Robust Stability Under a Class of Nonlinear Parametric Perturbations

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***Abstract*—This paper considers the robust stability verification of linear time-invariant systems admitting a class of nonlinear parametric perturbations. The general setting is one of determining the closed-loop stability of systems whose open-loop transfer functions consist of powers, products, and ratios of polytopes of polynomials. Apart from this general setting, two special cases of independent interest are also considered. The first special case concerns uncertainties in the open-loop gain and real poles and zeros, while the second special case treats uncertainties in the open-loop gain and complex poles and zeros. In light of the zero exclusion principle, robust stability is equivalent to zero exclusion of the value sets of the system characteristic function (a value set consists of the values of the characteristic functions at a fixed frequency). The main results of this paper are as follows. 1) The value set of the characteristic function at each fixed frequency is determined by the edges and some frequency-dependent internal line segments. 2) Consequently, Hurwitz invariance verification simplifies to that of checking certain continuous scalar functions for avoidance of the negative real axis. 3) For the case of real zero-pole-gain variations, the critical lines are all frequency independent, and therefore, the determination of the robust stability is even simpler. 4) For the case of complex zero-pole-gain variations, the critical internal lines are shown to be either frequency independent or to be confined in certain (two-dimensional) planes or (three-dimensional) boxes.**

I. INTRODUCTION

THE following problem is of interest in the robust stability verification of linear time-invariant control systems depicted in Fig. 1. Suppose we are given a stability region $\mathcal{D}$ and a family of open-loop transfer functions parameterized by a real vector γ:

$$T(\Gamma) := \{t(s, \gamma) : \gamma \in \Gamma\} \tag{1.1}$$

where $t(s, \gamma)$ is the transfer function of the plant and controller, and Γ is a connected set in $\boldsymbol{R}^N$. Determine as simply as possible if all members of the family of the corresponding characteristic functions

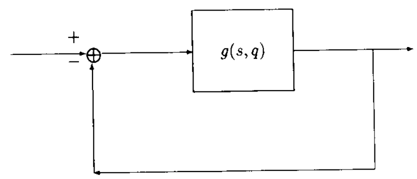

Fig. 1. Closed-loop uncertain system.

$$H(\Gamma) := \{h(s, \gamma) = 1 + t(s, \gamma) : \gamma \in \Gamma\} \tag{1.2}$$

have all zeros contained entirely in $\mathcal{D}$ (i.e., the family is $\mathcal{D}$-stable invariant). Generally, the transfer function coefficients depend nonlinearly on γ.

One approach to this problem is to treat it in its broadest generality, as is done in [1], [2] where a very broad class of $H(\Gamma)$ is considered. Alternatively, one can consider particular parameterizations reflecting specific forms of structural information supplied by the modeling process. This allows formulation of stability verification schemes which are computationally less demanding. Examples of this approach include [3], which considers a family of polynomials admitting independent variations in the coefficients; [4]–[6], which account for affinely dependent variations; and [7]–[9], which consider multilinear dependence (see [10], [11] for surveys). Each of [3]–[9], exploits the underlying structural information and demonstrates consequent simplifications.

This paper adopts the second approach by focusing on a special class of nonlinear parametric dependence. To keep the presentation simple, only Hurwitz invariance is investigated (i.e., $\mathcal{D}$ is the open left-half plane), although the results do, in fact, generalize to more general stability regions. Specifically, the family of characteristic functions to be considered in this paper admits the following form

$$h(s, \gamma) = 1 + g_0(s)\frac{\prod_{i=1}^{m}(p_{i0}(s) + \gamma_i^T P_i(s))^{\mu_i}}{\prod_{i=m+1}^{n}(p_{i0}(s) + \gamma_i^T P_i(s))^{\nu_i}} \tag{1.3}$$

where $g_0(s)$ and the $p_{i0}(s)$ are real scalar rational functions and polynomials in s, respectively, $\gamma_i \in \Gamma_i \subset \boldsymbol{R}^{N_i}$ represent a partition of γ, i.e.,

$$\gamma = (\gamma_1^T, \gamma_2^T, \cdots, \gamma_n^T)^T \tag{1.4}$$

and

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n \tag{1.5}$$

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$P_i(s)$ are real vector polynomials with dimension N_i, and μ_i and ν_i are fixed positive exponents. The quantities $g_0(s)$, $p_{i0}(s)$, $P_i(s)$, n, N_i, μ_i, ν_i, and Γ are assumed known. The jth element of γ_i [respectively, $P_i(s)$] is denoted by γ_{ij} [respectively, $P_{ij}(s)$]. Since Γ is an axis parallel box, each γ_{ij} varies independently of the others within an interval

$$\gamma_{ij}^- \leq \gamma_{ij} \leq \gamma_{ij}^+. \tag{1.6}$$

Thus, each factor $(p_{i0}(s) + \gamma_i^T P_i(s))$ forms a polytope of polynomials as γ_i varies in Γ_i. Notice that a polytope can simplify to model an uncertain gain in the open-loop transfer function. Furthermore, in the case where the exponents μ_i and ν_i are restricted to be $+1$, the robust stability verification of (1.3) is equivalent to that of a subclass of the multilinear problem, i.e., the characteristic polynomial associated with (1.3) depends on γ_{ij} in a multilinear fashion. To simplify our notation, we rewrite (1.3) as follows:

$$h(s, \gamma) = 1 + g_0(s)\prod_{i=1}^{n}(p_{i0}(s) + \gamma_i^T P_i(s))^{k_i} \tag{1.7}$$

where k_i are allowed to be nonzero integers (either positive or negative).

There are several situations in which the setting of (1.7) becomes important. By way of background, we cite the results of [4], where the setting considered translates to one involving an uncertain plant having numerator and denominator polynomials lying in polytopic sets. Essentially, [4] asks the following question: Given a fixed controller $g_0(s)$, how can one determine if the uncertain closed loop is stable? Thus, the problem considered in [4] can be viewed as a subclass of (1.7), with $n = 2$, $k_1 = 1$, and $k_2 = -1$. In many applications involving process control, the overall plant is itself a cascade of several subplants, such that physical uncertain parameters of a given subplant enjoy physical independence from those of the other subplants. Now, if one models each uncertain subplant individually as lying in sets analogous to those in [4], one readily obtains a special case of the structure exhibited in (1.7).

To further illustrate the scope of (1.7), two more examples are considered. The first example is of a plant with independent real zero, pole and gain variations and is as follows

$$h(s, \gamma) = 1 + dg_0(s)\prod_{i=1}^{n-1}(s + \lambda_i)^{k_i} \tag{1.8}$$

where d and λ_i are uncertain real parameters lying in given bounds. In this case, $g_0(s)$ can be a given controller, d the gain, and λ_i the zeros and poles of the plant whenever the k_i are, respectively, positive or negative. The uncertainty assumes that the λ_i vary independently within given bounds. The objective is to verify if $g_0(s)$ stabilizes the plant for all possible parameter variations. Therefore, under the assumption of no zero-pole cancellation (which is trivial to check in this example), the closed-loop system associated with (1.8) is robustly stable if and only if the corresponding $H(\Gamma)$ is Hurwitz invariant. Note that the terms involving cases where $k_i \neq \pm 1$ reflect structural preservation of multiplicities.

