Lack of Separation Principle for Quantized Linear Quadratic Gaussian Control

Minyue Fu

***Abstract*—This technical note studies the quantized linear quadratic Gaussian (LQG) control problem which is generalized from the classical LQG control but with the constraint that the feedback signal is quantized with a fixed bit rate. We show that state feedback control, state estimation and quantization can not be fully separated in general. Only a weak separation principle holds which converts the quantized LQG control problem into a quantized state estimation problem. Further separation of estimation and quantization is not possible in general. A concrete example is provided to demonstrate this fact. It is also shown that the so-called "whitening" approach to quantized state estimation is not optimal.**

***Index Terms*—Certainty equivalence, linear quadratic Gaussian control, networked control, quantized estimation, quantized feedback control, separation principle.**

I. Introduction

The well-celebrated separation principle (otherwise known as certainty equivalence principle) for linear quadratic Gaussian (LQG) control has two key "separation" properties: 1) state feedback control and state estimation can be designed separately and 2) the optimal state estimator over a given time horizon (finite or infinite) can be designed by separately minimizing the state estimation error at each time instant, a property which the well-known Kalman filter is based on. It is the combination of these two properties that allows the familiar recursive design of LQG controllers. What we intend to find out in this technical note is how these two separation properties generalize to the case where the feedback signal is quantized.

Quantized feedback control has received a lot of attention in recent years, due to the overwhelming need for network-based control systems. This raises many new challenges to the seemingly well-established linear control theory in questioning how to redesign control laws suitable for a networked environment. In this technical note, we study the so-called *quantized LQG control* problem which is generalized from the classical LQG problem in discrete time but with the constraint that the feedback channel is a digital link with a given bit rate. In doing so, we will also study a related problem of *quantized state estimation* which is an extension of the optimal state estimation problem but with a similar bit rate constraint. We first look back at the history of the research on the quantized LQG problem and discuss many attempts to generalize the separation principle, some dated back to early 1960's. We point out that many of these generalizations contain technical errors and/or misinterpretations. This leads us to the following results on quantized LQG control.

1) A *weak separation principle* holds which says that optimal quantized LQG control can be achieved by separately designing state feedback control and quantized state estimation. However, we provide a concrete example to demonstrate that further separation of quantized state estimation is not possible. The lack of a full separation principle implies that optimal design for quantized LQG control is very complex numerically and is in huge contrast with the classical separation principle where state estimation is independent of the state feedback control and that state estimation at each time instant can be done recursively without loss of optimality.
2) We then study the so-called *whitening approach* to quantized state estimation. This approach is often used in signal processing and has been proposed for quantized feedback control too. The basic idea is to pass the given measurement signal through a whitening filter to create an independent sequence before quantization by a memoryless quantizer, then pass the quantized signal through an inverse filter. We first show that if a given signal is a "white" signal, memoryless quantization is indeed optimal. This is due to the causality constraint in quantization and is in contrast with the standard quantization theory where vector quantization is known to be superior to memoryless quantization (or scalar quantization) when no causality constraint is imposed. However, we show that the whitening approach is not optimal for colored signals. This result should be of interest to a wide range of quantized feedback control problems.

Manuscript received July 20, 2011; revised July 23, 2011, and November 29, 2011; accepted January 31, 2012. Date of publication February 06, 2012; date of current version August 24, 2012. Recommended by Associate Editor J. Winkin.

The author was with the School of Control Science and Engineering, Zhejiang University, China. He is now with the School of Electrical Engineering and Computer Science, University of Newcastle, NSW, Australia.

Digital Object Identifier 10.1109/TAC.2012.2187010

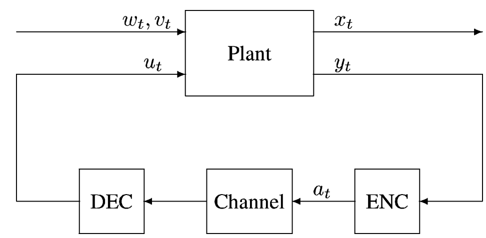

Fig. 1. Quantized LQG control system.

The rest of the technical note is organized as follows. Section II formulates the quantized LQG problem; Section III reviews the long history of this problem and related research; Section IV is devoted to the weak separation principle; Section V studies the quantized state estimation problem; and Section VI concludes the technical note. A preliminary version of this technical note appeared in [15].

II. Problem Formulation

Our quantized LQG problem is depicted in Fig. 1. The plant model is given by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $y_t \in \mathbb{R}^p$ is the measured output, $w_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^p$ are independent Gaussian random distributions with zero mean and covariances $W_t > 0$ and $V_t > 0$, respectively, and the initial state x_0 is also assumed to be an independent zero-mean Gaussian distribution with covariance Σ_0. In the sequel, we denote $z^t = \{z_0, z_1, \ldots, z_t\}$.

