; see, e.g., [8] for definitions and examples of P, NP-complete and NP-hard problems.

The Partition Problem [8]: Given an integer vector $c = (c_1, \ldots, c_p)^T$, determining if there exists a binary vector $x = (x_1, \ldots, x_p)^T \in \{-1, 1\}^p$ such that $c^T x = 0$.

Lemma 1: Denote two 2×2 matrices

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \tag{4}$$

and a scalar function $f(X, H) = \det(I + XH)$. Define

$$H_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$
$$H_3 = \frac{1}{3}\begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix} \quad H_4 = \begin{bmatrix} -1 & -1 \\ -2 & -4 \end{bmatrix}. \tag{5}$$

Then, the following simultaneous equations:

$$f(X, H_j) = 0, \qquad j = 1, 2, 3, 4 \tag{6}$$

have only two possible solutions

$$x \in \{-1, 1\} \quad y = -x \quad z = 1 + x \quad w = 1 + x. \tag{7}$$

Proof: It is straightforward to verify that

$$f(X, H) = 1 + ax + cy + bz + dw + (xw - yz)(ad - bc).$$

Substituting H_1 and H_2 into the previous equation, respectively, we obtain $z = 1 + x$ and $w = 1 - y$. Simplifying $f(X, H)$ gives

$$f(X, H) = (1 + b + d) + (a + b + ad - bc)x$$
$$+ (c - d - ad + bc)y - 2(ad - bc)xy$$

Substituting H_3 and H_4 into the previous equation, respectively, we obtain $x + y = 0$ and $xy = -1$. Hence, the only solutions are those given in (7). □

Proof of Theorem 1: We first note that, if we allow c to be a vector of rational numbers and replace $c^T x = 0$ by $c^T x + 1 = 0$, this modified partition problem is still NP-complete. This can be seen as follows. Given any instance of the original problem, if we set $x_p = 1$ and normalize c by dividing it by c_p (assumed to be nonzero, without loss of generality), then we obtain an instance of the modified partition problem with $p - 1$ variables $x_1, \ldots, x_{p-1}$. Similarly, if we set $x_p = -1$ and normalize c by dividing it by $-c_p$, we obtain another instance of the modified partition problem. Hence, the modified partition problem is NP-complete.

Next, given any instance of the modified partition problem, we want to construct an instance of the SOFPP problem such that its solution coincides with the solution of the former. To do so, we choose matrix $K \in \mathbf{R}^{2\times 2p}$ as follows:

$$K = [X_1 \; X_2 \; \ldots \; X_p] \tag{8}$$

where each X_i, is a 2×2 matrix. Next, we choose $4p + 1$ eigenvalues $\lambda_{i,j}$, $i = 1, \ldots, p$, $j = 1, \ldots, 4$, and λ_{4p+1} to be any distinct, real, nonzero rational values and require $G(s)$ to satisfy the following constraints:

$$G(\lambda_{i,j}) = \left[0 \; \ldots \; 0 \; H_j^T \; 0 \ldots \; 0\right]^T \in \mathbf{R}^{2p\times 2} \tag{9}$$

for $i = 1, \ldots, p$, $j = 1, \ldots, 4$. That is, $G(\lambda_{i,j})$ starts with $(i-1) 2 \times 2$ zero matrices, followed by H_j and some more 2×2 zero matrices. Also, $G(\lambda_{4p+1})$ is constructed as follows: The $(2i - 1)$th element in the first column equals to c_i, $i = 1, \ldots, p$, and all other elements are zero.

We then obtain $G(s)$ by interpolation as follows: Set $q = 4p + 1$ and order the eigenvalues $\lambda_{i,j}$ as $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}$. Then

$$G(s) = \sum_{i=1}^{q} \frac{\lambda_i^q}{s^q} \prod_{j\neq i} \frac{s - \lambda_j}{\lambda_i - \lambda_j} G(\lambda_i) \tag{10}$$

which is a simple polynomial interpolation. The extra term λ_i^q / s^q above is used to ensure the strict properness of $G(s)$. Obviously, $n \geq q$ because each entry of $G(s)$ has order q. The matrices A, B and C in (1) can be constructed from $G(s)$ using, e.g., the control canonical form. This construction procedure has a polynomial complexity.

By the construction of $G(\lambda_{i,j})$ and Lemma 1, we know that $\det(I + KG(\lambda_{i,j})) = 0$ at all i, j if and only if $x_i \in \{-1, 1\}$, where x_i is the upper left entry of X_i. Further, it is easy to verify that

$$\det(I + KG(\lambda_{4p+1})) = c^T x + 1.$$

Hence, we have done the required transformation. That is, the given instance of the modified partition problem has a solution x if and only if the instance of the corresponding SOFPP problem has a solution K.