To allow for complex zero and pole variations, one may include factors of the forms $(s^2+a_i s+b_i)^{k_i}$, with a_i and b_i also varying independently in intervals. The version of (1.7) given in (1.9) below will be referred to as the complex zero-pole-gain variation problem:

$$h(s, \gamma) = 1+dg_0(s)\prod_{i=1}^{\tau}(s+\lambda_i)^{k_i} \prod_{j=\tau+1}^{n-1} (s^2+a_j s+b_j)^{k_j} \tag{1.9}$$

with

$$\gamma = (\lambda_1, \lambda_2, \cdots, \lambda_\tau, a_{\tau+1}, b_{\tau+1}, \cdots, a_{n-1}, b_{n-1}, d)^T. \tag{1.10}$$

Throughout this paper, we adopt the following assumptions on the function $h(s, \gamma)$ in (1.7).

Assumption 1.1: The function $h(s, \gamma)$ has no unstable zero-pole cancellation for any $\gamma \in \Gamma$.

Assumption 1.2: Continuous variations of γ result in continuous changes in the zeros of $h(s, \gamma)$.

Assumption 1.3: The function $h(s, \gamma)$ has no purely imaginary poles for any $\gamma \in \Gamma$.

We note that Assumption 1.1 is essential to assure the internal stability of the closed-loop system. However, violation of this assumption will not cause any difficulty for constructing the value set $H(j\omega, \Gamma)$. Further, in light of the recent results of [20], the verification of this assumption is a relatively simple matter.

Assumption 1.2, on the other hand, is a standard one that requires the leading coefficient of the overall numerator polynomial in (1.7) not to equal zero within the specified uncertainty bounds.

Finally, Assumption 1.3 can be simply tested through a series of straightforward algebraic [18] techniques.

A. Approach and Main Results

As in [8], [9], [12], [13], we follow the so-called value set approach to robust stability analysis. For the family of rational functions (1.2), the value set at a frequency ω is defined as

$$H(\omega, \Gamma) := \{h(j\omega, \gamma) : \gamma \in \Gamma\}, \tag{1.11}$$

and its boundary will be denoted by $\partial H(\omega, \Gamma)$.

Our approach exploits a slight variation of the zero exclusion principle (see, e.g., [9]). Under assumptions (1.1), (1.2), and the fact that the value set changes continuously with ω, this principle reduces to the following conditions as being necessary and sufficient for the Hurwitz invariance of $H(\Gamma)$:

1) at least one member of $H(\Gamma)$ is Hurwitz, and

2) $0 \notin \partial(H(\omega, \Gamma))$, $\forall \omega \in \boldsymbol{R}$, where ∂ denotes "the boundary of."

There are several appealing qualities of the value set approach. First, it provides a unifying framework within which most of the currently known robust stability results can be understood. Second, it can be used to obtain simple and transparent proofs of these results (see, for example, [13], [14], [19] for proofs of [3], [4], [15]). Furthermore, while for certain special classes of uncertainties the robust stability of a family of polynomials reduces to that of some easily characterizable members of this family, the same does not hold for general parameterizations. For example, when the uncertainty set is one of multiaffinely parameterized polynomials, in general there are no easily characterizable internal subsets of the parameter box Γ, whose Hurwitzness implies the Hurwitz invariance of the entire set [23]. In the same vein, it is shown in [16] that in considering the Hurwitz invariance of the characteristic polynomials of a polytope of $(n \times n)$ matrices, one has to check all $(2n-4)$-dimensional boundaries of the parameter space, a task which is computationally prohibitive even for matrices of reasonably small sizes. In such a case, the zero exclusion condition 2) above provides a relatively simple frequency sweeping procedure for verifying Hurwitz invariance. Such a graphical approach has been advocated through successful application in [9], [12], [13], and provides the conditions of [8], [9]. Recently, the so-called finite zero exclusion principle has been developed by Rantzer [21] to avoid frequency sweeping, and to thus permit fast computation. This approach too relies on the ready calculation of pertinent value sets. Besides their utility for robust stability analysis, value sets play an important role in the determination of the frequency response of a family of transfer functions; see, for example, [17]. Consequently, they can be used in designing robust controllers which meet performance considerations that go beyond mere closed-loop stabilization; see [22], for example.

Accordingly, in this paper, we consider both the determination of value sets as well as the Hurwitz invariance of (1.3). The principle contribution is to show that for the general family (1.7), at each fixed frequency ω, each member of $\partial H(\omega, \Gamma)$ has preimages in certain line segments in the parameter set Γ. These critical line segments are simply characterized, vary with frequency, are independent of the exponents k_i [see (1.7)], and consequently the Hurwitz invariance of the family of functions in (1.7) becomes equivalent to checking a finite number of continuous and piecewise differentiable scalar functions in ω for avoidance of the negative real axis.

For the case of (1.8), we show that the critical segments are, in fact, the edges of Γ plus certain simply constructable, frequency-independent, 45-degree line segments in the parameter space. Further, the Hurwitz invariance of $H(\Gamma)$ is guaranteed by that of these frequency-independent line segments (including the edges).

For the case of (1.9), we show that the critical lines determining the value set boundaries are either frequency independent or, as frequency varies, vary on certain (two-dimensional) planes and certain (three-dimensional) boxes in Γ. To check for robust Hurwitzness, it then suffices to check these frequency-independent lines, planes, and boxes.

It is instructive to compare this paper with the work of [24], which considers the set of uncertain polynomials

$$\sum_{i=1}^{n} F_i(s)\left(\prod_{j=1}^{n_i} P_{ij}(s)\right) \tag{1.12}$$

where the $F_i(s)$ are fixed polynomials while the $P_{ij}(s)$ vary in independent interval sets. Several points of difference are noteworthy. First, the set in (1.12) is broader than ours in the sense that it allows the sum of more than two factors. It is narrower than (1.7) in that the $P_{ij}(s)$ vary in independent intervals and have unity power, whereas in (1.7), polytopic variations with arbitrary powers are allowed. However, the greatest difference lies in the approaches employed and the results derived in this paper and [24]. The latter employs a parameter space as opposed to the value set approach to Hurwitz invariance verification. Its result states that one needs to check the Hurwitz invariance of internal manifolds that are of the dimension

$$\max_i \{n_i\}. \tag{1.13}$$

Thus, even for the cases of (1.8) and (1.9), [24] requires checking Hurwitz invariance over manifolds that have dimensions that increase with the number of factors in (1.8) and (1.9).

Section II considers the general case. The special cases of (1.8) and (1.9) are addressed in Sections III and IV, respectively. Section V is the conclusion.

II. Value Set Boundaries

A major objective of this paper is to achieve the following. For a given frequency ω, identify a critical subset $\Gamma_c(\omega)$ of Γ having the property that for all

$$v \in \partial(H(\omega, \Gamma)) \tag{2.14}$$

there exists $\gamma \in \Gamma_c(\omega)$ such that

$$h(j\omega, \gamma) = v \tag{2.15}$$

i.e., every point on the boundary of the value set at this frequency has at least one preimage in $\Gamma_c(\omega)$. In characterizing such a $\Gamma_c(\omega)$, we will not attempt to extract the smallest possible such set, but will be content with one particular choice that enjoys the above properties, and at the same time has a relatively simple analytical description.