The communication channel we consider in this technical note is assumed to be a memoryless and error-free channel with a fixed transmission rate of R bits per (discrete-time) sample. The output signal y_t needs to be quantized by a causal encoder (the ENC block in Fig. 1) before transmission

$$a_t = \alpha_t(y_t | a^{t-1}) \tag{2}$$

where $\alpha(\cdot)$ takes values in a finite alphabet set $\mathcal{A}$ with size of 2^R. Similarly, the decoder (the DEC block in Fig. 1) is required to be a causal mapping from the received quantized signal, i.e.,

$$u_t = \beta_t(a_t|a^{t-1}). \tag{3}$$

The problem of *quantized LQG control* is to jointly design the quantizer and controller (or encoder and decoder) to minimize the following cost function J, under the given bit rate constraint

$$J = \mathcal{E}\left[x_T' Q_T x_T + \sum_{t=0}^{T-1} x_t' Q_t x_t + 2u_t' H_t x_t + u_t' S_t u_t\right] \tag{4}$$

where $\mathcal{E}[\cdot]$ is the expectation operator and $Q_t = Q_t'$, $S_t = S_t'$ and H_t are weighting matrices with

$$S_t > 0,\ \ Q_t - H_t S_t^{-1} H_t' \geq 0 \tag{5}$$

for all $t = 0, 1, \ldots, T-1$ and $Q_T = Q_T' \geq 0$.

III. Literature Review

Communication constrained control, also known as data-rate limited control, information limited control, networked control, is an area of research which attracts a lot of attention recently; see, e.g., Wong and Brockett [1], [2], Baillieul [3], Borkar and Mitter [4], Brockett and Liberzon [5], Nair and Evans [6], [7], Elia and Mitter [9], Tatikonda and Mitter [10], [11], Tatikonda *et al.* [12], Fu and Xie [13], [14], Matveev and Savkin [16], Braslavsky *et al.* [17], Fu and de Souza [18], and Bao *et al.* [19]. A comprehensive survey of recent research and historical background can be found in [8].

Research on quantized feedback control can be traced back to early days of modern control research. For example, Kalman [20] (1956) studied quantization effects in sampled-data systems; Widrow [21] (1961) conducted statistical analysis on quantization errors for sampled-data systems. It is interesting to know that the quantized LQG control problem, with a virtually identical problem formulation as in Section II, has been actively studied for a long time. Lewis and Tou [22] (1965), Meier [23] (1965) and a monograph by Tou [24] (1963) were perhaps the earliest attempts on this problem. Larson [25] (1967) for the first time claimed that the well-known separation principle for LQG control can be generalized to quantized LQG control. More specifically, he claimed that optimal control, estimation and quantization can be separately designed. Unfortunately, Marleau and Negro [26] (1972) came up with a counterexample to Larson's claim. Larson and Tse [27] (1972) then responded by pointing out an error in the counterexample but conceded that the separation claim in [25] is incorrect.

Fischer [28] (1982) revisited the quantized LQG control problem. He correctly pointed out that the optimal quantizer must be time-varying. By allowing time-varying quantization, Fisher claimed that the separation of control, estimation and quantization indeed holds. More specifically, he claimed that the optimal quantized LQG problem is solved by *separately*:

1) performing the optimal state estimation (Kalman filtering) to produce a state estimate $\hat{x}_t$;
2) generating the optimal control $u_t = K_t \hat{x}_t$ to the standard LQG problem;
3) quantizing u_t to minimize a weighted quantization distortion $\mathcal{E}[(u_t - u_t^q)'\Omega_t(u_t - u_t^q)]$ for some weighting matrix Ω_t dependant on the cost function, where u_t^q is the quantized u_t.

Apparently unaware of the works of Larson and Fischer, Borkar and Mitter [4] (1997) approached the quantized LQG control problem under a slightly different setting (by assuming the full state being measured without noise, but allowing a certain type of transmission errors). It is claimed in [4] that if, instead of the output, an "innovation process" is encoded and transmitted, control, estimation and quantization can be separately designed to achieve optimal performance. It should be noted that the separation is conditioned on the specific choice of the quantization scheme (i.e., quantization of a particular "innovation process"). The paper did not discuss whether such a choice is optimal or not. Also, this method does not apply to unstable systems [12].

In Tatikonda, Sahai and Mitter [12] (2004), the quantized LQG control problem is revisited once again also under the assumption of full state being measured and a separation result is given. In the work of Matveev and Savkin [16] (2004), a restrictive decoder, called memoryless decoder for which (3) reduces to $u_t = \beta_t(a_t)$, is used for quantized LQG control. It was shown that if using a memoryless decoder, separation of state feedback control, state estimation and quantization holds. Note that the use of memoryless decoders can degrade the control performance.

