Robust Relative Stability of Time-Invariant and Time-Varying Lattice Filters

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***Abstract*—We consider the relative stability of time-invariant and time-varying unnormalized lattice filters. First, we consider a set of lattice filters whose reflection parameters α_i obey $|\alpha_i| \leq \delta_i$ and provide necessary and sufficient conditions on the δ_i that guarantee that each time-invariant lattice in the set has poles inside a circle of prescribed radius $1/\rho < 1$, i.e., is relatively stable with degree of stability $\ln \rho$. We also show that the relative stability of the whole family is equivalent to the relative stability of a single filter obtained by fixing each α_i to δ_i and can be checked with only the real poles of this filter. Counterexamples are given to show that a number of properties that hold for stability of LTI Lattices do not apply to relative stability verification. Second, we give a diagonal Lyapunov matrix that is useful in checking the above pole condition. Finally, we consider the time-varying problem where the reflection coefficients vary in a region where the frozen transfer functions have poles with magnitude less than $1/\rho$ and provide bounds on their rate of variations that ensure that the zero input state solution of the time-varying lattice decays exponentially at a rate faster than $1/\rho_1 > 1/\rho$.**

***Index Terms*—Lattice filters, Lyapunov, robustness, stability, time-vaying filters.**

I. Introduction

THIS PAPER explores the relative stability of linear time-invariant (LTI) and linear time-varying (LTV) lattice filters. Lattice filters have been studied extensively in the last two decades. They bear a direct relationship to the celebrated Levinson–Durbin algorithm [1] and have been applied in speech processing and linear predictive coding [2].

An nth-order Lattice filter is depicted in Fig. 1, where q^{-1} is the unit delay element.

Here, the $\alpha_i(k)$ are called the reflection coefficients; the time index k used with these recognizes our intention to study the time-varying lattice, and q^{-1} is a unit delay. As is evident from this figure, the various signals obey, for $1 \leq i \leq n$

$$\begin{bmatrix} y_i(k) \\ w_i(k) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_i(k) \\ -\alpha_i(k) & 1-\alpha_i^2(k) \end{bmatrix} \begin{bmatrix} y_{i+1}(k) \\ u_i(k) \end{bmatrix} \tag{1.1}$$

for $1 \leq i \leq n-1$

$$u_{i+1}(k) = w_i(k-1) \tag{1.2}$$

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and

$$\begin{aligned} y(k) &= w_n(k) \\ y_{n+1}(k) &= u(k) \\ u_1(k) &= y_1(k-1). \end{aligned} \tag{1.3}$$

In the sequel, the LTI version of (1.1)–(1.3) will refer to the case in which each $\alpha_i(k)$ equals a constant α_i for all k. In this case, $G(z, \alpha_1, \cdots, \alpha_n)$ will be the transfer function $Y(z)/U(z)$.

Since we are interested in relative stability, we first make precise our notion of relative stability. As we deal with systems that are time varying, we use a state variable realization (SVR)-based approach to stability analysis.

Definition 1.1 (Relative Stability): Consider the LTV system with SVR $\{A(k), b(k), c(k), d(k)\}$, i.e., obeying

$$x(k+1) = A(k)x(k) + b(k)u(k) \tag{1.4}$$

$$y(k) = c(k)x(k) + d(k)u(k) \tag{1.5}$$

where $A(k)$, $b(k), c(k)$, and $d(k)$ are, respectively, $n \times n, n \times 1, 1 \times n$, and 1×1 bounded matrices, the $n \times 1$ state is $x(k)$, and $u(k)$ and $y(k)$ are the input and output signals, respectively. Then, (1.4)–(1.5) is relatively stable with a degree of stability $\ln \rho, \rho > 1$ if there exist constants $\beta_1 > 0, 0 \leq \beta_2 < 1$ such that the zero input state solution obeys for all k and initial time k_0

$$\rho^{k-k_0} ||x(k)|| \leq \beta_1 ||x(k_0)|| \beta_2^{k-k_0}. \tag{1.6}$$

where $||\cdot||$ denotes the standard 2-norm.

If $x(k_0) \neq 0$, (1.6) implies

$$\frac{||x(k)||}{||x(k_0)||} \leq \beta_1 \left(\frac{\beta_2}{\rho}\right)^{k-k_0}. \tag{1.7}$$

Thus, relative stability with degree of stability $\ln \rho$ ensures that the zero input state response decays at an exponential rate of at least $1/\rho$. If (1.6) holds, we will sometimes say that (1.4)–(1.5) is ρ stable. If, in (1.6), $\rho = 1$, then we simply call (1.4)–(1.5) stable.

In the LTI case (where A, b, c, and d are constant), (1.4)–(1.5) has transfer function

$$H(z) = c(zI - A)^{-1}b + d. \tag{1.8}$$

In this case, as long as $[A, b]$ is completely reachable and $[A, c]$ completely observable (see [13] for definitions), then (1.4)–(1.5) is ρ stable iff $H(z)$ has poles with magnitude less than $1/\rho$, i.e., $H(z/\rho)$ is stable.

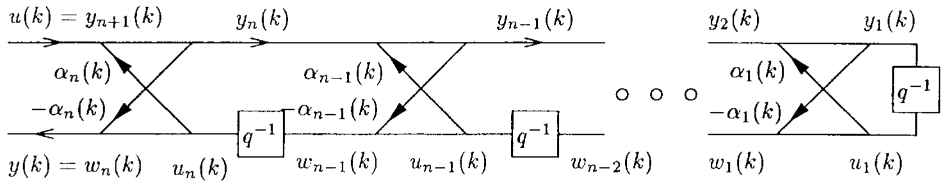

Fig. 1. Lattice filter.

It is known that in the LTI case, where $\alpha_i(k) = \alpha_i$ for all k, the lattice filter is stable iff

$$|\alpha_i| < 1 \quad \forall 1 \le i \le n. \tag{1.9}$$

Furthermore, under (1.2), the lattice transfer function

$$G(z, \alpha_1, \cdots, \alpha_n) = \frac{Y(z)}{U(z)} \tag{1.10}$$

is allpass, i.e., it obeys for all $\omega \in [0, 2\pi)$

$$|G(e^{j\omega}, \alpha_1, \cdots, \alpha_n)| = 1. \tag{1.11}$$

There are, however, two outstanding open issues in the understanding of lattice filters. The first of these concerns the issue of relative stability of the LTI lattice. Simply put, what are the conditions on the reflection coefficients that ensure the ρ stablity of the LTI lattice? Such relative stability, as opposed to mere stability, is important in most practical applications as it reduces the likelihood of quantization induced limit cycles. Further, as will become evident in the sequel, the relative stability of the LTI lattice is also critical to the stability of the LTV lattice.

The second concerns the relative stability of the LTV lattice. It is known that the normalized version of the above lattice [3], [4] is stable under arbitrary time variations in the reflection coefficients as long as they obey

$$|\alpha_i(k)| < 1 \quad \forall i \in \{1, \cdots, n\}, k. \tag{1.12}$$

However, to our knowledge, no nontrivial conditions exist that guarantee the stability, let alone the relative stability, of the LTV unnormalized lattice structure of Fig. 1. In fact, it is well known that the unnormalized LTV lattice could be unstable, despite the satisfaction of (1.12) [4].