To this end, we adopt a somewhat indirect approach. Specifically, given ω, we give necessary conditions on a parameter vector γ such that

$$h(j\omega, \gamma) \in \partial(H(\omega, \Gamma)). \tag{2.16}$$

Clearly, parameter vectors satisfying these necessary conditions together suffice to define a $\Gamma_c(\omega)$ meeting the requirements specified above.

In the sequel, a k-side of the parameter box Γ will refer to a subset of Γ in which only k parameters vary and all others are fixed at their extreme values. By the same token, a point γ in the interior of a k-side has exactly k-elements that do not take extreme values. A point on the boundary of a k-side lies on this k-side, but not in its interior. If γ is in the interior of a k-side of Γ, then the elements of γ not fixed at extreme values will be called the variables on this k-side. We note, that every k-side of Γ is also a hyperrectangle, that all its l-sides, $l \leq k$ are also l-sides of Γ, and that corners and edges of Γ are its respective 0- and 1-sides. Further, the $\sum_{i=1}^{n} N_i$-side of Γ (recall that this sum is the dimension of Γ) is Γ itself.

In characterizing $\Gamma_c(\omega)$, we will assume that all edges of Γ are automatically included in $\Gamma_c(\omega)$. Thereafter, for all

$$1 < k \leq \sum_{i=1}^{n} N_i \tag{2.17}$$

we will provide for each k-side of Γ necessary conditions for γ in its interior to obey (2.16).

The main results of this section can be summarized in the following way.

Result 1: Not all sides of Γ need contribute interior points to $\Gamma_c(\omega)$. In fact, certain sides are such that, irrespective of ω, their interior points need never be included in $\Gamma_c(\omega)$. In Section II-A, we give a result which characterizes what the sides that potentially contribute interior points to $\Gamma_c(\omega)$ are. A feature of this characterization is that, for each side which in Section II-A is identified as a potential interior point contributor, all the sides of this side are also similar contributors.

Result 2: Having eliminated a vast majority of sides by virtue of Result 1, we restrict attention to an arbitrary side Q of Γ whose interior points have been ascertained by Result 1 as being potential members of $\Gamma_c(\omega)$. For every such Q, at a given frequency ω, we associate a unique, possibly frequency-dependent affine line $L_a(Q)$ such that a γ in the interior of Q obeys (2.16) only if γ belongs to this line. Consequently, together with the edges of Γ, the union of the intersections of all $L_a(Q)$ with the interior of the corresponding contributing sides Q comprise $\Gamma_c(\omega)$. The affine line $L_a(Q)$ associated with a given contributing boundary Q is characterized in Section II-B.

The results to be presented will be illustrated through the following example.

$$h(j\omega,\gamma)=1+\frac{[(\omega^2-2)\gamma_{11}+(j\omega+1)\gamma_{12}][\omega^2+\gamma_{31}+\gamma_{32}]}{[(1-\omega^2+j\omega)\gamma_{21}+j\omega\gamma_{22}][0.5j\omega+\gamma_{41}]}. \tag{2.18}$$

To proceed with the development, observe first that as far as the determination of $\Gamma_c(\omega)$ is concerned, one need only consider the transfer function

$$g(s,\gamma)=\prod_{i=1}^{n}(p_{i0}(s)+\gamma_i^T P_i(s))^{k_i}. \tag{2.19}$$

For unless $g_0(j\omega)$ in (1.3) is zero, at any frequency ω, $g(j\omega,\gamma)\in\partial(G(\omega,\Gamma))$ (with $G(\omega,\Gamma)$, obviously defined), iff $h(j\omega,\gamma)\in\partial(H(\omega,\Gamma))$. Of course, if $g_0(j\omega)=0$, then at this ω, $H(\omega,\Gamma)$ collapses to a single point, and any $\gamma\in\Gamma$, including any corner, describes $\partial(H(\omega,\Gamma))$. Thus, in this case, $\Gamma_c(\omega)$ can be trivially constituted by a solitary corner of Γ.

Thus, here onwards, attention will be restricted to the sets $g(j\omega,\gamma)$, $\partial(G(\omega,\Gamma))$ instead of $h(j\omega,\gamma)$, $\partial(H(\omega,\Gamma))$, respectively.

A. Result 1

We have the following theorem.

Theorem 2.1: For a given ω and every $v\in\partial(H(\omega,\Gamma))$, there exists a γ obeying (2.15), such that [see (1.4)], for each i, at most one γ_{ij} is a variable. Consequently, a boundary Q of Γ need contribute internal points to $\Gamma_c(\omega)$ only if, for each i, at most one γ_{ij} is a variable in the interior of Q.

Thus, for example, in (2.18), any side on which both γ_{11}, γ_{12} are variables is not included in $\Gamma_c(\omega)$. The proof of this theorem relies on the following lemma, proved in Appendix A, which involves boundary determination of the power and the product of sets of complex numbers.

Lemma 2.1: Let $D_1, D_2,\cdots,D_\sigma$ be bounded and closed sets of complex numbers. Define, for integers k_i,

$$D_i^{(k_i)}:=\{d_i^{k_i}: d_i\in D_i: i=1,2,\cdots,\sigma\}, \tag{2.20}$$

and

$$\prod_{i=1}^{\sigma}D_i^{(k_i)}:=\left\{\prod_{i=1}^{\sigma}d_i^{k_i}: d_i\in D_i: i=1,2,\cdots,\sigma\right\}. \tag{2.21}$$

Then

$$\prod_{i=1}^{\sigma}(\partial D_i)^{(k_i)}\supset\partial\left(\prod_{i=1}^{\sigma}D_i^{(k_i)}\right). \tag{2.22}$$

We can now prove the theorem.

Proof of Theorem 2.1: In Lemma 2.1, identify

$$D_i:=\{p_{i_0}(j\omega)+\gamma_i^T P_i(s):\gamma_i\in\Gamma_i\} \tag{2.23}$$

and $\sigma=n$. Then the result follows by noting that every element in $\partial(D_i)$ has at least one preimage in an edge of Γ_i [14].

B. Result 2

From here onwards, in determining contributions to $\Gamma_c(\omega)$ from the interior of a given side of Γ, attention need only be restricted to sides Q of Γ on which *at most one* element of each Γ_i is a variable.

Call such a prototype side Q. Lump the variables defining this side into the vector $q=[q_1,\cdots,q_k]$, $k\le n$. Suppose the coefficient polynomial of each q_i in (2.19) [i.e., the corresponding $P_{ij}(s)$], evaluated at $s=j\omega$, is nonzero. (If the coefficient polynomial is evaluated to be zero, then the corresponding q_i need not be included in q; see discussion later.) Then, through a suitable extraction of the frequency-dependent coefficients of q_i in (2.19), at every ω, the image of the side Q under the mapping defined in (2.19), i.e., $G(\omega,Q)$, can be described by a set of complex numbers

$$f(Q):=\left\{f(q)=f_0\prod_{i=1}^{k}(q_i+\alpha_i+j\beta_i)^{k_i}: q\in Q\right\} \tag{2.24}$$

where f_0 is a complex constant, α_i and β_i, $i=1,2,\cdots,k$ are real constants, and $k_i\ne 0$ for all $i=1,2,\cdots,k$.