Despite all these claims, we will show via an example that the separation principle breaks down for quantized LQG control, i.e., the optimal control problem can not be separated into *independent* state feedback control, state estimation and quantization problems. We point out that the lack of separation for quantized LQG has been recognized by, Bao *et al.* [19] (2008) and [8] (2007), but no explicit examples are shown there.

IV. Separation Principle and Lack of It

The core of the classical LQG control theory is the well-known separation principle which states that the optimal controller is given by $u_t = K_t \hat{x}_t$, where $\hat{x}_t$ is the optimal estimate of the state x_t based on y^t, and K_t is the optimal control gain assuming that the true state is known and is given by

$$\begin{aligned} K_t &= -(S_t + B'P_{t+1}B)^{-1}(B'P_{t+1}A + H_t) \\ P_t &= Q_t + A'P_{t+1}A - K_t'(S_t + B'P_{t+1}B)K_t \end{aligned} \tag{6}$$

with $t = T-1, T-2, \ldots, 0$ and $P_T = Q_T$. The optimal cost of J, in the case where x_t is available for feedback, is given by

$$J_{\mathrm{LQ}} = \mathrm{tr}(P_0\Sigma_0) + \sum_{t=0}^{T-1} \mathrm{tr}(W_t P_{t+1}). \tag{7}$$

The result above is slightly generalized from [30] to allow time-varying matrices in J. It is well known (see, e.g., [31]) that the optimal state estimate $\hat{x}_t$ is given by a Kalman filter:

$$\begin{aligned} \hat{x}_t &= \hat{x}_{t|t-1} + E_t C' V_t^{-1}(y_t - C\hat{x}_{t|t-1}) \\ \hat{x}_{t+1|t} &= A\hat{x}_t + Bu_t \end{aligned} \tag{8}$$

with $\hat{x}_{0|-1} = \mathcal{E}[x_0] = 0$, where

$$E_t = \mathcal{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'] \tag{9}$$

is the state estimation error covariance, computed recursively as follows:

$$\begin{aligned} E_t &= E_{t|t-1} - E_{t|t-1}C'(CE_{t|t-1}C' + V_t)^{-1}CE_{t|t-1} \\ E_{t+1|t} &= AE_tA' + W_t \end{aligned} \tag{10}$$

with $t = 0, 1, \ldots, T-1$ and $E_{0|-1} = \Sigma_0$. It is clear that the optimal state estimator is independent of the optimal control problem. The optimal cost of J, when $u_t = K_t \hat{x}_t$ is used, becomes [31]

$$J_{\mathrm{LQG}} = J_{\mathrm{LQ}} + \sum_{t=0}^{T-1} \mathrm{tr}(K_t' \Omega_t K_t E_t) \tag{11}$$

where $\Omega_t = S_t + B' P_{t+1} B$.

A. Weak Separation of Quantized LQG Control

Returning to the quantized LQG control problem, we have the following weak separation result, which was first established in [28].

Theorem 4.1: Consider the quantized LQG control problem for the system (1), the cost function (4) and R-bit fixed-rate quantization. Denote

$$z_t = K_t \hat{x}_t \tag{12}$$

and its quantized version by z_t^q. Then, optimal quantized LQG control is achieved by choosing the encoder (sequence) $\{\alpha_t\}$ to minimize the following distortion function:

$$D = \sum_{t=0}^{T-1} \mathcal{E}[(z_t - z_t^q)' \Omega_t (z_t - z_t^q)]. \tag{13}$$

The corresponding optimal controller and minimum cost function are given by, respectively

$$u_t = z_t^q \tag{14}$$
$$\min J = J_{\mathrm{LQG}} + \min D. \tag{15}$$

Moreover, given any encoder $\{\alpha_t\}$ in (2), the optimal solution to u_t is given by

$$u_t = \mathcal{E}[K_t \hat{x}_t | a^t] = K_t \mathcal{E}[\hat{x}_t | a^t] \tag{16}$$

where the expectation above is done over the distribution of $K_t \hat{x}_t$ conditioned on a^t.

Remark 4.1: The result above suggests that optimal quantized LQG control can be achieved by first constructing the optimal estimate $\hat{x}_t$, which is independent of the cost function, then generating the optimal control $z_t = K_t \hat{x}_t$ and then quantizing it. However, the result does not suggest that each z_t can be independently quantized (a technical error made in [28]). Instead, the encoder sequence $\alpha_0, \alpha_1, \ldots, \alpha_{T-1}$ needs to be chosen jointly to minimize the distortion function D in (13), which depends on the cost function J. □

The next result shows that the quantized LQG problem is equivalent to a quantized state estimation problem. For this purpose, we consider the open-loop system of (1) as follows:

$$\begin{aligned} \tilde{x}_{t+1} &= A \tilde{x}_t + w_t \\ \tilde{y}_t &= C \tilde{x}_t + v_t. \end{aligned} \tag{17}$$