This paper considers relative stablity of both the LTI and the LTV lattices depicted in Fig. 1. The two problems addressed are as follows.

Problem 1.1: In Fig. 1, the α_i's are all time invariant, and for some δ_i

$$|\alpha_i| < \delta_i \quad \forall i \in \{1, \cdots, n\}. \tag{1.13}$$

Find necessary and sufficient conditions on the δ_i so that $G(z, \alpha_1, \cdots, \alpha_n)$ are ρ stable with $\rho > 1$, for all α_i, as in (1.13).

Problem 1.2: Suppose for some $\rho > 1$, every LTI system $G(z, \alpha_1, \cdots, \alpha_n)$ obeying (1.13) is ρ stable. Now, suppose the reflection coefficients in Fig. 1 vary with time and obey for some arbitrarily small $\epsilon_i > 0$

$$|\alpha_i(k)| + \epsilon_i < \delta_i < 1, \quad \forall i \in \{1, \cdots, n\}, k. \tag{1.14}$$

Find sufficient conditions on the rate of variations in the $\alpha_i(k)$ such that for some $1 < \rho_1 < \rho$, this LTV lattice is ρ_1 stable.

In analyzing LTV systems, it is, in general, unreasonable to assume complete knowledge of the nature of the time variations. Normally, the knowledge we have is limited to the extent and rate of parameter variations. Equation (1.14) characterizes the extent of variation. Problem 2 then calls for specification of the variation rate.

Observe that effectively, the statement of Problem 2 ensures the ρ stability of all possible frozen LTI lattices corresponding to the LTV lattice being analyzed. Problem 1 addresses the condition under which all such frozen LTI lattices will be ρ stable. Subject to this condition on the frozen LTI lattices, Problem 2 then calls for determining the parameter variation rates that ensure relative stability with a smaller degree $\ln \rho_1$. Although it is important in its own right, Problem 1 is therefore also useful to the analysis of LTV lattices. In Problem 2, there is clearly a natural tradeoff between ρ, ρ_1, and the allowable rate of time variation for which ρ_1 stability is preserved. For a given ρ_1, the larger the ρ, the greater the permissible rate. Our solution to Problem 2 captures this tradeoff, very much in the spirit of [5]. Translated to the digital filter framework, [5] considers digital filters in the direct form. It gives bounds on the logarithmic rate of variation of the filter coefficients that guarantee the relative stability of the underlying LTV system, subject to a relative stability assumption on the frozen systems.

In particular, [5] assumes that the denominator coefficients of the frozen LTI system transfer functions are $a_i, 1 \le i \le n$ and that the time-varying values of these a_i obey, for some $\epsilon_i > 0$

$$a_i^- + \epsilon_i < a_i(k) < a_i^+ - \epsilon_i \qquad 1 \le i \le n \tag{1.15}$$

and that all LTI frozen systems defined by

$$a_i^- < a_i < a_i^+ \qquad 1 \le i \le n \tag{1.16}$$

are ρ stable for some $\rho > 1$. Recall that the a_i directly appear in the direct-form implementations.

Then, with

$$\gamma_i(k) = \frac{a_i(k) - a_i^-}{a_i^+ - a_i(k)} \tag{1.17}$$

the LTV filter is shown in [5] to be ρ_1 stable with $1 < \rho_1 < \rho$ if there exist $N > 0$ and $0 < \beta < 1$ such that

$$\sup_{k \ge 0} \frac{1}{N} \sum_{l=k}^{k+N-1} \sum_{i=1}^{n} \left[\ln \frac{\gamma_i(l+1)}{\gamma_i(l)}\right]^+ < 2 \ln\left(\frac{\rho\beta}{2\rho_1}\right). \tag{1.18}$$

Here

$$[a]^+ = \begin{cases} a & a \ge 0 \\ 0 & \text{else.} \end{cases} \tag{1.19}$$

Note the tradeoff between the degree of the frozen system relative stability $\ln \rho$, the LTV filter degree of stability $\ln \rho_1$, and the *average* rate of variation in the parameters γ_i directly related to the filter coefficients a_i; the γ_i monotonically increase with the a_i. Further, only increases in $\gamma_i(k)$ and, hence, $a_i(k)$ are of concern. Diminishing $a_i(k)$ carry no destabilizing influence. A result of this nature is sought here for the LTV lattice of Fig. 1.

In Section II, we provide some preliminaries. Section III gives a series of results connected to Problem 1. Section IV develops a Lyapunov matrix needed in the solution to Problem 2. Section V then solves Problem 2. Section VI is the conclusion.

II. Preliminaries

This section derives a number of preliminary results and definitions. First, we define the technical concept of uniform complete observability (UCO) [9], which is needed for some of our analysis.

Definition 2.1: The pair of matrix sequences $A(k)$ and $Q(k)$, respectively, $n \times n$ and $m \times n$, is called UCO if there exist $\mu_1, \mu_2 > 0$ and integer N such that for all k

$$\mu_1 I \leq \sum_{i=k}^{k+N-1} \left(\prod_{l=k}^{i} A(l)\right)' Q'(i)Q(i)\left(\prod_{l=k}^{i} A(l)\right) \leq \mu_2 I. \tag{2.1}$$

Here, the products are identity should the lower index exceed the upper, and the order is exemplified by

$$\prod_{l=k}^{i-1}(A(k)) = (A(i-1))(A(i-2))\cdots(A(k)).$$

We next recount a fact from stability theory that provides the principle tool to be used in out LTV analysis.

Theorem 2.1 [6]: Consider (1.4)–(1.5) with the various quantities defined in Definition 1.1. Then, (1.6) holds iff there exists a symmetric $n \times n$ Lyapunov matrix satisfying

$$\mu_3 I \geq P(k) = P'(k) \geq \mu_4 I > 0 \qquad \forall k \tag{2.2}$$

for which

$$\rho^2 A'(k)P(k+1)A(k) - P(k) \leq -Q'(k)Q(k) \tag{2.3}$$

with $Q(k)$ real and $[\rho A(k), Q(k)]$ UCO. In the LTI case of constant A, P will be constant as well.

Much of our analysis relies on the concept of bounded real (BR) transfer functions defined below.

Definition 2.2: An LTI system with transfer function $G(z)$ is BR if $G(z)$ is stable and for all $\omega \in [0, 2\pi)$

$$|G(e^{j\omega})| \leq 1. \tag{2.4}$$

An important tool in robust stability analysis of LTI systems is the *zero exclusion principle* [11], which is presented below.