Fact 1: Notice that, as ω varies, f_0, α_i, and β_i vary in a rational fashion with ω. Further, $f(Q)$ is bounded because of Assumption 1.3 and the fact that the coefficient polynomial of each q_i in (2.19) evaluated at $s=j\omega$ is nonzero.

Thus, for example, in (2.18), consider the interior of the 2-side defined by $\gamma_{12}=\gamma_{12}^+$, $\gamma_{32}=\gamma_{32}^-$, $\gamma_{21}=\gamma_{21}^+$, $\gamma_{41}=\gamma_{41}^+$, $\gamma_{22}=\gamma_{22}^-$. In this case, $q_1=\gamma_{11}$, $q_2=\gamma_{31}$:

$$Q:=\{q=[q_1,q_2]^T: q_i^-\le q_i\le q_i^+,\ i=1,2\}. \tag{2.25}$$

Further

$$f(q)=\frac{(\omega^2-2)}{[(1-\omega^2+j\omega)\gamma_{21}^+ + j\omega\gamma_{22}^-][0.5j\omega+\gamma_{41}^+]}\cdot\left[q_1+\frac{(j\omega+1)\gamma_{12}^+}{(\omega^2-2)}\right][q_2+\omega^2+\gamma_{32}^-]. \tag{2.26}$$

Observe that, at $\omega^2 = 2$, the coefficient of $\gamma_{11} (= q_1)$ is zero. Consequently, the representation in (2.26) is infeasible. However, we argue now that at this ω, no interior point of the Q under consideration need be included in $\Gamma_c(\omega)$. This is because, at this ω, $H(\omega, \Gamma)$ is independent of $\gamma_{11} = q_1$. Thus, if we select γ_{11} at an extreme value, without changing the variable $\gamma_{31} = q_2$, one does not alter the value of $h(j\omega, \gamma)$. Thus, corresponding to each point q in the interior of Q, there lies a point q^* on a boundary of Q having precisely the same image in the value set space as does q. Then, in determining contributors to $\Gamma_c(\omega)$, one need not consider any point in the interior of Q, as these points are covered by points on the boundary of Q. This observation leads to the following formal fact.

Fact 2: Suppose Q is a side of Γ that conforms to the requirements of Theorem 2.1. Further suppose, with q_i the variables on Q, at some frequency ω, and some j, the coefficient polynomial of q_j in (1.3) is zero. Then, $\Gamma_c(\omega)$ will not contain any points in the interior of Q. Also, if the coefficient polynomial of each q_i is nonzero, then f_0 in (2.24) is nonzero.

Finally, observe, from the foregoing that the basis for limiting the sides that contribute to $\Gamma_c(\omega)$ is that the edges of each individual factor by themselves cover the value set boundary of that factor. Although, in general, an N-dimensional polytope has $N2^{N-1}$ edges, at a given frequency, at most $2N$ of these edges need be considered for constructing the value set boundary of the polytope. These special $2N$ edges are easily characterized (see [18]). Thus, the number of contributing sides is even smaller than that specified in Theorem 2.1. However, to prevent notational encumbrances, henceforth we will adhere to the somewhat conservative characterization given in Theorem 2.1.

Provided in (2.24), $\beta_i \neq 0$, $\forall i \in \{1, \cdots, k\}$, we will call the affine line in (2.27), below, the line associated with Q

$$L_a(Q) := \{(q_1\, q_2 \cdots q_k)^T = \rho(\beta_1\, \beta_2 \cdots \beta_k)^T - (\alpha_1\, \alpha_2 \cdots \alpha_k)^T : -\infty < \rho < \infty\}. \quad (2.27)$$

The situation where one or more of the β_i equal zero will be dealt with later. Observe that the intersection of $L_a(Q)$ with the *interior* of Q is given by the segment

$$L(Q) = \{q = (q_1 \cdots q_k)^T : q_i = \rho\beta_i - \alpha_i, \forall\, i \in \{1, \cdots, k\},\ \rho^-(Q) < \rho < \rho^+(Q),\ q \in Q\} \quad (2.28)$$

where

$$\rho^-(Q) = \max_{i \in \{1, \cdots, k\}} \min\left\{\frac{q_i^+ + \alpha_i}{\beta_i}, \frac{q_i^- + \alpha_i}{\beta_i}\right\} \quad (2.29)$$

and

$$\rho^+(Q) = \min_{i \in \{1, \cdots, k\}} \max\left\{\frac{q_i^+ + \alpha_i}{\beta_i}, \frac{q_i^- + \alpha_i}{\beta_i}\right\} \quad (2.30)$$

where q_i^+ and q_i^- are the extreme values of the variable q_i. Note the following important facts.

Fact 3: If $\rho^+(Q) \leq \rho^-(Q)$, then this set is empty, in which case, as will be shown soon, the interior of Q contributes nothing to $\Gamma_c(\omega)$.

Fact 4: As the α_i, β_i depend on ω (Fact 1), so does the line $L(Q)$. Equally, while at some frequencies, $L(Q)$ may not be empty, at others, it may well be.

Fact 5: As noted earlier, whenever Q conforms to the requirements of Theorem 2.1, so do all its boundaries. It is easy to see from the foregoing definitions that the line associated with a boundary of Q is, in fact, a projection of the above line onto this boundary.

We now present the main result of this section, proved in Appendix B.

Theorem 2.2: Consider Q a boundary of Γ obeying the conditions of Theorem 2.1, with the various quantities as defined in the foregoing and $f_0 \neq 0$ (see Fact 2).

i) Suppose in (2.24) that, for some $i \in \{1, \cdots, k\}$, $\beta_i = 0$. Then for every q in the interior of Q such that $f(q) \in \partial(f(Q))$, there exists a q^* on an edge of Q such that $f(q) = f(q^*)$. ii) Suppose $\beta_i \neq 0$, $\forall i \in \{1, \cdots, k\}$, and there exists$q$ in the interior of Q such that $f(q) \in \partial(f(Q))$. Then $q \in L(Q)$. iii) Suppose, in addition to the conditions set out in ii)

$$\sum_{i=1}^{k} k_i = 0. \quad (2.31)$$

Then there exists q^* on a boundary of Q such that for all $q \in L(Q)$

$$f(q) = f(q^*). \quad (2.32)$$

Before illustrating this theorem with the example of (2.18), we highlight some of its features. First of all, if any $\beta_i = 0$, then no point in the interior of Q is included in the formation of $\Gamma_c(\omega)$. For, given any γ in the interior of Q obeying (2.16), there is a parameter vector on an *edge* of Γ that has the same image in the value set space as does γ. Since $\Gamma_c(\omega)$ already includes the edges of Γ, the image of this γ stands covered from the outset.

Likewise, if (2.31) holds, then all interior points of Q mapping to the value set boundary have the same image on this boundary, an image also shared by another parameter vector lying in the interior of a boundary of Q. Thus, these points need not be included as they will be covered when one considers the boundaries of Q.

Further, in the event that i), iii) of the theorem do not hold, then the critical subset contributed by interior points of Q lies exclusively on the associated line segment. Thus, indeed, at every ω, $\Gamma_c(\omega)$ comprised exclusively of line segments, at most one for each side conforming to the requirements of Theorem 2.1. Of course, these segments in general vary with ω, and should their intersection with an associated Q be empty, then the interior of that Q fails to contribute elements to $\Gamma_c(\omega)$.