It is clear that x_t and $\tilde{x}_t$ are related by

$$x_t = \tilde{x}_t - \sum_{i=0}^{t-1} B u_i. \tag{18}$$

Consider the following distortion function

$$\tilde{D} = \sum_{t=0}^{T-1} \mathcal{E}(\tilde{x}_t - \tilde{x}_t^q)' \Pi_t (\tilde{x}_t - \tilde{x}_t^q) \tag{19}$$

where $\Pi_t \geq 0$, $\tilde{x}_t^q$ is the quantized $\tilde{x}_t$ and the associated encoder is given by

$$a_t = \tilde{\alpha}_t(\tilde{y}_t | a^{t-1}) \tag{20}$$

which is a rate-R encoder. The quantized state estimation problem is to find an encoder (20) such that $\tilde{D}$ is minimized.

Theorem 4.2: Consider the quantized LQG problem for the system (1) with the cost function (4) and fixed bit rate R. Define $\Pi_t = K_t' \Omega_t K_t$ and let $\{\tilde{\alpha}_t(\cdot)\}$ be the optimal encoder that minimizes the distortion function (19) for the associated quantized state estimation problem. Then, the optimal encoder (2) for the quantized LQG problem is given by

$$\alpha_t(y_t | a^{t-1}) = \tilde{\alpha}_t(\tilde{y}_t | a^{t-1}) \tag{21}$$

with

$$\tilde{y}_t = y_t - \sum_{i=0}^{t-1} C B u_i. \tag{22}$$

Moreover, the minimum cost for the quantized LQG problem is given by

$$\min J = J_{\mathrm{LQG}} + \min \tilde{D} \tag{23}$$

and

$$x_t^q = \tilde{x}_t^q + \sum_{i=0}^{t-1} B u_i = \mathcal{E}[\tilde{x}_t | a^t] + \sum_{i=0}^{t-1} B u_i. \tag{24}$$

Proof: The proof follows from Theorem 4.1 and two simple facts below: 1) y_t is linear in u^{t-1}; 2) $u_t = K_t x_t^q$ with x_t^q fully determined by a^t. These two facts collectively mean that $y^t | a^t$ and $\tilde{y}_t | a^t$ possess the same information. Hence, we can encode $\tilde{y}_t | a^t$ instead of $y_t | a^t$ and construct x_t^q from $\tilde{x}_t^q$ as in (24) without affecting the optimality of the distortion. □

B. Lack of Separation for Quantization

From Theorems 4.1-4.2, we understand that the quantized LQG control problem boils down to quantizing the sequence $\{z_t\}$ (or $\{\hat{x}_t\}$). Denoting

$$D_t = \mathcal{E}[(z_t - u_t)' \Omega_t (z_t - u_t)] \tag{25}$$

then, for any $t \geq 0$, the distortion function can be split into two terms

$$D = \sum_{\tau=0}^{t-1} D_\tau + \sum_{\tau=t}^{T-1} D_\tau. \tag{26}$$

At each time t, it is clear that α_t needs to be designed to minimize the second term above, which is called the *distortion-to-go*. **The specific separation question we ask is whether the optimal encoder sequence $\{\alpha_t\}$ for minimizing D can be obtained by separately**

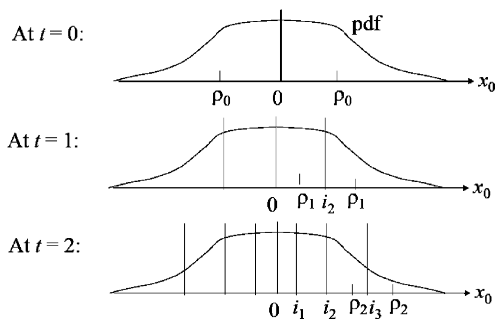

Fig. 2. Example of quantized LQG.

choosing each α_t to minimize the distortion D_t only. If this property held, a Kalman-like forward recursive formula could be derived for the quantizer design. This property, together with the weak separation principle, would constitute what we call a *full separation principle* for quantized LQG.

Unfortunately, this type of separation is not possible, as demonstrated by the example below.

1) Example 4.1: We consider a scalar system without process and measurement noises

$$\begin{aligned} x_{t+1} &= x_t + u_t, \quad x_0 \sim N(0,1) \\ y_t &= x_t \end{aligned} \tag{27}$$

where $N(0,1)$ is the normalized Gaussian distribution (with zero mean and unity variance). The cost function is given by

$$J = \mathcal{E}[Q_1 x_1^2 + Q_2 x_2^2 + Q_3 x_3^2 + S_0 u_0^2 + S_1 u_1^2 + S_2 u_2^2]. \tag{28}$$

A single-bit quantizer is to be used.

