Consider the general case where the zeros are to be restricted in p regions D_i, $i = 1, \cdots, p$ where $D_i \subset \{D_m, D\}$ in a given fashion. Let d_{ij}; $i = 1, \cdots, p; j = 1, \cdots, k_i$ be the shortest distances from t in (1) to each of the surfaces defined by the boundaries of D_i, $i = 1, \cdots, p$.

We now state a general theorem for the polynomial $P(z)$ with perturbation Δt where all the zeros of $P(z)$ lie within or on the regions D_i in a given fashion.

Theorem 2: The largest hypersphere H_D centered at $t \in D_t$ and containing polynomials whose zeros lie within or on the regions D_i, $i = 1, \cdots, p$ in a given fashion has a radius R given by

$$R^2 = \min\{d_{ij}^2; i=1, \cdots, p; j=1, \cdots, k_i\}.$$

Proof: The proof is geometrically obvious because the hypersphere of radius R is contained within those of radii d_{ij}^2; $i = 1, \cdots, p; j = 1, \cdots, k_i$.

III. An Example

To provide an illustration of the approach, consider the polynomial

$$P(z) = z^4 + 48z^3 + 752z^2 + 4480z + 12800 \tag{6}$$

which has two zeros in the region $D_1 \subset D$ where $f(x, y^2) = (x + 20)^2 + y^2 = 25$ and two zeros in the regions $D_2 \subset D_m$ where $f(x, y) = (x + 4) + y^2 = 1$ and $a = 4$. Find the maximum allowable perturbation Δt_m which confines the zeros of (6) to be in D_1 and D_2 as specified above. In accordance with Theorem 2, we find the shortest distance to the hyperplane defined by $P(-15)$ [2], that is

$$d_{11}^2 = 1.02.$$

Similarly, the shortest distance to the hyperplane defined by $P(-25)$ [2] is given by

$$d_{12}^2 = 0.53.$$

The shortest distance to the hypersurface defined by the movement of zeros on the boundary of D_1 is obtained by numerical minimization [2], where $E_{13} = x^2 + 25 - (x + 20)^2$ and $x \in (-25, -15)$, that is

$$d_{13}^2 = 0.72.$$

Finally, the shortest distance to the hypersurface defined by the movement of zeros on the boundaries D_2 is again obtained numerically as given by Theorem 1. In this case we need to consider two minimizations, that is

$$E_{21}^+ = x^2 + (4 + (1 - (x+4)^2)^{1/2})^2$$

and

$$E_{21}^- = x^2 + (4 - (1 - (x+4^2))^{1/2})^2$$

where

$$x \in (-5, -3).$$

The shortest distance is then given by the smaller magnitude of these minimizations, that is

$$d_{21}^2 = 105.95.$$

Using Theorem 2

$$\Delta t_m^T \cdot \Delta t_m = \min\{1.02, 0.53, 0.72, 105.95\} = 0.53.$$

Acknowledgment

The author would like to thank Dr. W. Soh for fruitful discussions.

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Polytopes of Polynomials with Zeros in a Prescribed Set

MINYUE FU and B. ROSS BARMISH

Abstract—**In Bartlett, Hollot, and Lin [2], a fundamental result is established on the zero locations of a family of polynomials. It is shown that the zeros of a polytope P of nth-order real polynomials are contained in a simply connected set D if and only if the zeros of all polynomials along the edges of P are contained in D. This note is motivated by the fact that the requirement of simple connectedness of D may be too restrictive in applications such as dominant pole assignment and filter design where the separation of zeros is required. In this note, we extend the "edge criterion" in [2] to handle any set D whose complement D^c has the following property: every point $d \in D^c$ lies on some continuous path which remains within D^c and is unbounded. This requirement is typically verified by inspection and allows for a large class of disconnected sets. We also allow for polynomials with complex coefficients.**

I. Introduction

In this note we address a special case of the following problem. Given a family of nth-order polynomials P (real or complex) and a set D in the complex plane, determine whether all polynomials $p(s)$ in P have all their zeros interior to D. When this is the case, P is said to be *D-stable*. A first seminal result on this problem is given in a paper by Kharitonov [1] for the special case when P corresponds to a family of real interval polynomials and D is the left-half plane. More precisely, bounding intervals $[\alpha_i, \beta_i]$ are specified *a priori* and polynomials $p(s) \in P$ are of the form

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

with $a_i \in [\alpha_i, \beta_i]$ for $i = 1, 2, \cdots, n$. Subsequently, Kharitonov's theorem indicates that D-stability of only four extreme polynomials (generated using the α_i and β_i) are sufficient to guarantee the D-stability of P.