Finally, we note that the equations describing the internal line segments do not depend on the powersμ_i and ν_i.

We now illustrate these results by invoking the example of (2.18). In the example, assume $\omega = 2$, $\gamma_{ij}^- = 0$, and $\gamma_{ij}^+ = 1$.

Consider first any side on which γ_{31} or γ_{32} are variables [recall that if both were variables, then such a side will not contribute an interior point to $\Gamma_c(\omega)$]. Then i) of Theorem 2.2 holds, and no point in the interior of this boundary is included in $\Gamma_c(\omega)$.

Next, consider the 2-side defined by $\gamma_{31} = \gamma_{41} = 0$ and $\gamma_{32} = \gamma_{12} = \gamma_{22} = 1$. Then, on this boundary, with $q_1 = \gamma_{11}$ and $q_2 = \gamma_{21}$, (2.31) holds. Consequently, this side too contributes no interior points to $\Gamma_c(\omega)$.

Finally, consider the 3-side $\gamma_{12} = \gamma_{21} = 1$ and $\gamma_{31} = \gamma_{32} = 0$. Choose $q_1 = \gamma_{11}$, $q_2 = \gamma_{22}$, and $q_3 = \gamma_{41}$. Then, we have

$$f(q) = 4j\frac{q_1 + 0.5 + j}{[q_2 + 1 + 1.5j][q_3 + j]}. \tag{2.33}$$

Then the interior points of such a 3-side to be included in $\Gamma_c(\omega)$ are given by the segment

$$L(Q) = \{[q_1, q_2, q_3] = \rho[1, 1.5, 1] - [0.5, 1, 0] : 2/3 < \rho < 1\}. \tag{2.34}$$

C. Hurwitz Invariance

Having shown that $\partial(H(j\omega, \Gamma))$ is mapped from the critical line segments in Γ, we now turn to verifying the Hurwitz invariance of (1.7). Essentially, $\partial(H(j\omega, \Gamma))$ must be checked for zero exclusion.

In view of the definition of $\Gamma_c(\omega)$, the Hurwitz invariance of (1.7) is equivalent to the requirement that: i) at least one member of (1.7) is Hurwitz, and ii) that at every ω and all $\gamma \in \Gamma_c(\omega)$

$$h(j\omega, \gamma) \neq 0. \tag{2.35}$$

Since at every ω, $\Gamma_c(\omega)$ comprises the edges of Γ and certain ω-specific line segments, to check for Hurwitz invariance, it suffices to check that all transfer functions corresponding to the edges of (1.7) are Hurwitz, and that the image, in the value set space, of each aforementioned ω-specific line segment is zero exclusive. The principal contribution of this section is to demonstrate that the zero exclusivity of these segments can be verified by checking certain piecewise continuous and piecewise differentiable functions of ω for avoidance of the negative real axis.

To avoid notational complexities, we show this fact in a somewhat informal fashion. Consider a side Q of Γ, which meets the requirements implicit in Theorem 2.1. Recall that for each such Q, potentially there exists an internal line segment that contributes to $\Gamma_c(\omega)$.

For a prototype Q, the internal line segment is as given in (2.28)–(2.30), with the value set $G(\omega, \Gamma)$ as in (2.24). Observe that the various quantities in these equations depend on ω, and that at certain frequencies, this segment could be empty (Facts 3 and 4), or may not be a member of $\Gamma_c(\omega)$ (e.g., when i) and/or iii) of Theorem 2.2 holds, or when f_0 (see (2.24) and Fact 2) or $g_0(j\omega)$ is zero. Call the set of frequencies at which $L(Q)$ is a nonempty subset of $\Gamma_c(\omega)$, $\Omega(Q)$. Then we need to check that the image in the value set space of $L(Q)$ is zero exclusive at all $\omega \in \Omega(Q)$. Direct substitution of (2.28)–(2.30) into (2.24), together with the relation among $G(\omega, \Gamma)$, $H(\omega, \Gamma)$, and (2.24), shows that at all $\omega \in \Omega(Q)$, the image of $L(Q)$ is given by (2.36)–(2.38). In these equations, to avoid confusion, unlike (2.28)–(2.30) and (2.24) the various quantities have been expressed explicitly as functions of ω and Q

$$h(j\omega, \gamma) = 1 + F(Q, \omega)(j + \rho)^{M(Q)}; \quad \rho^-(Q, \omega) < \rho < \rho^+(Q, \omega) \tag{2.36}$$

where

$$F(Q, \omega) = g_0(j\omega) f_0(Q, \omega) \prod_{i=1}^{k(Q)} \beta_i(Q, \omega)^{k_i(Q)} \tag{2.37}$$

and

$$M(Q) = \sum_{i=1}^{k(Q)} k_i(Q). \tag{2.38}$$

Recall that the choice of Q (see (2.31) and the discussion subsequent to Theorem 2.2) ensures that $M(Q) \neq 0$. Moreover, from the definition of $\Omega(Q)$,

$$F(Q, \omega) \neq 0, \qquad \forall \omega \in \Omega(Q) \tag{2.39}$$

and

$$\rho^-(Q, \omega) < \rho^+(Q, \omega), \qquad \forall \omega \in \Omega(Q). \tag{2.40}$$

We claim that the bounds $\rho^+(Q, \omega)$ and $\rho^-(Q, \omega)$ are continuous and piecewise differentiable in these frequency ranges. To see this, we note that $\alpha_i(Q, \omega)$ and $\beta_i(Q, \omega)$ [see (2.24)] are continuous and differentiable. The minmax functions in (2.29)–(2.30) preserve continuity, and the lack of differentiability occurs at the isolated frequencies where the minmax selections "change."

Thus, from (2.36), one can see that (2.35) holds for all γ in the interior of this prototype Q and belonging to $\Gamma_c(\omega)$ iff

$$\mu(Q, \omega) := (-F(Q, \omega))^{-1/M(Q)} - j \notin (\rho^-(Q, \omega), \rho^+(Q, \omega)), \qquad \forall \omega \in \Omega(Q). \tag{2.41}$$

In the above, if $M(Q) \neq \pm 1$, then all the roots of $(\cdot)^{-1/M(Q)}$ should be considered. Define the functions

$$\xi(Q, \omega) = \begin{cases} \frac{\mu(Q, \omega) - \rho^+(Q, \omega)}{\mu(Q, \omega) - \rho^-(Q, \omega)} & \forall \omega \in \Omega(Q) \\ 1 & \text{otherwise.} \end{cases} \tag{2.42}$$

The functions $\xi(Q, \omega)$ defined above are piecewise continuous and differentiable. Moreover, $\forall \omega \in \{(-\infty, \infty) - \Omega(Q)\}$, $\xi(Q, \omega) = 1$. Recall that, at these frequencies, there are no contributions from the interior of Q to $\Gamma_c(\omega)$, and (2.41) need not be checked. Furthermore, the required zero exclusion of $h(j\omega, \gamma)$ for all γ in the interior of this prototype Q and belonging to $\Gamma_c(\omega)$ is equivalent to

$$\xi(Q, \omega) \notin (-\infty, 0). \tag{2.43}$$

We therefore have the following theorem.