Proposition 2.1: Consider the set of polynomials

$$\mathcal{A}(z) = \left\{ a(\lambda, z) = \sum_{i=0}^{n} a_i(\lambda) z^{n-i} | \lambda = [\lambda_1, \cdots, \lambda_M]' \right.$$
$$\left. \lambda_i^- \leq \lambda_i \leq \lambda_i^+ \right\} \tag{2.5}$$

with $a_i(\lambda)$ continuous functions of λ and for all $\lambda_i^- \leq \lambda_i \leq \lambda_i^+, a_0(\lambda) \neq 0$. Then, all members of $\mathcal{A}(z)$ are Schur (have zeros strictly inside the unit circle) iff one member is Schur and for all $\omega \in [0, 2\pi)$

$$a(\lambda, e^{j\omega}) \neq 0. \tag{2.6}$$

We next present a recursive formula for determining the transfer function of a lattice filter.

In the sequel (see Fig. 1), we will define

$$G_0(z) = 1 \tag{2.7}$$

and for $1 \leq i \leq n$

$$G_i(z, \alpha_1, \cdots, \alpha_i) = \frac{W_i(z)}{Y_{i+1}(z)}. \tag{2.8}$$

Thus

$$G_n(z, \alpha_1, \cdots, \alpha_n) = G(z, \alpha_1, \cdots, \alpha_n) \tag{2.9}$$

which is the overall transfer function of the lattice. Then, we have, from [12], that for all $0 \leq i \leq n-1$

$$G_{i+1}(z, \alpha_1, \cdots, \alpha_{i+1}) = \frac{z^{-1}G_i(z, \alpha_1, \cdots, \alpha_i) - \alpha_{i+1}}{1 - z^{-1}\alpha_{i+1}G_i(z, \alpha_1, \cdots, \alpha_i)}. \tag{2.10}$$

Further, we will define the transfer function sets $1 \leq i \leq n$

$$\mathcal{G}_0(z) = \{1\} \tag{2.11}$$
$$\mathcal{G}_i(z) = \{G_i(z, \alpha_1, \cdots, \alpha_i) || \alpha_j| \leq \delta_j < 1, 1 \leq j \leq i\}. \tag{2.12}$$

Finally, we present a similar result for LTV lattice SVR's. In the sequel, unless necessary, we will drop the explicit dependence on the α_i.

Define $\{A_i(k), b_i(k), c_i(k), d_i(k)\}$ to be an SVR of the system with input $y_{i+1}(k)$ and output $w_i(k)$, as in Fig. 1 [i.e., the system that in the LTI case has transfer function $G_i(z, \alpha_1, \cdots, \alpha_i)$]. The state vector $x_i(k)$ is the output of the delay elements appearing in the system and is given by

$$x_i(k) = [u_1(k), u_2(k), \cdots, u_i(k)]'. \tag{2.13}$$

In Theorem 2.1, we provide recursions that relate $\{A_{i+1}(k), b_{i+1}(k), c_{i+1}(k), d_{i+1}(k)\}$ to $\{A_i(k), b_i(k), c_i(k), d_i(k)\}$ for $1 \leq i \leq n-1$. The recursion is initiated with the nondynamic system corresponding to $G_0(z)$ in (2.7), i.e., $A_0(k), b_0(k)$, and $c_0(k)$ are zero dimensional objects, and $d_0(k) = 1$.

Theorem 2.2: Consider, with $0 \leq i \leq n-1$, the SVR $\{A_i(k), b_i(k), c_i(k), d_i(k)\}$ of the system with input $w_i(k)$ and output $y_{i+1}(k)$ in Fig. 1, with state vector $x_i(k)$, as in (2.13). Then

$$A_{i+1}(k) = \begin{bmatrix} A_i(k) & b_i(k)\alpha_{i+1}(k) \\ c_i(k) & d_i(k)\alpha_{i+1}(k) \end{bmatrix} \tag{2.14}$$

$$b_{i+1}(k) = \begin{bmatrix} b_i(k) \\ d_i(k) \end{bmatrix} \tag{2.15}$$

$$c_{i+1}(k) = (1-\alpha_{i+1}^2(k))e'_{i+1} \tag{2.16}$$

$$d_{i+1}(k) = -\alpha_{i+1}(k) \tag{2.17}$$

where the $i \times 1$ vector e_i obeys

$$e_i = [0, \cdots, 0, 1]'. \tag{2.18}$$

Proof: By definition

$$x_i(k+1) = A_i(k)x_i(k) + b_i(k)y_{i+1}(k) \tag{2.19}$$

$$w_i(k) = c_i(k)x_i(k) + d_i(k)y_{i+1}(k). \tag{2.20}$$

Further, from (1.1)

$$y_{i+1}(k) = \alpha_{i+1}(k)u_{i+1}(k) + y_{i+2}(k). \tag{2.21}$$

From Fig. 1, (2.20), and (2.21)

$$\begin{aligned} u_{i+1}(k+1) &= w_i(k) \\ &= c_i(k)x_i(k) + d_i(k)y_{i+1}(k) \\ &= c_i(k)x_i(k) + d_i(k)\alpha_{i+1}(k)u_{i+1}(k) \\ &\quad + d_i(k)y_{i+2}(k). \end{aligned} \tag{2.22}$$

Thus, substituting into (2.19), we have

$$\begin{bmatrix} x_i(k+1) \\ u_{i+1}(k+1) \end{bmatrix} = \begin{bmatrix} A_i(k) & b_i(k)\alpha_{i+1}(k) \\ c_i(k) & d_i(k)\alpha_{i+1}(k) \end{bmatrix} \begin{bmatrix} x_i(k) \\ u_{i+1}(k) \end{bmatrix} + \begin{bmatrix} b_i(k) \\ d_i(k) \end{bmatrix} y_{i+2}(k). \tag{2.23}$$

Since, by the definition (2.13), $x_{i+1}(k) = [x'_i(k), u_{i+1}(k)]'$, this proves (2.14), (2.15). Further, from (1.1)

$$\begin{aligned} w_{i+1}(k) &= -\alpha_{i+1}(k)y_{i+2}(k) + (1-\alpha_{i+1}^2(k))u_{i+1}(k) \\ &= -\alpha_{i+1}(k)y_{i+2}(k) + (1-\alpha_{i+1}^2(k))e'_{i+1} \\ &\quad \cdot x_{i+1}(k). \end{aligned} \tag{2.24}$$

This proves (2.16)–(2.17) because of (2.18). ■

An important consequence of this theorem is that if $\alpha_i(k) \neq 0$, then

$$b_i(k) = \frac{1}{\alpha_i(k)} A_i(k) e_i(k). \tag{2.25}$$

Together with the initiating process stated just before the theorem statement, this provides the SVR $\{A_n(k), b_n(k), c_n(k), d_n(k)\}$, of the lattice filter in Fig. 1. As an illustration, observe that

$$\begin{aligned} &\{A_1(k), b_1(k), c_1(k), d_1(k)\} \\ &\quad = \{\alpha_1(k), 1, 1-\alpha_1^2(k), -\alpha_1(k)\}. \end{aligned} \tag{2.26}$$

III. Robust Relative Stability of the LTI Lattice

We call a set of transfer functions *stable invariant* if all its members are stable. In this section, we provide a necessary and sufficient condition for $\mathcal{G}_n(z/\rho)$ to be stable invariant, given $\rho > 1$. Thus, this solves the problem of determining whether each member of $\mathcal{G}_n(z/\rho)$ has degree of stability $\ln \rho$. In addition, where appropriate, we will point out certain salient points on which ρ-stability properties differ from mere stability. A third contribution of this section is to answer the following question. Are there any distinguished members of $\mathcal{G}_n(z)$ whose ρ stability implies the ρ stability of all members of $\mathcal{G}_n(z)$?