From a system theoretic point of view, there are two fundamental limitations of Kharitonov's theorem. The first fundamental limitation stems from the assumption that D is the left-half plane. Hence, the result does not apply to discrete-time systems where D is the open unit disk or to problems where specifications on pole locations must be satisfied. For example, for a so-called dominant pole location problem, it is desirable to have two closed-loop poles within some prescribed ϵ-neighborhoods of a given target $\alpha \pm j\beta$ ($\alpha < 0$) with the remaining poles having real part less than some specified $\sigma \ll \alpha$. A second example is the Butterworth filtering problem where the set of ideal poles should be uniformly distributed on the circle with radius ω_c where ω_c is the cutoff frequency of the filter. In view of the fact that variations in the filter parameters may lead to perturbations in the pole locations, the following robustness problem is of interest. Given a prescribed $\epsilon > 0$ and a range of variations for the filter

Manuscript received October 29, 1987; revised May 18, 1988. This work was supported by the National Science Foundation under Grant ECS-8612948.

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IEEE Log Number 8926754.

0018-9286/89/0500-0544$01.00

parameters, determine if the poles of the perturbed filter stay within the ϵ-neighborhoods of their ideal locations.

The second fundamental limitation of Kharitonov's theorem stems from the assumption that coefficients vary within prescribed intervals $[\alpha_i, \beta_i]$. This assumption is tantamount to "independence" between coefficient variations and is rarely met in practice. For example, in a mechanical system, perturbations in a coefficient of friction typically enter into more than one coefficient in the transfer function of the system.

An important result aimed at overcoming the limitations of Kharitonov's theorem is given in Bartlett, Hollot, and Lin [2]. These authors take D to be simply connected and allow for *linearly dependent* coefficient perturbations by taking P to be a polytope of real nth-order monic polynomials. That is, they consider a polytope of monic nth-order polynomials P generated by polynomials $p_1(s), p_2(s), \cdots, p_m(s)$. Hence, P is described by

$$P=\left\{p(s)=\sum_{i=1}^{m} r_i p_i(s) : \sum_{i=1}^{m} r_i=1;\ r_i\geq 0,\ i=1, 2, \cdots, m\right\}. \quad (1.1)$$

Subsequently, it is shown that P is D-stable if and only if all edges of P are D-stable. Hence, to determine if P is D-stable, it suffices to show that $rp_i(s)+(1-r)p_j(s)$ is D-stable for all $i, j \in \{1, 2, \cdots, m\}$ and all $r \in [0, 1]$. This result is further refined (see, for example, [3] and [4]) where it is shown that the r-sweep associated with the D-stability test above can be replaced by a "one-shot" test if D is the open left-half plane.

The main motivation for this note is derived from the fact that the assumption of simple connectedness of D might be too restrictive in many applications. Recalling the examples (dominant pole specification and Butterworth filter design) given above, notice that although D violates the simple connectedness requirement in [2], its complement D^c satisfies the following condition: through every point D^c, there is an unbounded continuous path which remains within D^c. More precisely, we say that D^c is *pathwise connected on the Riemann sphere*. This will be the fundamental property of D which we exploit in the derivation of our main result. Indeed, we extend the "edge criterion" in [2] to accommodate this class of D-sets. For examples of practical interest, it is not hard to see that simple connectedness of D implies pathwise connectedness of D^c on the Riemann sphere; i.e., this theory not only handles disconnected sets but also those considered in [2]. Other (perhaps less important) differences between this paper and [2] are that we do not require the generating polynomials $p_i(s)$ for P to be monic and that we allow for polynomials with complex coefficients.