Theorem 2.3: The family of transfer functions $H(\Gamma)$ described in (1.7) is Hurwitz invariant if and only if the following conditions hold:

a) $h(s, \gamma)$ is Hurwitz for all γ in the edges of Γ; b) For each $\omega \in \boldsymbol{R}$, and all Q satisfying the requirements of Theorem 2.1, the piecewise continuous and differentiable functions $\xi(Q, \omega)$ defined in (2.42) avoid the negative real axis.

III. Real Zero-Pole-Gain Variations

We now consider the special case of (1.2) where the uncertain parameters are real poles, zeros, and gains.

Consider the family of transfer function $H(\Gamma)$ described by (1.8) and (1.2). The parameters d and λ_i vary independently within given bounds, i.e.,

$$d^- \le d \le d^+;\ \lambda_i^- \le \lambda_i \le \lambda_i^+, \qquad i = 1, 2, \cdots, n-1 \tag{3.44}$$

and

$$\gamma \in \Gamma = [\lambda_1^-, \lambda_1^+] \times \cdots \times [\lambda_{n-1}^-, \lambda_{n-1}^+] \times [d^-, d^+]. \tag{3.45}$$

Notice that Assumption 1.3 implies that if a given k_i is negative, the corresponding interval of λ_i cannot include zero.

Recall, from Section II, that apart from the edges of Γ, we need to consider an internal segment associated with each side of Γ, which conforms to the requirements of Theorem 2.1. Observe that every k-side Q of Γ, $k \ge 1$ conforms to this requirement. Consider, now, two possible cases of such k-sides Q.

Case I: The interior of Q has $d$as a variable.

Observe that, with $q_1 = d$ in (2.24), $\beta_1 = 0$. Thus, because of i) of Theorem 2.2, no parameter vector in the interior of such a Q is included in $\Gamma_c(\omega)$. Consequently, for each $\gamma \in \Gamma_c(\omega)$, d is at an extreme value.

Case II: The variables of Q exclude d

Assume the variables in Q are for some $S \subset \{1, \cdots, n-1\}$, λ_i, $i \in S$. Observe, in (2.24), that each $\beta_i = \omega$ and $\alpha_i = 0$. From i) of Theorem 2.2, one concludes that at $\omega = 0$, $\Gamma_c(\omega)$ comprises only the edges of Γ. Also from Theorem 2.2, when $\omega \ne 0$, a parameter vector in the interior of Q is in $\Gamma_c(\omega)$ only if it obeys

$$\lambda_i = \rho\omega, \forall \rho^-(Q) < \rho < \rho^+(Q), \qquad i \in S \tag{3.46}$$

where

$$\rho^-(Q) = \max_{i\in S} \lambda_i^- \tag{3.47}$$

and

$$\rho^+(Q) = \min_{i \in S} \lambda_i^+. \tag{3.48}$$

In addition [ii) of Theorem 2.2], we must have

$$\sum_{i\in S} k_i \ne 0. \tag{3.49}$$

Thus, we have the following theorem.

Theorem 3.1: Consider the parameter box Γ in $\boldsymbol{R}^n$ given in (3.45). Then $\Gamma_c(\omega)$ is frequency invariant, and comprises the edges of Γ, and all line segments of the form in (3.51)–(3.53), for every $S \subset \{1, \cdots, n-1\}$, obeying (3.49) and

$$\max_{i\in S} \lambda_i^- \le \lambda_j \le \min_{i\in S} \lambda_i^+, \qquad \forall j \in S. \tag{3.50}$$

$$\lambda_i = \lambda_j, \qquad \forall i, j \in S \tag{3.51}$$

$$\lambda_j \in \{\lambda_j^+, \lambda_j^-\}, \qquad \forall j \notin S \tag{3.52}$$

$$d \in \{d^+, d^-\}. \tag{3.53}$$

Furthermore, the family of functions $H(\Gamma)$ is Hurwitz invariant if and only if $h(s, \gamma)$ is Hurwitz invariant on these line segments and the edges of Γ.

A. A Special Case

Notice that if some of the λ_i vary in nonoverlapping intervals, then the number of critical segments reduces, as can be easily seen from (3.51). A case of special interest is when all the poles and zeros vary in intervals that do not overlap. In such an event, the projections mentioned above are empty, and edges suffice for value set boundary and Hurwitz invariance. One thus obtains the corollary below, itself a variation of the results (see Remark 3.3 for comparison) of [25], [26].

Corollary 3.1 (An Edge Theorem): Consider the parameter box Γ in (3.45) and the family of function $H(\Gamma)$ described by (1.8). Suppose $(\lambda_i^-, \lambda_i^+) \cap (\lambda_j^-, \lambda_j^+)$ are empty for all $1 \le i < j < n$. Then the boundary of the value set $H(\omega, \Gamma)$ at any frequency ω is mapped from the edges of Γ. Furthermore, $H(\Gamma)$ is Hurwitz invariant if and only if all the edges of $H(\Gamma)$ are Hurwitz invariant.

Remark 3.1: For overlapping intervals of λ_i, however, the 45-degree line segments are indeed necessary. To show this, we provide the following simple example. Consider

$$\begin{aligned} &h(s, \lambda_1, \lambda_2)\\ &\quad = 1 + \frac{0.1(0.8s^2 + 0.8s + 4.5)(s + \lambda_1)(s + \lambda_2)}{s^4 + 10s^3 + 11.8s^2 + 11.8s + 0.2},\\ &\quad\ \lambda_1, \lambda_2 \in [-30, 0]. \end{aligned} \tag{3.54}$$

In this example, there are four edges with the associated transfer functions given by

$$h(s, 0, \lambda_2), \lambda_2 \in [-30, 0];$$

$$h(s, -30, \lambda_2), \lambda_2 \in [-30, 0];$$

$$h(s, \lambda_1, 0), \lambda_1 \in [-30, 0];$$

$$h(s, \lambda_1, -30), \lambda_1 \in [-30, 0].$$

There is only one 45-degree line segment given by

$$h(s, \lambda, \lambda), \lambda \in [-30, 0].$$

It is straightforward to verify that the transfer functions on all the edges are Hurwitz, but some on the 45-degree line segment are not. For example, at $\lambda = -15$, $h(s, \lambda, \lambda)$ has the unstable zeros $0.2424 \pm 1.8914j$.

Remark 3.2: In actual fact, the result in [26] is stronger. It requires that only $2n$ edges be considered. This fact is also recoverable from [27], which also employs the Jacobian rank deficiency approach underlying our development.

Remark 3.3: Neither [26] nor [25] deals with the overlapping root situation considered in Theorem 3.1 above; nor do [26] and [25] permit the structural preservation of pole-zero multiplicities.

IV. Complex Zero-Pole-Gain Variations

In this section, the complex zero-pole-gain variations case of (1.9) is treated. As before, $\Gamma_c(\omega)$ comprise line segments.

We note that the case $\omega = 0$ is trivial. At this frequency, for every Q obeying Theorem 2.1, all the β_i in (2.24) are zero. Thus, from Theorem 2.2, $\Gamma_c(0)$ comprises the edges of Γ only, and $\omega \neq 0$ will be assumed in the sequel.