It is known that for any $\alpha_1, \cdots, \alpha_n$, $G_n(z, \alpha_1, \cdots, \alpha_n)$ is stable iff for all $1 \leq i \leq n, G_i(z, \alpha_1, \cdots, \alpha_i)$ is stable. Example 3.1 shows this not to be the case for ρ stability in general.

Example 3.1: Consider the lattice filter as in Fig. 1 and (1.1) with

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] = [-0.9, -0.8, -0.4, -0.6, -0.5].$$

Then, we can verify that $G_5(z/\rho, \alpha_1, \cdots, \alpha_5)$ is stable for all $\rho < 1.099$. Yet, $G_4(z/\rho, \alpha_1, \cdots, \alpha_4)$ is unstable for all $\rho > 1.061$.

Nonetheless, Lemma 3.1 below shows that when it comes to verifying the *stable invariance* of the *entire set* $\mathcal{G}_n(z)$, an order reductibility property does hold.

Lemma 3.1: The set $\mathcal{G}_n(z/\rho)$ is stable invariant iff $\mathcal{G}_i(z/\rho)$ is stable invariant for all $1 \leq i \leq n$.

Proof: Sufficiency is clear. To prove necessity, assume that for some $1 \leq i \leq n-1$, $\mathcal{G}_i(z/\rho)$ is not stable invariant. Then, there exists $G_i(z, \alpha_1, \cdots, \alpha_i) \in \mathcal{G}_i(z)$ such that $G_i(z/\rho, \alpha_1, \cdots, \alpha_i)$ is unstable. Now, observe $0 < \delta_{i+1}$. Thus

$$G_{i+1}(z, \alpha_1, \cdots, \alpha_i, 0) \in \mathcal{G}_{i+1}(z). \tag{3.1}$$

Observe from (2.10) that

$$G_{i+1}(z, \alpha_1, \cdots, \alpha_i, 0) = z^{-1} G_i(z, \alpha_1, \cdots, \alpha_i). \tag{3.2}$$

Thus, $G_{i+1}(z/\rho, \alpha_1, \cdots, \alpha_i, 0)$ is unstable. Hence, we have the result. □

The fact that the order reductability property applies to stable invariance of sets such as $\mathcal{G}_n(z/\rho)$ even for $\rho > 1$ is crucially dependent on the fact that the sets $\mathcal{G}_i(z/\rho)$ contain elements involving $\alpha_i = 0$.

Henceforth, we consider the stable invariance of all the $\mathcal{G}_i(z/\rho)$. We are now in a position to state the main result of this section. This result requires that

$$f_0 = 1 \tag{3.3}$$

$$f_i = \frac{\rho f_{i-1} - \delta_i}{1 - \rho \delta_i f_{i-1}}, \qquad i = 1, \cdots, n \tag{3.4}$$

be considered. Then, the necessary and sufficient condition for the stable invariance of the $\mathcal{G}_i(z/\rho), 1 \leq i \leq n$, is as follows.

Theorem 3.1: Consider the sets $\mathcal{G}_i(z), 1 \leq i \leq n$, as defined in (2.8)–(2.12). Then, with $\rho > 1$, $\mathcal{G}_n(z/\rho)$ is stable invariant iff the f_i defined in (3.3)–(3.4) exist and obey, for all $1 \leq i \leq n$

$$0 < \rho f_{i-1}\delta_i < 1. \tag{3.5}$$

Further, under (3.5), for all $1 \leq i \leq n$

$$f_i > \rho f_{i-1}. \tag{3.6}$$

■

The proof of this result is developed in the sequel. However, before embarking on this proof, we make a few pertinent observations.

Note that with $\rho = 1$, the recursion in (3.3)–(3.4) gives $f_i = 1$ for all $1 \leq i \leq n$, and (3.5) boils down to

$$\delta_i < 1 \tag{3.7}$$

which is a fact well known about lattice filters. Note, however, that (3.7) is necessary and sufficient for stability of any $G_n(z, \alpha_1, \cdots, \alpha_n)$, whereas (3.5) is not necessary for the stability of $G_n(z/\rho, \alpha_1, \cdots, \alpha_n)$. Indeed, return to the filter in Example 3.1. $G_5(z/\rho, \alpha_1, \cdots, \alpha_5)$ is stable for $\rho = 1.08$. Yet, for this value of ρ, taking $\delta_i = |\alpha_i|$

$$f_1 = 6.4286$$

and

$$\rho\delta_2 f_1 = 5.5543 > 1.$$

Example 3.1 illustrates a further departure from the $\rho = 1$ case. Despite the fact that for the given α_i, $G_5(z/\rho, \alpha_1, \cdots, \alpha_5)$ is stable for all $\rho < 1.099$, $G_5(z/\rho, |\alpha_1|, \cdots, |\alpha_5|)$ is unstable for $\rho > 1.0004$. Thus, although the stability of a solitary lattice filter is determined entirely by the magnitude of the reflection coefficients, this is not the case for the relative stability of a solitary lattice filter.

Observe that (3.6) implies that

$$f_i > \rho^i \tag{3.8}$$

whence we have that a necessary, although not sufficient, condition for stable invariance of $\mathcal{G}_n(z/\rho)$ is

$$\delta_i < \frac{1}{\rho^i}, \qquad \forall 1 \leq i \leq n. \tag{3.9}$$

Finally, observe that the number of computations needed to check the condition in question grows only linearly with n.

We now turn to proving this theorem through a series of lemmas. The first of these concerns a sequence related to the f_i

$$g_0 = 1 \tag{3.10}$$

$$g_i(|\alpha_1|, \cdots, |\alpha_i|) = \frac{\rho g_{i-1}(|\alpha_1|, \cdots, |\alpha_{i-1}|) - |\alpha_i|}{1 - \rho g_{i-1}(|\alpha_1|, \cdots, |\alpha_{i-1}|)|\alpha_i|}. \tag{3.11}$$

Lemma 3.2: Suppose the set $\mathcal{G}_n(z/\rho)$ is stable invariant. Then, for all $1 \leq i \leq n$ and $|\alpha_i| \leq \delta_i$, $g_i(|\alpha_1|, \cdots, |\alpha_i|)$ exist

$$\rho|\alpha_i|g_{i-1}(|\alpha_1|, \cdots, |\alpha_{i-1}|) < 1 \tag{3.12}$$

and

$$g_i(|\alpha_1|, \cdots, |\alpha_i|) \geq \rho g_{i-1}(|\alpha_1|, \cdots, |\alpha_{i-1}|). \tag{3.13}$$