Using (12), (13), and (15), we rewrite $J = J_{\text{LQG}} + D$ with

$$D = \mathcal{E}[\Pi_0(x_0 - x_0^q)^2 + \Pi_1(x_1 - x_1^q)^2 + \Pi_2(x_2 - x_2^q)^2].$$

Note that $\{\Pi_t\}$ is a function of $\{Q_t\}$ and $\{S_t\}$. Conversely, we can choose $\{Q_t\}$ and $\{S_t\}$ to make any positive $\{\Pi_t\}$ we want. In particular, we will take $\Pi_0 = \Pi_1 = 1$ and leave $\Pi_2 > 0$ as a free parameter. Defining

$$\rho_0 = x_0^q;\ \rho_1 = x_1^q - K_0 x_0^q;\ \rho_2 = x_2^q - K_1 x_1^q - K_0 x_0^q$$

and using $x_{t+1} = x_t + K_t x_t^q$, D can be rewritten as

$$D = \mathcal{E}[(x_0 - \rho_0)^2 + (x_0 - \rho_1)^2 + \Pi_2(x_0 - \rho_2)^2]. \tag{29}$$

Therefore, for this particular example, the quantized LQG problem becomes a quantization problem for $x_0 \sim N(0,1)$ with the distortion function in (29). We can interpret $\{\rho_t\}$ as a sequence of successive quantized estimates of x_0, i.e., $\rho_t = \mathcal{E}[x_0|a^t]$.

Fig. 2 shows how the quantization works: At $t=0$, the range of x_0 is split into two quantization intervals: $(-\infty, 0]$ and $(0, \infty)$. At $t=1$, each of the above intervals is divided into two intervals: The interval $(0,\infty)$ is divided into $(0, i_2]$ and (i_2, ∞), and $(-\infty, 0]$ divided into $(-\infty, -i_2]$ and $(-i_2, 0]$. At $t=2$, each of the above intervals is further divided into two. Thus, we will have four intervals on the positive side: $(0, i_1], (i_1, i_2], (i_2, i_3], (i_3, \infty)$, and the negative side is mirror imaged. It can be shown that this symmetric structure is necessary for minimizing D in (29) for any $\Pi_2 > 0$.

The separation question we asked before becomes the following: Suppose, at time t, y_t is encoded to minimize $\mathcal{E}[(x_0 - \rho_t)^2|a^t]$ at each time t. Will such an encoder lead to the optimal D? The answer turns out to be negative. If $\Pi_2 \to 0$, to minimize D, the optimal value for $i_2 = 0.9816$. This value is also optimal for minimizing D_1 (the second term in (29)). The corresponding optimal values for i_1 and i_3 are 0.4709 and 1.6942, respectively. If $\Pi_2 \to \infty$, to minimize D, the optimal value for $i_2 \to 1.05$, and the corresponding optimal values for i_1 and i_3 are 0.5006 and 1.7470, respectively. If Π_2 is very large and we use the optimal value of i_2 for minimizing D_1, there will be some increase in the distortion (around 0.8%).

Remark 4.2: From this example, we see that an optimal encoder α_t which yields the optimal quantized state estimate at time t may not lead to an optimal total distortion. In other words, in designing α_t, its influence to "future" distortions must be considered. This implies that optimal encoder design a very nontrivial task, as it can not be done recursively as in the Kamlan filtering case. The example above uses a scalar system without measurement noise, so it is interesting that the separation principle can fail even in such a simple case.

V. Quantized State Estimation

We now study the quantized state estimation problem. The system we consider is given by

$$\begin{aligned} x_{t+1} &= A x_t + w_t \\ y_t &= C x_t + v_t \end{aligned} \tag{30}$$

with $x_0, \{w_t\}, \{v_t\}$ being independent Gaussian random variables as before. Let $\hat{x}_t$ be the optimal (Kalman) estimate of x_t and consider $z_t = K_t \hat{x}_t$ for some given K_t. The task of quantized state estimation is to encode $\{y_t\}$ (or $\{z_t\}$ indirectly) using (3) with fixed bit rate R to minimize the following distortion function:

$$D = \sum_{t=0}^{T-1} \mathcal{E}[(z_t - z_t^q)'\Omega_t(z_t - z_t^q)] \tag{31}$$

for some given Ω_t, where z_t^q is the quantized z_t.

The quantized state estimation problem above resembles the traditional vector quantization problem in the sense that both consider quantizing a sequence of input signal $\{z_t\}$ to minimize some distortion function. However, in our problem the quantizer has the additional constraint of causality. That is, the encoding-decoding pair at time t is not allowed to "see" the "future" values of z_τ and a_τ, $\tau > t$. The detailed connection to vector quantization can be seen in [15].