The characterization of the line segments follows as in Theorems 2.1 and 2.2. In this section, we will focus mainly on showing that these frequency-dependent line segments obey the confinement rules stated in the Introduction. To this end, we consider the various possible Q obeying Theorem 2.1, and consider how the associated line segments change with frequency. Observe, as in iii) of Theorem 2.2, that certain sides Q can be eliminated according to the combinations of powers of their defining factors. In the sequel, we will only consider Q on which (2.31) does not hold. Then the only restriction that Theorem 2.2 places on Q is that for no i can both a_i and b_i simultaneously be variables. Consider, now, the following possible cases of Q.

Case 1: d is a variable on Q As in Section III, such a Q does not contribute interior points to $\Gamma_c(\omega)$.

Here onwards, we assume d is at an extreme value. In the sequel, without loss of generality, all nonvariables will be denoted with a superscript "$\pm$," and the λ_i will always be considered as potential variables.

Case 2: No a_i nor any b_i is a variable. As in Section III, an interior point of Q is in $\Gamma_c(\omega)$ only if it is on a frequency-invariant 45-degree line segment similar to that in Section III. The set of all such segments will be called L_1.

Case 3: No a_i is a variable, but some b_i are. Suppose, as before, that q_i are the variables. Then if $q_i = \lambda_i$, in the corresponding factor in (2.24), $\alpha_i = 0$ and

$$\beta_i = \omega. \tag{4.55}$$

Further, if $q_i = b_i$, then in the corresponding factor in (2.24)

$$\alpha_i = -\omega^2 \tag{4.56}$$

and

$$\beta_i = \omega a_i^{\pm}. \tag{4.57}$$

Thus, on the line segment associated with Q, the variable λ_i and b_i obey

$$\lambda_i = 0 + \rho\omega \tag{4.58}$$

$$b_i = -\omega^2 + \rho\omega a_i^{\pm}. \tag{4.59}$$

Defining $\rho_1 = \rho\omega$ and $\rho_2 = \omega^2$, one finds that all interior points of Q to be included in $\Gamma_c(\omega)$ lie on the two-dimensional plane,

$$\gamma = \rho_1 C_1 + \rho_2 C_2 \tag{4.60}$$

where $C_1, C_2 \in \boldsymbol{R}^N$ are frequency-independent constant vectors.

Case 4: No b_i is a variable. If $q_i = a_i$, then in the corresponding factor in (2.24)

$$\alpha_i = 0 \tag{4.61}$$

and

$$\beta_i = \omega - \omega^{-1} b_i^{\pm}. \tag{4.62}$$

Then, as long as all the $\beta_i \neq 0$, on the line segment associated with Q, the variable λ_i obey (4.58) and a_i obey

$$a_i = 0 + \rho(\omega - \omega^{-1} b_i^{\pm}). \tag{4.63}$$

Thus, again, this segment is confined to a plane of the form of (4.60) with different C_1 and C_2. The corresponding ρ_1 and ρ_2 in this case should be defined by $\rho_1 = \rho\omega$ and $\rho_2 = \rho\omega^{-1}$.

Case 5: For some i, a_i is a variable, for some others, b_i is a variable. As before, on the line segment associated with Q, the variable λ_i, b_i, and a_i, respectively, obey (4.58), (4.59), and (4.63). Then with $\rho_1 = \rho\omega$, $\rho_2 = \omega^2$, $\rho_3 = \rho\omega^{-1}$, and C_i, $i = 1, 2, 3$ frequency-independent constant vectors (C_1, C_2 different from those in Case 3), this segment can be seen to lie on a three-dimensional plane of following form

$$\gamma = \rho_1 C_1 + \rho_2 C_2 + \rho_3 C_3. \tag{4.64}$$

The set of two-dimensional planes covered by Cases 3 and 4 will be denoted L_2, while the set of boxes in Case 5 will be called L_3. The zero exclusion principle then immediately yields the following theorem.

Theorem 4.1: With L_1, L_2, and L_3 defined as above, the family of transfer functions $H(\Gamma)$ described by (1.9) is Hurwitz invariant if and only if all the edges, the internal segments in L_1, the rectangles in L_2, and the boxes in L_3 are Hurwitz invariant.

We note that the sets in L_1, L_2, and L_3 are easily characterized from the critical frequency-dependent segments they contain. As in Section III, many of these segments, rectangles, and boxes will be empty.

V. Conclusions

In this paper, we have considered robust stability verification of linear time-invariant systems characterized by the class of nonlinear parametric perturbations given in (1.7). In light of the zero exclusion principle, our focus has been on both the verification of Hurwitz invariance and the construction of value sets for the system characteristic function. The main result on construction of value sets shows that for the class of nonlinear parametric perturbations given in (1.7), the value set boundary of the characteristic function at each fixed frequency is determined by the edges and some frequency-dependent internal line segments in the parameter box. This result greatly simplifies the construction of the value sets, and considerably eases the task of robust stability verification. Indeed, a piecewise continuous and differentiable frequency sweeping function is found such that Hurwitz invariance of the set in question is equivalent to this function's avoidance of the negative real axis. For the special case of real zero-pole gain variations, the critical line segments are all frequency independent; hence, the determination of robust stability is even simpler. For the case of complex zero-pole gain variations, the critical internal lines are either frequency independent or vary in certain (two-dimensional) planes or (three-dimensional) boxes.

The key device used in our development concerns a Jacobian function which helps isolate certain critical subsets of the parameter box whose elements collectively determine the value set boundary. This Jacobian-based technique may provide an effective tool for the robust stability analysis of sets which are even more general than the ones considered here. Indeed, a similar device is featured in [7], [28], [27] in relation to the multilinear problem.

Appendix A
Proof of Lemma 2.1

We prove this result in two parts. First, we show that for D a bounded and closed set of complex numbers,

$$(\partial D)^{(k)} \supset \partial(D^{(k)}). \tag{A.65}$$

We then show that for $D_1, D_2, \cdots, D_\sigma$ bounded and closed sets of complex numbers, with

$$\prod_{i=1}^{\sigma} D_i := \left\{ \prod_{i=1}^{\sigma} d_i : d_i \in D_i : i = 1, 2, \cdots, \sigma \right\} \tag{A.66}$$

$$\prod_{i=1}^{\sigma}(\partial D_i) \supset \partial(\prod_{i=1}^{\sigma} D - i).$$

Together, these two parts prove the result.

Proof of Part 1: Given any complex number $z \in \partial(D^{(k)})$, we need to show that $z \in (\partial D)^{(k)}$. By the boundedness of $D^{(k)}$ (from that of D), there exists a sequence of complex numbers $\{z_j\}$ outside $D^{(k)}$ such that $z_j \to z$ as $j \to \infty$. Since $D^{(k)}$ is closed (as D is closed), there exists some $d \in D$ such that $d^k = z$. Define $d_{1j}, d_{2j}, \cdots, d_{kj}$ to be the kth roots of z_j. Then for all i, j, $d_{ij} \notin D$ because $z_j = (d_{ij})^k \notin D^{(k)}$. On the other hand, the sequence $\{d_{ij}\}$ has a subsequence converging to $d (\in D)$. It follows that $d \in \partial D$, or equivalently, $z = d^k \in (\partial D)^{(k)}$.