II. Preliminary Notation

A complex nth-order polynomial $p(s)$ is described by

$$p(s)=a_0s^n+a_1s^{n-1}+\cdots+a_n; \qquad a_0\neq 0 \quad (2.1)$$

with $a_i=\alpha_i+j\beta_i$; α_i and $\beta_i \in \boldsymbol{R}$. We denote the coefficient vector of $p(s)$ by

$$\boldsymbol{p}=[\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \cdots\ \alpha_n\ \beta_n]^T. \quad (2.2)$$

Given a polytope of nth-order polynomials P (not necessarily monic, with $n\geq 1$) generated by $p_1(s), p_2(s), \cdots, p_m(s)$, we denote the set of coefficients by

$$\boldsymbol{P}=\left\{\sum_{i=1}^{m} r_i\boldsymbol{p}_i : \sum_{i=1}^{m} r_i=1; \qquad r_i\geq 0,\ i=1, 2, \cdots, m\right\} \quad (2.3)$$

where $\boldsymbol{p}_i$ is the coefficient vector for $p_i(s)$. Note that if P is a polytope of real polynomials, then P is nth order if and only if all the generating polynomials $p_i(s)$ are nth order with the same sign of their highest order coefficients. In general, a polytope of polynomials P is nth order if and only if the highest order coefficients of all the generating polynomials $p_i(s)$ stay within any half plane which does not include the origin. We denote the affine hull of $\boldsymbol{P}$ by aff $(\boldsymbol{P})$. We call s a zero of $\boldsymbol{P}$ if s is a zero of some polynomial $p(s) \in P$. Equivalently, there exists some $\boldsymbol{p} \in \boldsymbol{P}$ such that

$$K(s)\boldsymbol{p}=0$$

where

$$K(s) \triangleq \begin{bmatrix} \mathrm{Re}\,(s^n) & -\mathrm{Im}\,(s^n) & \mathrm{Re}\,(s^{n-1}) & -\mathrm{Im}\,(s^{n-1}) \cdots \mathrm{Re}\,(s) & -\mathrm{Im}\,(s) & 1 & 0 \\ \mathrm{Im}\,(s^n) & \mathrm{Re}\,(s^n) & \mathrm{Im}\,(s^{n-1}) & \mathrm{Re}\,(s^{n-1}) \cdots \mathrm{Im}\,(s) & \mathrm{Re}\,(s) & 0 & 1 \end{bmatrix} \in \boldsymbol{R}^{2\times 2(n+1)}. \quad (2.4)$$

III. Main Result

Theorem 3.1: Consider a polytope of nth-order (real or complex) polynomials P and a set D in the complex plane such that D^c is pathwise connected on the Riemann sphere. Then, P is D-stable if and only if the edges of P are D-stable.

Proof: Throughout the proof we use $E(\Omega)$ to denote the edges of a polytope Ω.

(Necessity): Suppose P is D-stable. Then it follows trivially that $E(P)$ is D-stable because $E(P)$ is a subset of P.

(Sufficiency): We assume that $E(P)$ is D-stable and must show that P is D-stable. First, we dispose with the trivial case when dim aff $(\boldsymbol{P})=1$ because $E(P)=P$ in this situation. Hence, we assume dim aff $(\boldsymbol{P})\geq 2$ and proceed by contradiction. Indeed, assume that P is not D-stable. Then, there exists some $\boldsymbol{p} \in \boldsymbol{P}$ and some $\alpha \in D^c$ such that

$$K(\alpha)\boldsymbol{p}=0. \quad (3.1)$$

To obtain the desired contradiction, we need to show that there exists some $\boldsymbol{q} \in E(\boldsymbol{P})$ and some $\beta \in D^c$ such that

$$K(\beta)\boldsymbol{q}=0. \quad (3.2)$$

To this end, we consider two cases. In Case 1, we assume dim aff $(\boldsymbol{P})=2$. Subsequently, for the case when dim aff $(\boldsymbol{P})>2$, we argue that the proof can be reduced to Case 1.