Proof: Use induction. First, observe that (3.12) guarantees the existence of $g_i(|\alpha_1|, \cdots, |\alpha_i|)$. Now, g_0 clearly exists. Suppose for some $1 \leq l \leq n$ and all $1 \leq i \leq l$, $g_{i-1}(|\alpha_1|, \cdots, |\alpha_{i-1}|)$ exists. Then, as (3.12) holds at $i = l$ and for $\alpha_l = 0$, its violation will imply that for some $|\alpha_i| \leq \delta_i, 1 \leq i \leq l$

$$\rho|\alpha_l|g_{l-1}(|\alpha_1|, \cdots, |\alpha_{l-1}|) = 1. \tag{3.14}$$

For this choice of $\alpha_i, 1 \leq i \leq l$

$$\rho g_i(|\alpha_1|, \cdots, |\alpha_{l-1}|) \neq |\alpha_l| \tag{3.15}$$

as otherwise, $|\alpha_l|^2 = 1$, which violates $\delta_l < 1$. Thus, for this choice of $|\alpha_i| < \delta_i, 1 \leq i \leq l$, from (3.11)

$$g_l(|\alpha_1|, \cdots, |\alpha_l|) = \infty. \tag{3.16}$$

Observe from (2.7), (2.10), (3.10), and (3.11)

$$g_l(|\alpha_1|, \cdots, |\alpha_l|) = G_l(1/\rho, |\alpha_1|, \cdots, |\alpha_l|). \tag{3.17}$$

Hence $G_l(z/\rho, |\alpha_1|, \cdots, |\alpha_l|)$ has a pole at 1, and $\mathcal{G}_l(z/\rho)$ is not stable invariant. Then, from Lemma 3.1, $\mathcal{G}_n(z/\rho)$ is not stable invariant. The contradiction proves (3.12).

To prove (3.13), again use induction. Suppose it holds for some $1 \leq i \leq l$. Then, because of (3.10), $g_l(|\alpha_1|, \cdots, |\alpha_l|) > 1$

$$\begin{aligned} g_{l+1}&(|\alpha_1|, \cdots, |\alpha_{l+1}|) \\ &= \frac{\rho g_l(|\alpha_1|, \cdots, |\alpha_l|) - |\alpha_{l+1}|}{1 - \rho g_l(|\alpha_1|, \cdots, |\alpha_l|)|\alpha_{l+1}|} \\ &= \frac{\rho g_l(|\alpha_1|, \cdots, |\alpha_l|)(1 - \rho g_l(|\alpha_1|, \cdots, |\alpha_l|)|\alpha_{l+1}|)}{1 - \rho g_l(|\alpha_1|, \cdots, |\alpha_l|)|\alpha_{l+1}|} \\ &\quad + \frac{\rho^2 g_l^2(|\alpha_1|, \cdots, |\alpha_l|)|\alpha_{l+1}| - |\alpha_{l+1}|}{1 - \rho g_l(|\alpha_1|, \cdots, |\alpha_l|)|\alpha_{l+1}|} \\ &\geq \rho g_l(|\alpha_1|, \cdots, |\alpha_l|). \end{aligned} \tag{3.18}$$

■

Clearly, the satisfaction of (3.12)–(3.13) is a necessary condition for the stable invariance of $\mathcal{G}_n(z/\rho)$. From (3.11), it is also sufficient for the existence of the $g_i(|\alpha_1|, \cdots, |\alpha_i|)$ for all $|\alpha_i| \leq \delta_i$. Comparing (3.10)–(3.11) with (3.3)–(3.4), we find that for all $1 \leq i \leq n$

$$f_i = g_i(\delta_1, \cdots, \delta_i) \tag{3.19}$$

should, of course, $g_i(\delta_1, \cdots, \delta_i)$ exist. Thus, we have shown the following.

Lemma 3.3: The set $\mathcal{G}_n(z/\rho)$ is stable invariant only if the f_i in (3.3)–(3.4) exist and obey (3.5). Further, (3.6) also holds.

Remark 3.1: An interesting consequence of Lemma 3.3 is the fact that the violation of (3.5) is equivalent to the requirement that for some i, $\mathcal{G}_i(z)$ has a member with pole at $1/\rho$. In view of this, $\mathcal{G}_n(z)$ must also have a member with a pole at $1/\rho$.

Henceforth, we will assume that (3.5), and thus (3.6), holds.

Lemma 3.4: Consider (3.3)–(3.4) and (3.10)–(3.11). Suppose the f_i exist and that (3.5) holds. Then, for all $0 \le i \le n$ and $|\alpha_i| \le \delta_i$, the $g_i(|\alpha_1|, \cdots, |\alpha_i|)$ exist

$$g_i(|\alpha_1|, \cdots, |\alpha_i|) \le f_i \tag{3.20}$$

and (3.12)–(3.13) holds.

Proof: Clearly, should (3.20) hold and the f_i exist, then the g_i must exist, and as $|\alpha_i| \le \delta_i$, (3.12) must hold. We use induction to prove (3.20). This clearly holds at $i = 0$. Now, suppose it holds at some $0 \le i \le n-1$. Then

$$\rho|\alpha_{i+1}|g_i(|\alpha_1|, \cdots, |\alpha_i|) \le \rho\delta_{i+1}f_i < 1 \tag{3.21}$$

whence from (3.11), $g_{i+1}(|\alpha_1|, \cdots, |\alpha_{i+1}|)$ exists.

Further, $|\alpha_{i+1}| \le |\delta_{i+1}|$, from whence

$$\begin{aligned} g_{i+1}(|\alpha_1|, \cdots, |\alpha_{i+1}|) &= \frac{\rho g_i(|\alpha_1|, \cdots, |\alpha_i|) - |\alpha_{i+1}|}{1 - \rho|\alpha_{i+1}|g_i(|\alpha_1|, \cdots, |\alpha_i|)} \\ &\le \frac{\rho f_i - |\alpha_{i+1}|}{1 - \rho|\alpha_{i+1}|f_i}. \end{aligned}$$

Further, observe that as $\rho f_i > 1$ and $|\alpha_{i+1}| < 1$

$$\frac{\partial}{\partial|\alpha_{i+1}|}\left(\frac{\rho f_i - |\alpha_{i+1}|}{1 - \rho|\alpha_{i+1}|f_i}\right) = \frac{\rho^2 f_i^2 - 1}{(1 - \rho|\alpha_{i+1}|f_i)^2} > 0.$$

Thus, as $|\alpha_{i+1}| \le |\delta_{i+1}|$

$$g_{i+1}(|\alpha_1|, \cdots, |\alpha_{i+1}|) \le \frac{\rho f_i - \delta_{i+1}}{1 - \rho\delta_{i+1}f_i} = f_{i+1}. \tag{3.22}$$

The fact that (3.13) holds follows similarly to the proof of Lemma 3.2. ■

The next lemma points to a BR result.