Proof of Part 2: Given any complex number $z \in \partial(\prod_{i=1}^{\sigma} D_i)$, we need to show $z \in \prod_{i=1}^{\sigma}(\partial D_i)$. This clearly holds if one of the D_i is just $\{0\}$. So, assume every D_i contains at least one nonzero element. This implies that every ∂D_i has at least one nonzero element. By the boundedness of $(\prod_{i=1}^{\sigma} D_i)$ (from that of D_i), there exists a sequence of complex numbers $\{z_j\}$ outside $(\prod_{i=1}^{\sigma} D_i)$ such that $z_j \to z$ as $j \to \infty$. Since $(\prod_{i=1}^{\sigma} D_i)$ is closed (as D_i is closed), there exist some $d_i \in D_i$, $i = 1, 2, \cdots, \sigma$ such that $z = d_1 d_2 \cdots d_\sigma$. If z is zero, it follows that all d_i, except for one which is set to zero, can be chosen to be a nonzero boundary point of ∂D_i. In the sequel, we assume that at most one d_i is zero. We claim that $d_i \in \partial D_i$, $i = 1, 2, \cdots, \sigma$. Without loss of generality, consider d_1. From the foregoing, one can choose $d_1 \in \partial D_1$ if one of the other d_i is zero. On the other hand, if the remaining d_i are nonzero, we define

$$d_{1j} := \frac{z_j}{d_2 \cdots d_\sigma}, \qquad j = 1, 2, \cdots. \tag{A.67}$$

Then, d_{1j} converge to d_1 as $j \to \infty$ because z_j converge to z. Yet $d_{1j} \notin D_1$ for all j because $z_j = d_{1j} d_2 \cdots d_\sigma \notin \prod_{i=1}^{\sigma} D_i$. Therefore, $d_1 \in \partial D_1$. Similarly, it can be shown that $d_i \in \partial D_i$ for all other i; hence, the result holds.

Appendix B
Proof of Theorem 2.2

To prove the theorem, we need three lemmas. The first of these is well known (see, e.g., [28], [27]), and hence its proof is omitted.

Lemma B.1: Consider the hyperrectangle

$$Q := \{q = (q_1\, q_2 \cdots q_m)^T : \\ q_i^- \le q_i \le q_i^+,\ i = 1, 2, \cdots, m\} \subset \boldsymbol{R}^m \tag{B.68}$$

and a differentiable complex valued function $f(\cdot): Q \to \mathcal{C}$ with all its first derivatives continuous. In the sequel, we will denote

$$f(Q) := \{f(q) : q \in Q\}. \tag{B.69}$$

Then a point q in the interior of Q obeys $f(q) \in \partial f(Q)$ only if the following Jacobian matrix, evaluated at q, has rank less than two

$$J_f(q) := \begin{bmatrix} \mathrm{Re}\left(\frac{\partial f(q)}{\partial q_1}\right) & \mathrm{Re}\left(\frac{\partial f(q)}{\partial q_2}\right) & \cdots & \mathrm{Re}\left(\frac{\partial f(q)}{\partial q_m}\right) \\ \mathrm{Im}\left(\frac{\partial f(q)}{\partial q_1}\right) & \mathrm{Im}\left(\frac{\partial f(q)}{\partial q_2}\right) & \cdots & \mathrm{Im}\left(\frac{\partial f(q)}{\partial q_m}\right) \end{bmatrix}. \tag{B.70}$$

It should be noted that the lemma above is of little value when $f(Q)$ degenerates to a real segment. For when $f(\cdot)$ is real, the second row of the matrix in (B.70) is identically zero, and hence rank deficiency occurs for all q. For such a case, we have the following result.

Lemma B.2: Consider the hyperrectangle Q in (B.68) and a real continuous function $f(\cdot): Q \to \boldsymbol{R}$. Then $f(Q)$ can be mapped from the edges of Q if and only if the two extreme points (minimum and maximum) of $f(Q)$ can be mapped from the edges of Q.

Proof: Necessity is obvious. To prove sufficiency, suppose q^1 and q^2 are the edge points corresponding to the extreme points of $f(Q)$. Observe that q^1 and q^2 can be connected by a path entirely in the edges of Q. By continuity of $f(\cdot)$, the image of this path, which is a subset of the edges, covers the whole of $f(Q)$.

The final lemma needed is given below.

Lemma B.3: Consider the hyperrectangle Q and the bounded set $f(Q)$ in (2.24) with $\beta_i = 0, \forall i$. Then each point in $f(Q)$ has at least one preimage in the edges of Q.

Proof: Observe that the result will not be affected by the value of f_0. So choose $f_0 = 1$. Using Lemma B.2, we simply need to show that both the minimum and the maximum of $f(Q)$ have preimages in the edges of Q. Take the minimum, for example, as the maximum can be dealt with in the same manner. Denote the minimum by f_m and consider the two cases: 1) $f_m = 0$; and 2) $f_m \neq 0$. Case 1) implies that some $q_i + \alpha_i$ is zero with $k_i > 0$. In this case, it is obvious that setting the other q_i at extreme values does not change f_m. In Case 2), we claim that all the q_i must take their extreme value. Indeed, if some q_i were not at its extreme, f_m would not be minimum because we can decrease the value of the function by increasing or decreasing this q_i. Therefore, in both cases, f_m can be achieved at an edge point.

Proof of Theorem 2.2: Consider a point q in the interior of Q such that

$$f(q) \in \partial f(Q). \tag{B.71}$$

Proof of i): Suppose first that $\beta_i = 0$, $\forall i$. Then, Lemma B.3 proves the conclusions of i). Next, suppose that at least one β, without loss of generality β_1, is nonzero. Suppose also, that for some q in the interior of Q, $f(q) \in \partial f(Q)$. Then from Lemma B.1, $\forall i, j \in \{1, \cdots, k\}$, there exist real scalars x and y, not both zero, such that

$$x\frac{\partial f(q)}{\partial q_i} = y\frac{\partial f(q)}{\partial q_j}. \tag{B.72}$$

Thus, from (2.24), the above gives

$$x\frac{k_i f(q)}{q_i + \alpha_i + j\beta_i} = y\frac{k_j f(q)}{q_j + \alpha_j + j\beta_j} \tag{B.73}$$

which simplifies to

$$(q_i + \alpha_i)\beta_j = (q_j + \alpha_j)\beta_i. \tag{B.74}$$

Now, suppose that at least one β_i, $i \neq 1$. without loss of generality β_2, equals zero. Then with $i = 1$, $j = 2$, one has from (B.74) that

$$q_2 + \alpha_2 = 0. \tag{B.75}$$

Thus, $f(q) = 0$. Moreover, this holds no matter what value the q_i, $i \neq 2$ take. Setting these q_i, $i \neq 2$ to their respective extreme values, one proves i).

Proof of ii): Follows from (B.74).

Proof of iii): Direct substitution of (B.74) into (2.24) yields

$$f(q) = c(\rho + j)^M \tag{B.76}$$

where c is a suitable complex constant and the integer M is given by

$$M = \sum_{i=1}^{k} k_i. \tag{B.77}$$

Thus, when $M = 0$, $f(q)$ has the same value on the whole segment $L(Q)$. Thus, by continuity, the image of this entire segment is covered by any one of its endpoints which is on a boundary of Q.

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