Case 1: dim aff $(\boldsymbol{P})=2$. First we express aff $(\boldsymbol{P})$ as

$$\mathrm{aff}(\boldsymbol{P})=\{\boldsymbol{p}+Ax : x \in \boldsymbol{R}^2\}$$

for some appropriate $2(n+1)\times 2$ dimensional matrix A. We now consider two subcases.

Subcase 1A: rank $(K(\alpha)A)\leq 1$. Notice that the set of coefficients of polynomials associated with aff $(\boldsymbol{P})$ having α as a zero is

$$\begin{aligned}\boldsymbol{P}_\alpha&=\{\boldsymbol{p}+Ax : K(\alpha)(\boldsymbol{p}+Ax)=0;\ x \in \boldsymbol{R}^2\}\\&=\{\boldsymbol{p}+Ax : K(\alpha)Ax=0;\ x \in \boldsymbol{R}^2\}.\end{aligned}$$

Furthermore, $\boldsymbol{P}_\alpha$ is contained in aff $(\boldsymbol{P})$ and since $\boldsymbol{P}_\alpha$ has dimension 1 or 2, it follows that $\boldsymbol{P}_\alpha$ intersects $E(\boldsymbol{P})$. Choosing $\boldsymbol{q} \in \boldsymbol{P}_\alpha \cap E(\boldsymbol{P})$, we obtain the desired contradiction with $\beta=\alpha$.

Subcase 1B: rank $(K(\alpha)A)=2$. Now, since D^c is pathwise connected on the Riemann sphere, there exists some unbounded continuous path Γ in D^c passing through α. Furthermore, by compactness of $\boldsymbol{P}$, there must exist some $\gamma \in \Gamma$ which is not a zero of any polynomial in P. Now let $f(\cdot)$:[0, 1] $\rightarrow \Gamma$ be a continuous function associated with the segment of Γ between α and γ, i.e., $f(0)=\alpha$ and $f(1)=\gamma$. Furthermore, we define

$$\lambda^* \triangleq \sup\,\{\lambda \in [0, 1] : \mathrm{rank}\,(K(f(\zeta))A)=2 \quad \text{for all } \zeta \in [0, \lambda)\}.$$

By definition of λ^*, the equation

$$K(f(\lambda))(\boldsymbol{p}+Ax)=0$$

has a unique solution

$$x_\lambda=-[K(f(\lambda))A]^{-1}K(f(\lambda))\boldsymbol{p}$$

for all $\lambda \in [0, \lambda^*)$. This solution generates a continuous path in aff (P) described by

$$p_\lambda = p + Ax_\lambda; \qquad \lambda \in [0, \lambda^*).$$

There are two possibilities. The first possibility is that $p_\zeta \notin P$ for some $\zeta \in (0, \lambda^*)$. In this situation, there must exist some $\delta \in [0, \zeta)$ such that $p_\delta \in E(P)$. Hence, we obtain the desired contradiction with $q = p_\delta$ and $\beta = f(\delta)$.

The second possibility is that $p_\lambda \in P$ for all $\lambda \in [0, \lambda^*)$. By compactness of P and continuity of p_λ, there must exist some sequence $\{\lambda_n\}$ in $[0, \lambda^*)$ converging to λ^* and some $p^* \in P$ such that

$$p^* = \lim_{n \to \infty} p_{\lambda_n}.$$

Furthermore, we have

$$K(f(\lambda^*))p^* = 0 \tag{3.3}$$

because

$$K(f(\lambda^*))p^* = \lim_{n \to \infty} K(f(\lambda_n))p_{\lambda_n}$$

and

$$K(f(\lambda_n))p_{\lambda_n} = 0$$

for each n. Since $f(1) = \gamma$ is not a zero of any polynomial in P, (3.3) implies that $\lambda^* < 1$. In view of the openness of the set of nonsingular matrices and the fact that $\lambda^* < 1$, it follows that the supremum in the definition of λ^* is not achievable. Hence,

$$\text{rank}\,(K(f(\lambda^*))A) \le 1.$$

Now, by repeating the analysis used in Subcase 1A [with $p = p^*$ and $\alpha = f(\lambda^*)$], we obtain some $q \in E(P)$ and $\beta \in D^c$ such that $K(\beta)q = 0$.