It is easy to verify that this transformation is done in a polynomial time. It follows that the SOFPP problem is NP-hard. □

III. Conclusion

We have given a negative result for the static output feedback pole placement problem. This explains why there is lack of effective solutions to such a fundamental problem. However, it does not imply that the SOFS problem is NP-hard. In fact, it seems that neither problem reduces to the other in an obvious way. Nevertheless, it seems to strengthens the conjecture that the SOFS problem is NP-hard as well.

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A Remark on "Disturbance Decoupling for Linear Time-Invariant Systems: A Matrix Pencil Approach"

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***Abstract*—In the above paper, an approach was proposed for the disturbance decoupling for the linear system. Some details in the proof of the sufficiency of Theorem 2 for the existence of the matrix F_1 are omitted. In this note, we will give a detailed proof.**

***Index Terms*—Generalized inverse, orthogonal matrix transformation, stabilization.**

The authors of [1] presented a systematic analysis of disturbance decoupling problem for standard linear systems based on the theory of the matrix pencils. However, some details in the proof of the sufficiency of Theorem 2 for the existence of the matrix $F_1 \in R^{m\times n_1}$ are omitted. In this remark, we will give a detailed proof and present a method for computation of F_1. Based on the disturbance decoupling and closed-loop stability requirement given by [1], it is necessary to construct matrix F_1 such that

$$B_3F_1 + A_{31} = 0 \tag{1}$$

holds and the matrix

$$G = \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix}^T \begin{bmatrix} A_{11}+B_1F_1 \\ A_{21}+B_2F_1 \end{bmatrix} \tag{2}$$

is stable, where $E_{11} \in R^{\tilde{n}_1\times n_1}$, $E_{21} \in R^{\tilde{n}_2\times n_1}$, $[\, E_{11}^T \quad E_{21}^T \,]^T$ is orthogonal, and $A_{11},\ A_{21},\ B_1,\ B_2$ are matrices of appropriate dimensions. From [1], the matrix B_3 is of full-row rank, thus we have

$$F_1 = -B_3^+A_{31} + (I_m - B_3^+B_3)Z \tag{3}$$

where B_3^+ is the Moore–Penrose inverse [2] of $B_3 \in R^{\tilde{n}_3\times m}$, $Z \in R^{m\times n_1}$ is an arbitrary matrix. In addition, the requirement that the matrix G in (2) must be stable implies that matrix F_1 should satisfy that

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11}-B_1F_1 \\ sE_{21}-A_{21}-B_2F_1 \end{bmatrix} = \tilde{n}_1+\tilde{n}_2 \qquad \forall s \in \bar{\mathbf{C}}^+. \tag{4}$$

Substituting (3) for F_1 in (4) gives

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11}+B_1B_3^+A_{31}-B_1(I_m-B_3^+B_3)Z \\ sE_{21}-A_{21}+B_2B_3^+A_{31}-B_2(I_m-B_3^+B_3)Z \end{bmatrix} = \tilde{n}_1+\tilde{n}_2 \qquad \forall s \in \bar{\mathbf{C}}^+. \tag{5}$$

It is clear that if

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11}+B_1B_3^+A_{31} & B_1(I_m-B_3^+B_3) \\ sE_{21}-A_{21}+B_2B_3^+A_{31} & B_2(I_m-B_3^+B_3) \end{bmatrix} = \tilde{n}_1+\tilde{n}_2 \qquad \forall s \in \bar{\mathbf{C}}^+ \tag{6}$$

holds, i.e., the triple of matrices

$$\left(\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix}, \begin{bmatrix} A_{11}-B_1B_3^+A_{31} \\ A_{21}-B_2B_3^+A_{31} \end{bmatrix}, \begin{bmatrix} B_1(I_m-B_3^+B_3) \\ B_2(I_m-B_3^+B_3) \end{bmatrix}\right)$$

is R-stabilizable [3], then there must exist a constant real matrix Z, such that (5) holds. Next, we prove that (6) is true. On one hand, noticing that the matrix B_3 is of full-row rank, we have

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11} & B_1 \\ sE_{21}-A_{21} & B_2 \\ -A_{31} & B_3 \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 \\ B_3^+A_{31} & I_m-B_3^+B_3 \end{bmatrix} \le \tilde{n}_1+\tilde{n}_2 \qquad \forall s \in \bar{\mathbf{C}}^+. \tag{7}$$

On the other hand, since the following equation:

$$\operatorname{rank}(I_m-B_3^+B_3) = m-\tilde{n}_3$$

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