Lemma 3.5: Consider (3.3), (3.4), (3.10), and (3.11), with $\rho > 1$. Suppose (3.5) holds for all $1 \le i \le n$. Then, for all $0 \le i \le n, G_i(z, \alpha_1, \cdots, \alpha_i) \in \mathcal{G}_i(z)$, and for all $\omega \in [0, 2\pi)$

$$1 \le |G_i(e^{j\omega}/\rho, \alpha_1, \cdots, \alpha_i)| \le g_i(|\alpha_1|, \cdots, |\alpha_i|) \le f_i. \tag{3.23}$$

Proof: That the first two inequalities imply (3.23) is a consequence of Lemma 3.4. Observe also from this Lemma that $g_i(|\alpha_1|, \cdots, |\alpha_i|)$ exist and obey

$$g_i(|\alpha_1|, \cdots, |\alpha_i|) \ge \rho^i > 0. \tag{3.24}$$

Now, use induction. From (2.7), (3.23) clearly holds for $i = 0$. Now, suppose it holds at $i = l - 1$ for some $1 \le l \le n$. Then, dropping the arguments α_i at any $\omega \in [0, 2\pi)$, there exist θ and r such that

$$G_{l-1}(e^{j\omega}/\rho) = re^{j\theta} \tag{3.25}$$

with

$$1 \le r \le g_{l-1}. \tag{3.26}$$

Now, at this ω from (2.10), with $\beta = \theta - \omega$

$$\begin{aligned} |G_l(e^{j\omega}/\rho)|^2 &= \left|\frac{\rho re^{j\beta} - \alpha_l}{1 - \rho r\alpha_l e^{j\beta}}\right|^2 \\ &= \frac{\rho^2 r^2 + \alpha_l^2 - 2\alpha_l\rho r\cos\beta}{1 + \alpha_l^2 r^2\rho^2 - 2\alpha_l\rho r\cos\beta} = \sigma(\beta). \end{aligned} \tag{3.27}$$

Now, observe under (3.26)

$$\rho^2 r^2 + \alpha_l^2 - (1 + \alpha_l^2 r^2\rho^2) = (\rho^2 r^2 - 1)(1 - \alpha_l^2) > 0 \tag{3.28}$$

because of (3.26) and the facts that $\rho > 1, \delta_l < 1$. Thus, the lower bound in (3.23) holds at $i = l$. Consider next the maximum of $\sigma(\beta)$ with respect to β

$$\frac{\partial\sigma}{\partial\beta} = \frac{2\alpha_l r\sin\beta(1 - r^2\rho^2)(1 - \alpha_l^2)}{(1 + \rho^2 r^2\alpha_l^2 - 2\rho r\alpha_l\cos\beta)^2}. \tag{3.29}$$

Thus, because of (3.26), $\rho > 1$, and $\delta_l < 1$, the maximum occurs according to the following rule: At

$$\beta = \begin{cases} 0 & \text{if } \alpha_i \ge 0 \\ \pi & \text{if } \alpha_i < 0. \end{cases} \tag{3.30}$$

In either case, because of (3.26)

$$|G_l(e^{j\omega}/\rho)|^2 \le \left(\frac{\rho r - |\alpha_l|}{1 - \rho r|\alpha_l|}\right)^2 \le \left(\frac{\rho g_l - |\alpha_l|}{1 - \rho g_l|\alpha_l|}\right)^2. \tag{3.31}$$

Then, (3.23) follows from (3.11). ■

We can now prove the sufficiency part of the theorem.

Lemma 3.6: Consider the sets $\mathcal{G}_i(z), 1 \le i \le n$ as defined in (2.8)–(2.12) and the sequence f_i as in (3.3)–(3.4). Suppose the f_i exist and for all $1 \le i \le n$ obey (3.5). Then, for all $1 \le i \le n$, $\mathcal{G}_i(z/\rho)$ is stable invariant.

Proof: We use induction. Clearly, $\mathcal{G}_0(z/\rho)$ is stable invariant. Now, suppose $\mathcal{G}_l(z/\rho)$ is stable invariant for some $0 \le l \le n-1$. Observe that all elements of $\mathcal{G}_{l+1}(z/\rho)$ have degree $l+1$. Further, from (3.2)

$$G_{l+1}(z/\rho, \alpha_1, \cdots, \alpha_l, 0)$$

is stable for all $|\alpha_i| \le \delta_i, 1 \le i \le l$. Thus, from Proposition 2.1, $\mathcal{G}_{l+1}(z/\rho)$ is not stable invariant only if there exists $|\alpha_i| \le \delta_i, 1 \le i \le l$ such that for some $\omega \in [0, 2\pi)$

$$G_{l+1}(e^{j\omega}/\rho, \alpha_1, \cdots, \alpha_{l+1}) = \infty \tag{3.32}$$

i.e., because of (2.10)

$$\rho\alpha_{l+1}G_l(e^{j\omega}/\rho, \alpha_1, \cdots, \alpha_l) = e^{j\omega} \tag{3.33}$$

i.e., because of Lemma 3.5

$$1 = \rho|\alpha_{l+1}||G_l(e^{j\omega}/\rho, \alpha_1, \cdots, \alpha_l)| \le \rho\delta_{l+1}f_l \tag{3.34}$$

violating (3.12). ■

Thus, Lemmas 3.3 and 3.6 prove Theorem 3.1. We conclude this section with two results of independent interest. The proof of the first follows from Lemma 3.5 and the fact that BR systems are stable.

Theorem 3.2: The set $\mathcal{G}_n(z/\rho)$ is stable invariant iff for all $1 \le i \le n$ and $|\alpha_i| \le \delta_i$

$$\frac{1}{f_i}G_i(z/\rho, \alpha_1, \cdots, \alpha_i) \tag{3.35}$$

and

$$\frac{1}{g_i(|\alpha_1|, \cdots, |\alpha_i|)}G_i(z/\rho, \alpha_1, \cdots, \alpha_i) \tag{3.36}$$

are BR. Further, for all $\omega \in [0, 2\pi)$

$$1 \le |G_i(e^{j\omega}/\rho, \alpha_1, \cdots, \alpha_i)|. \tag{3.37}$$

Compare this with the allpass property when $\rho = 1$.

The next theorem relates the stable invariance of $\mathcal{G}_n(z/\rho)$ to the stability, in fact, the real poles, of a "worst" member.

Theorem 3.3: Given $\rho > 1, 0 < \delta_i < 1, i = 1, \cdots, n$. The following are equivalent.

1) The set $\mathcal{G}_n(z/\rho)$ is stable invariant.
2) $G_n(z/\rho, \delta_1, \cdots, \delta_n)$ is stable.
3) $G_n(z/\rho, \delta_1, \cdots, \delta_n)$ has no poles on $z \in [1, \rho)$.

The proof of the Theorem relies on the following Lemma.