A. Special Case: "White" Signal

We first study a special case where $\{z_t\}$ is an independent sequence of random variables, but different distributions are allowed at different t. We may consider this sequence to be generated by (30) with $A = 0$. In this case, $z_t = K_t \hat{x}_t$ with the optimal Kalman state estimate given by

$$\hat{x}_t = \Sigma_t C'(C\Sigma_t C' + V_t)^{-1} y_t. \tag{32}$$

We have the following simple result which shows that a memoryless quantizer is optimal, i.e., each z_t can be quantized independently, **which implies that a full separation principle holds in this case**. The result is somewhat surprising because in the standard quantization setting where there is no causality constraint, it is well known that memoryless quantization is only suboptimal (hence the need for vector quantization).

Theorem 5.1: Suppose $\{z_t\}$ is an independent sequence of random variables with probability density functions $\{f_t\}$. Then, the optimal quantizer that minimizes D has the following memoryless structure:

$$a_t = \alpha_t(z_t);\ \ z_t^q = \mathcal{E}[z_t|a_t] \tag{33}$$

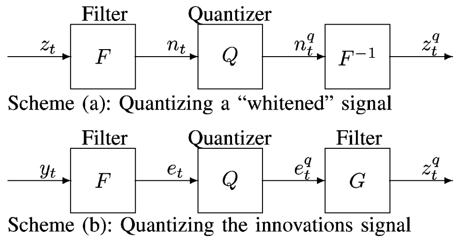

Fig. 3. "Whitening" approach to quantized state estimation.

with α_t chosen to minimize

$$D_t = \mathcal{E}[(z_t - z_t^q)' \Omega_t (z_t - z_t^q)] \tag{34}$$

where the expectation is taken over f_t.

Proof: The independence of $\{z_t\}$ implies that the *a posteriori* probability density function of z_t is still f_t, not altered upon receiving a^{t-1}. For the same reason, quantization of z_t has no effect to the future distortion terms $D_\tau, \tau > t$. Hence, the optimal quantizer must be the one that minimizes D_t only, thus it is a memoryless quantizer. □

B. General Case: Colored Signal

We return to the general case where $\{z_t\}$ is not an independent sequence. Motivated by Theorem 5.1, we may be tempted to consider a "whitening" approach, as depicted in Fig. 3. In this approach, z_t is "whitened" first before quantization. This can be done by passing $\{z_t\}$ through a whitening filter F to generate a white sequence $\{n_t\}$ for quantization [Scheme (a)] or, alternatively, by quantizing the *innovation* signal (or prediction error) $e_t = z_t - \hat{z}_{t|t-1}$ from the Kalman filter for $\{z_t\}$ directly [Scheme (b)]. The quantized signal is then used to "reconstruct" the intended signal by ignoring the quantization error.

We remark that this approach has been proposed for quantized LQG control. In Borkar and Mitter [4], under the assumption that the full state is available (hence no need for state estimation), it was suggested to quantize the process noise w_t (an innovation signal), instead of the state or control signal, using a memoryless quantizer.

We first show via an example that the whitening approach is, unfortunately, not optimal.

1) Example 5.1: Consider $z_0 = n_0$ and $z_1 = n_1 + n_0$, where n_0 and n_1 independent and uniformly distributed in [0, 1]. Let

$$D = \mathcal{E}[(z_0 - z_0^q)^2 + (z_1 - z_1^q)^2]. \tag{35}$$

An L-level quantizer is to be used for a large L.

It is easy to "whiten" z_t to get $n_0 = z_0;\ n_1 = z_1 - z_0$. If we quantize $\{n_t\}$, then, following Theorem 5.1, n_t can be quantized independently. Since n_t is uniformly distributed, a uniform quantizer is known to be optimal and the distortion is given by $\mathcal{E}[(n_t - n_t^q)^2] = \delta^2/12$ with $\delta = 1/L$ (see [32]). The corresponding distortion D equals

$$\begin{aligned} D &= \mathcal{E}[(n_0 - n_0^q)^2 + (n_1 + n_0 - n_1^q - n_0^q)^2] \\ &= \mathcal{E}[2(n_0 - n_0^q)^2 + (n_1 - n_1^q)^2] = \frac{\delta^2}{4}. \end{aligned} \tag{36}$$

Now consider the alternative quantization scheme where $z_0 = n_0$ is quantized using a uniform quantizer as before which gives $D_0 = \delta^2/12$, but z_1 is quantized differently as follows. At time $t = 1$, $z_0^q = n_0^q$ is known to lie uniformly in $[n_0^q - \delta/2,\ n_0^q + \delta/2]$. Then, $z_1 = n_1 + n_0$ has probability density function shown in Fig. 4 (without the offset of n_0^q). Let $[n_0^q - \delta/2,\ n_0^q + \delta/2]$ and $[1 + n_0^q - \delta/2,\ 1 + n_0^q + \delta/2]$ to be two quantization intervals and take the remaining $L - 2$ intervals