Case 2: dim aff $(P) = r > 2$. In view of Case 1, it suffices to prove the following: there exists an $(r - 1)$-dimensional face F of P, some $f \in F$, and some $\gamma \in D^c$ such that

$$K(\gamma)f = 0.$$

Once F and f are found, it is apparent that this argument can be repeated (note F is a polytope) until we obtain a two-dimensional face of P containing the coefficient vector for a polynomial which is *not* D-stable. Then Case 1 applies. Indeed, let $\mathcal{P}$ denote any two-dimensional affine set passing through p and notice that

$$P' \triangleq P \cap \mathcal{P}$$

is a subpolytope of P of dimension 2 containing p. Hence, from Case 1, it follows that there exists some $f \in E(P')$ and some $\gamma \in D^c$ such that $K(\gamma)f = 0$. The proof is completed by noting that $E(P')$ is contained in some $(r - 1)$-dimensional face F of P. □

IV. CONCLUSION

The next step in this line of research is to develop stability criteria for a more general family of polynomials. The polytopic assumption on P clearly restricts the class of physical perturbations which can be handled. Another important point to note is that the edge criterion given here does not easily degenerate into Kharitonov's theorem for the special case when the polytope corresponds to a family of real interval polynomials, i.e., in this special case, it is not obvious (from the theory given here) why it suffices to test four polynomials in lieu of all the edges. This leaves open the possibility that for polytopes of polynomials, there is some alternative to the edge criterion which specializes to Kharitonov's theorem in the "correct manner." Besides having aesthetic appeal, such an alternative would be desirable for two reasons. First, as the number of extreme points of P increases, one might be able to avoid the "combinatoric explosion" in computation associated with checking stability of all convex combinations of extreme points taken two at a time. Second, such an alternative for the polytopic case might suggest approaches to stability analysis for more general families of polynomials.

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Strict Aperiodic Property of Polynomials with Perturbed Coefficients

C. B. SOH AND C. S. BERGER

Abstract—**Let a family of polynomials be $P(s) = t_0 s^n + t_1 s^{n-1} + \cdots + t_n$ where $0 < a_j \le t_j \le b_j$. Recently, Kharitonov [2] derived a necessary and sufficient condition for (1) to have only zeros in the open left-half plane. This note derives some similar results for (1) to be strictly aperiodic (distinct real roots).**

I. INTRODUCTION

Consider the characteristic polynomial of a linear continuous-time system

$$P(s) = t_0 s^n + t_1 s^{n-1} + \cdots + t_n \tag{1}$$

or the characteristic polynomial of a linear discrete-time system

$$Q(z) = t_0 z^n + t_1(-1)^1 z^{n-1} + \cdots + t_n(-1)^n \tag{2}$$

where

$$t^T = [t_0 \cdots t_n].$$

The real vector t can be represented by a point in $(n + 1)$-dimensional Euclidean space. The polynomial (1) is said to be asymptotically stable if its zeros lie in the open left-half complex plane. Let G^n be the set of all asymptotically stable polynomials [of the form (1)] of order n. Denote by S^n the set of all polynomials (1) and (2), satisfying $\alpha_i \le t_i \le \beta_i$, $i = 0, \cdots, n$ and let S_1^n be the family of polynomials in S^n in which each coefficient (t_i) is equal to either α_i or β_i. Recently, Kharitonov [2] has proved the following theorem.

Theorem 1: In order that $S^n \subset G^n$, it is necessary and sufficient that $S_1^n \subset G^n$.

This note derives some similar results for S^n to be strictly aperiodic. The problem of aperiodicity arises in obtaining a response that has no oscillations or that has oscillations of a finite number only. It is customary

Manuscript received July 24, 1985; revised November 18, 1986, October 29, 1987, December 11, 1987, and June 2, 1988.

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IEEE Log Number 8927065.

0018-9286/89/0500-0546$01.00