Lemma 3.7: Suppose $G_n(z/\rho, \delta_1, \cdots, \delta_n)$ has no poles in $z \in [1, \rho), \rho > 1, 0 < \delta_i < 1, i = 1, \cdots, n$. Then

$$G_n(1/\rho, \delta_1, \cdots, \delta_n) > 0.$$

Proof: Since $0 < \delta_i < 1$, $G_n(z, \delta_1, \cdots, \delta_n)$ is stable allpass. Thus, with $|p_i| < 1$

$$G_n(z, \delta_1, \cdots, \delta_n) = \prod_{i=1}^{n} \frac{z^{-1} - p_i^*}{1 - z^{-1}p_i}.$$

Since $G_n(z, \delta_1, \cdots, \delta_n)$ has real coefficients, it readily follows that

$$G_n(\rho/\rho, \delta_1, \cdots, \delta_n) > 0.$$

Suppose $G_n(1/\rho, \delta_1, \cdots, \delta_n) \le 0$. Since $G_n(z/\rho, \delta, \cdots, \delta_n)$ has no poles on $z \in [1, \rho)$, the only way $G_n(z/\rho, \delta_1, \cdots, \delta_n)$ changes sign as z travels from ρ to 1 is if there exists some $1 < \rho_0 < \rho$ such that

$$G_n(\rho_0/\rho, \delta_1, \cdots, \delta_n) = 0.$$

Using (2.10), we have

$$\frac{\rho}{\rho_0}G_{n-1}(\rho_0/\rho, \delta_1, \cdots, \delta_{n-1}) = \delta_n$$

or

$$G_{n-1}(\rho_0/\rho, \delta_1, \cdots, \delta_{n-1}) = \frac{\rho_0}{\rho}\delta_n \in (0, 1). \tag{3.38}$$

We proceed to show that (3.38) implies $G_0 \in (0, 1)$, which contradicts $G_0 = 1$. To see this by induction, we assume, for some $1 \le i \le n-1$, that

$$G_i(\rho_0/\rho, \delta_1, \cdots, \delta_{i-1}) \in (0, 1).$$

Indeed, (2.10) gives

$$G_{i-1}(\rho_0/\rho, \delta_1, \cdots, \delta_n) = \frac{\rho_0}{\rho}\frac{G_i + \delta_i}{1 + \delta_i G_i} \in (0, 1)$$

because $\rho_0 < \rho$, and

$$G_i + \delta_i < 1 + \delta_i G_i.$$

■

Proof of Theorem 3.3: Since 1) implies 2) and 2) implies 3), it suffices to show that 3) implies 1). In view of Theorem 3.1, it suffices to show that 3) implies that $f_i, i = 1, \cdots, n$ exist and obey (3.5). We proceed by contradiction. Obviously, $f_0 = 1$ exists. Suppose 3) holds. Assume that f_i exists, $f_i \ge 1$, and $\rho\delta_i f_{i-1} < 1$, for all $i = 0, \cdots, l-1$ and some $1 \le l \le n$ but that $\rho\delta_l f_{l-1} \ge 1$.

Case I: $\rho\delta_l f_{l-1} > 1$. Then

$$f_l = \frac{\rho f_{l-1} - \delta_l}{1 - \rho\delta_l f_{l-1}} < 0.$$

It follows from induction that $f_{l+1}, \cdots, f_n$ are all negative.

Case II: $\rho\delta_l f_{l-1} = 1$. Consequently, $f_l = \infty$. If $l < n$, then $f_{l+1} = -1/\delta_l < 0$, and $f_n < 0$ follows from induction.

In either case, $f_n < 0$ or $f_n = \infty$. Note that $f_n = G_n(1/\rho, \delta_1, \cdots, \delta_n)$. By Lemma 3.7, this cannot happen. Therefore, $\rho\delta_l f_{l-1} < 1$ must hold. Hence, $f_i, i = 1, \cdots, n$, all exist and obey (3.5). ■

Thus, Theorem 3.3 shows that the stability invariance of the whole set $\mathcal{G}_n(z/\rho)$ boils down to the stability of a single corner Lattice filter. Recall that when $\rho = 1$, the set of $G(z/\rho, \delta_1, \cdots, \delta_n)$ stability preserving lattice coefficients form a convex set ($|\delta_i| < 1$). Therefore, it is intuitive to conjecture that the result in Theorem 3.3 can be generalized to the case where the set of reflection coefficients lie in a nonsymmetric interval, i.e.,

$$\alpha_i^- \le \alpha_i \le \alpha_i^+.$$

We show via the following example that when the parameter set becomes nonsymmetric, relative stability of corner filters will not imply the relative stability of the whole set.

Example 3.2: $n = 5, (\alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-0.5, 0.1, 0, -0.1), \alpha_1 \in [-0.8, 0.8], \rho = 1.25$. It is straightforward to verify that $G_n(z/\rho)$ is stable at $\alpha_1 = \pm 0.8$ but unstable at $\alpha_1 = -0.45$. Note from this example that α_1 even lies in a symmetric interval, although $\alpha_2, \cdots, \alpha_5$ does not.

Remark 3.2: Condition 3) in Theorem 3.3 offers a simple way of determining the maximum $\rho, \rho_{\max}$ for which relative stability of $G_n(z/\rho)$ is guaranteed for all $1 \le \rho < \rho_{\max}$. Indeed, $\rho_{\max}^{-1}$ is the smallest pole of $G_n(z, \delta_1, \cdots, \delta_n)$ on the positive real axis, which can be checked easily by solving the real eigenvalues of A_n in Theorem 2.2.

IV. Lyapunov Matrix for Relatively Stable LTI Lattices

In order to address the LTV problem considered in Section V, we need to determine a Lyapunov matrix that proves the stable invariance of $\mathcal{G}_n(z/\rho)$. It is known [7] that with

$$P = \operatorname{diag}\{(1-\alpha_1^2)\cdots(1-\alpha_{n-1}^2), (1-\alpha_2^2)\cdots(1-\alpha_{n-1}^2), \cdots, 1\} \tag{4.1}$$

and $\{A_n, b_n, c_n, d_n\}$, the SVR of $G_n(z, \alpha_1, \cdots, \alpha_n)$

$$A_n' P A_n - P = -(1-\alpha_n^2) e_n e_n'. \quad (4.2)$$

However, for the stable invariance of $\mathcal{G}_n(z/\rho)$, we need to find a positive definite symmetric Π_n that obeys

$$\rho^2 A_n' \Pi_n A_n - \Pi_n \le -Q_n' Q_n \quad (4.3)$$

for $|\alpha_i| \le \delta_i, 1 \le i \le n$, where $[\rho A_n, Q_n]$ is a completely observable pair. The main result of this section, which is presented below, solves this problem.

Theorem 4.1: Suppose $\mathcal{G}_n(z/\rho)$ is stable invariant with $\rho > 1$. Then, with $\{A_n, b_n, c_n, d_n\}$, the SVR of $G_n(z, \alpha_1, \cdots, \alpha_n) \in \mathcal{G}_n(z)$, and Π_n defined by

$$\begin{aligned}\Pi_n = \mathrm{diag}\,\{&\rho^{n-1}(1-\alpha_{n-1}^2)\cdots(1-\alpha_1^2) g_n/g_0\\ &\rho^{n-2}(1-\alpha_{n-1}^2)\cdots(1-\alpha_2^2) g_{n-1}/g_1, \cdots\\ &\rho(1-\alpha_{n-1}^2) g_{n-1}/g_{n-2}, 1\}\end{aligned} \quad (4.4)$$

for all $|\alpha_i| \le \delta_i, 1 \le i \le n$, we have $1-\rho^2 g_{n-1}^2 \alpha_n^2 > 0$ and

$$\rho^2 A_n' \Pi_n A_n - \Pi_n \le -(1-\rho^2 g_{n-1}^2 \alpha_n^2) e_n e_n'. \quad (4.5)$$

Here, the g_i are as in (3.10)–(3.11). We have dropped the arguments $|\alpha_i|$ in g_i, A_n, and Π_n.