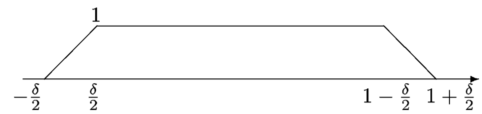

Fig. 4. Probability density function for z_1 in Example 5.1.

to be uniform in $[n_0^q + \delta/2,\ 1 - n_0^q - \delta/2]$. Then, the distortion for z_1 becomes

$$D_1 = \mathcal{E}[(z_1 - z_1^q)^2] = \frac{(1-\delta)^2}{12(L-2)^2}(1-\delta) + \frac{\delta^2}{18}\delta. \tag{37}$$

Combining it with the distortion for z_0, we get $D = \delta^2/6 + O(\delta^3)$, where $O(\delta^3)$ involves only third-order terms of δ. It is clear that when L is large, this distortion is smaller than (36).

Remark 5.1: We should not confuse the whitening approach with the so-called linear predictive coding (LPC) approach [32]. The latter quantizes the prediction error $e_t = z_t - \hat{z}_{t|t-1}$, where $\hat{z}_{t|t-1}$ is the prediction of z_t based on past quantized values of $e_\tau, \tau < t$, i.e., $\hat{z}_{t|t-1} = \mathcal{E}[z_t | e_\tau^q,\ 0 \le \tau < t]$. The key to the LPC approach is that the previous quantization values are used in constructing the input to the quantizer, which is not the case in the whitening approach. Note that the alternative approach in the example uses LPC.

VI. Conclusion

We have offered and discussed several results on the quantized LQG control problem. A weak separation principle holds for this problem, which allows this problem to be converted into a quantized state estimation problem. The latter is unfortunately difficult to solve. An example is given to show that a full separation principle does not hold in general. It is also shown that the "whitening" approach to quantized state estimation is not optimal and is even inferior to the LPC approach. It would be desirable to study suboptimal approaches to quantized state estimation.

References

[1] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints I: State estimation problems," *IEEE Trans. Autom. Control*, vol. 42, no. 9, pp. 1294–1299, Sep. 1997.

[2] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1049–1053, May 1999.

[3] J. Baillieul, "Feedback designs in information-based control," in *Proc. Stochastic Theory and Control Workshop*, Lawrence, KS, 2001, pp. 35–57.

[4] V. Borkar and S. Mitter, "LQG control with communication constraints," in *Communications, Computation, Control and Signal Processing: A Tribute to Thomas Kailath*. Norwell, MA: Kluwer, 1997.

[5] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[6] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: Tightest attainable bounds," *Syst. and Control Lett.*, vol. 41, pp. 49–56, 2000.

[7] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[8] G. N. Nair *et al.*, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, no. 1, pp. 108–137, Jan. 2007.

[9] N. Elia and K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.

[10] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.
[11] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1196–1201, Jul. 2004.
[12] S. Taikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1549–1561, Sep. 2004.
[13] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, Nov. 2005.
[14] M. Fu and L. Xie, "Finite-level quantization feedback control for linear systems," *IEEE Trans. Autom. Control*, vol. 54, no. 5, pp. 1165–1170, May 2009.
[15] M. Fu, "Quantized linear quadratic Gaussian control," in *Proc. American Control Conf.*, St. Louis, MI, 2009.
[16] A. S. Matveev and A. V. Savkin, "The problem of LQG optimal control via a limited capacity communication channel," *Syst. Control Lett.*, vol. 53, pp. 51–64, 2004.
[17] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1391–1403, Aug. 2007.
[18] M. Fu and C. E. de Souza, "State estimation using quantized measurements," *Automatica*, vol. 45, no. 12, pp. 2937–2945, 2009.
[19] L. Bao, M. Skoglund, and K. H. Johansson, "On iterative system design and separation in control over noisy channels," in *Proc. 17th IFAC World Congr.*, Seoul, Korea, Jul. 2008.
[20] R. E. Kalman, "Nonlinear aspects of sampled-data control systems," in *Proc. Symp. Nonlinear Circuit Theory*, Brooklyn, NY, 1956, vol. VII.
[21] B. Widrow, "Statistical analysis of amplitude-quantized sampled-data systems," *Trans. AIEE*, vol. 79, pt. 2, pp. 555–567, Jan. 1961.
[22] J. B. Lewis and J. T. Tou, "Optimum sampled-data systems with quantized control signals," *Trans. AIEE*, vol. 82, pt. 2, pp. 195–201, Jul. 1965.
[23] L. Meier, "Combined optimal control and estimation," in *Proc. Allerton Conf. Circuit and System Theory*, 1965, pp. 109–120.
[24] J. T. Tou, *Optimum Design of Digital Control Systems*. New York: Academic, 1963.
[25] R. E. Larson, "Optimum quantization in dynamic systems," *IEEE Trans. Autom. Control*, vol. AC-12, no. 2, pp. 162–168, Jan. 1967.
[26] R. S. Marleau and J. E. Negro, "Comments on 'Optimum quantization in dynamic systems'," *IEEE Trans. Autom. Control*, vol. AC-17, no. 2, pp. 273–274, Apr. 1972.
[27] R. E. Larson and E. Tse, "Authors' reply," *IEEE Trans. Autom. Control*, vol. AC-17, no. 2, pp. 274–276, Apr. 1972.
[28] T. R. Fischer, "Optimal quantized control," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 996–998, 1982.
[29] V. Borkar, S. Mitter, and S. Tatikonda, "Optimal sequential vector quantization of Markov sources," *SIAM J. Control Optim.*, vol. 40, no. 1, pp. 135–148, 2001.
[30] M. H. Davis and M. Zervos, "A new proof of the discrete-time LQG optimal control theorems," *IEEE Trans. Autom. Control*, vol. 40, no. 8, pp. 1450–1453, Aug. 1995.
[31] D. Bertsekas, *Dynamic Programming and Optimal Control*. Belmont, MA: Athena Scientific, 2000.
[32] R. M. Gray and D. L. Neuhoff, "Quantization," *IEEE Trans. Inform. Theory*, vol. 44, no. 6, pp. 2325–2383, Jun. 1998.

Reduced-Order Iterative Learning Control and a Design Strategy for Optimal Performance Tradeoffs

Goele Pipeleers and Kevin L. Moore

***Abstract*—When iterative learning control (ILC) is applied to improve a system's tracking performance, the trial-invariant reference input is typically known or contained in a prescribed set of signals. To account for this knowledge, we propose a novel ILC structure that only responds to a given set of trial-invariant inputs. The controllers are called reduced-order ILCs as their order is less than the discrete-time trial length. Exploiting all knowledge available on the input signals is instrumental in facing the fundamental performance limitations in ILC: an ILC is bound to amplify trial-varying inputs and reducing this trial-varying performance degradation invokes a slower learning transient. We present a novel optimal ILC design strategy that allows for a quantitative and systematic analysis of this tradeoff. The merit of reduced-order ILCs in view of this tradeoff is demonstrated by numerical results.**

***Index Terms*—Iterative learning control, optimal control.**

I. Introduction

Iterative learning control (ILC) is an open-loop control strategy that improves the performance of a system executing the same task over and over again by learning from previous iterations/trials [1], [2]. Most current ILCs comply with the following trial-domain description [2], [3]:

$$\mathbf{u}_{j+1} = \mathbf{Q}(\mathbf{u}_j + \mathbf{L}\mathbf{e}_j) \tag{1}$$

for some $N \times N$ matrices $\mathbf{Q}$ and $\mathbf{L}$, with N the discrete-time trial length. The index $j = 0, 1, \ldots$ labels the trials, while the N-dimensional vector signals $\mathbf{u}_j$ and $\mathbf{e}_j$, correspond to the supervectors of the control signal, respectively the tracking error, as described in more detail in Section II-A.

It is well-known in the ILC literature that for linear discrete-time systems, ILC (1) achieves perfect asymptotic rejection/tracking of *any* trial-invariant input if and only if $\mathbf{Q} = I_N$, where I_N denotes the $N \times N$ identity matrix. This is a direct implication of the Internal Model Principle [4], [5], which states that in order to achieve perfect asymptotic rejection/tracking of disturbance/reference signals corresponding to the output of an autonomous system, this system must be embedded in a stable feedback loop. Fig. 1 shows the trial-domain implementation of (1) with $\mathbf{Q} = I_N$, where $\mathbf{q}$ denotes the one-trial-advance operator:[1] $\mathbf{q}\mathbf{u}_j = \mathbf{u}_{j+1}$. This figure reveals that the corresponding ILC embeds

Manuscript received October 19, 2010; revised April 28, 2011 and July 26, 2011; accepted August 23, 2011. Date of publication September 01, 2011; date of current version August 24, 2012. This work was supported by K. U. Leuven—BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), and by IWT-Flanders through project IWT-SBO 80032 (LeCoPro). Recommended by Associate Editor D. Arzelier.

G. Pipeleers is with the Department of Mechanical Engineering, Katholieke Universiteit Leuven, Celestijnenlaan 300B, B-3001 Heverlee, Belgium (e-mail: goele.pipeleers@mech.kuleuven.be).
K. L. Moore is with the Division of Engineering, Colorado School of Mines, Golden, CO 80401 USA (e-mail: kmoore@mines.edu).

Digital Object Identifier 10.1109/TAC.2011.2166690

[1]Notice that the boldfaced $\mathbf{q}$ notation is equivalent to the w-operator introduced in [6] and developed in [3].