Observe from (3.10), Lemma 3.2, and (2.12) that the stable invariance of $\mathcal{G}_n(z/\rho)$ ensures that Π_n is positive definite for all $|\alpha_i| \le \delta_i, 1 \le i \le n$. Further, from Lemma 3.2

$$0 \le \rho g_{n-1} \alpha_n < 1. \quad (4.6)$$

Thus, Q_n in (4.3) is

$$Q_n = \sqrt{1-\rho^2 g_{n-1}^2 \alpha_n^2}\, e_n'. \quad (4.7)$$

Further, observe that

$$\sum_{i=0}^{n} (\rho A_n')^i Q_n' Q_n (\rho A_n)^i = W_n' W_n \quad (4.8)$$

where

$$W_n = \begin{bmatrix} Q_n \\ \rho Q_n A_n \\ \vdots \\ \rho^{n-1} Q_n^{n-1} A_n \end{bmatrix}. \quad (4.9)$$

Then, it is readily verified (see the Appendix) that $W_n' W_n$ is positive definite throughout $\mathcal{G}_n(z)$.

Observe that as $g_i = 1$ for all $1 \le i \le n$, whenever $\rho = 1$, we recover the result of [7] when $\rho = 1$. A few further comments on the nature of the derived Lyapunov matrix are in order. In the setting of [5] involving direct-form realization, the Lyapunov matrix was multiaffine in the coefficients of the transfer function denominator. This fact considerably simplified the LTV analysis conducted in [5]. The Lyapunov matrix in (4.4) is clearly *not* multiaffine. There is, however, one vast simplification in the form of (4.4) over its counterpart in [5], namely, that it is diagonal. As will be shown in Section V, this diagonal nature aids the LTV analysis conducted there. Two other points to be exploited in Section V are as follows. First, Π_n is independent of α_n. Further, because of (3.10)–(3.11), the Lyapunov matrix in (4.4) depends only on $|\alpha_i|, 1 \le i \le n-1$ as opposed to depending on α_i directly. The rest of this section is devoted to proving Theorem 4.1.

With A_i, b_i, c_i, and d_i as defined in Theorem 2.2 (we are assuming time invariance here), the transfer function

$$\begin{aligned}&\frac{1}{g_p} G_p(z/\rho, \alpha_1, \cdots, \alpha_p)\\ &\quad = \frac{d_p}{g_p} + c_p((z/\rho)I - A_p)^{-1} \frac{b_p}{g_p}\\ &\quad = \frac{d_p}{g_p} + (\rho c_p)(zI - \rho A_p)^{-1} \frac{b_p}{g_p}.\end{aligned} \quad (4.10)$$

In other words, $G_p(z/\rho, \alpha_1, \cdots, \alpha_p)/g_p$ has SVR $\{\rho A_p, b_p/g_p, \rho c_p, d_p/g_p\}$. Accordingly, we will call the realization matrix of $G_p(z/\rho, \alpha_1, \cdots, \alpha_p)/g_p$

$$S_p = \begin{bmatrix} \rho A_p & b_p/g_p \\ \rho c_p & d_p/g_p \end{bmatrix}. \quad (4.11)$$

Observe from Theorem 2.2 that

$$\rho A_{p+1} = S_p \begin{bmatrix} I & 0 \\ 0 & \rho g_p \alpha_{p+1} \end{bmatrix}. \quad (4.12)$$

Our proof of Theorem 4.1 will use induction. To this end, note from Theorem 3.2 that the stable invariance of $\mathcal{G}_n(z/\rho)$ implies that for all $|\alpha_i| \le \delta_i, 1 \le i \le n$ and all $1 \le p \le n$, $(1/g_{p-1}) G_{p-1}(z/\rho, \alpha_1, \cdots, \alpha_{p-1})$ is BR. Consequently, from [14], it follows that there exists a $(p-1)\times(p-1)$ matrix

$$P_{p-1} = P_{p-1}' > 0 \quad (4.13)$$

such that with

$$\Pi_p = \begin{bmatrix} P_{p-1} & 0 \\ 0 & 1 \end{bmatrix} \quad (4.14)$$

$$S_{p-1}' \Pi_p S_{p-1} - \Pi_p \le 0. \quad (4.15)$$

Observe that $\Pi_p = \Pi_p' > 0$. Then, the next Lemma shows that Π_p acts as a Lyapunov matrix for the stability verification of $G_p(z/\rho, \alpha_1, \cdots, \alpha_p)$.

Lemma 4.1: Suppose that $\mathcal{G}_n(z/\rho)$ is stable and for some $1 \le p \le n$, Π_p is as in (4.13)–(4.15), with (4.11) in force. Then

$$\rho^2 A_p' \Pi_p A_p - \Pi_p \le -(1-(\rho g_{p-1}\alpha_p)^2) e_p e_p'. \quad (4.16)$$

Proof: Because of (4.15) and (4.13)

$$\begin{aligned}&\rho^2 A_p' \Pi_p A_p - \Pi_p\\ &\quad \le \begin{bmatrix} I & 0 \\ 0 & \rho g_{p-1}\alpha_p \end{bmatrix} \Pi_p \begin{bmatrix} I & 0 \\ 0 & \rho g_{p-1}\alpha_p \end{bmatrix} - \Pi_p\\ &\quad = -\begin{bmatrix} 0 & 0 \\ 0 & (1-\rho g_{p-1}\alpha_p)^2 \end{bmatrix}.\end{aligned} \quad (4.17)$$

Hence, we have the result. ■

Clearly, the stable invariance of $\mathcal{G}_n(z/\rho)$ and Lemma 3.2 ensures that the left-hand side of (4.16) is negative semidefinite.

The next step of the induction argument must relate Π_{p+1} to Π_p. To this, we need two intermediate Lemmas. The first of these relates the positive semidefiniteness of a special class of matrices to the definiteness of certain 2×2 matrices.

Lemma 4.2: Consider scalars $\beta_1, \beta_2, \beta_3$, and e_p as in (2.18). Then

$$\begin{bmatrix} \beta_1 e_p e_p' & \beta_2 e_p \\ \beta_2 e_p' & \beta_3 \end{bmatrix} \geq 0 \tag{4.18}$$

iff

$$\begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{bmatrix} \geq 0. \tag{4.19}$$

Proof: Consider $\xi = [\xi_1', \xi_2], \xi_1 \in \mathcal{R}^p, \xi_2$ to be scalar. Then, (4.16) is equivalent to

$$\beta_1(\xi_1' e_p)^2 + 2\xi_2(\xi_1' e_p)\beta_2 + \beta_3^2 \xi_2 \geq 0 \tag{4.20}$$

for all ξ. This in turn is equivalent to

$$[\xi_1' e_p, \xi_2] \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} \xi_1' e_p \\ \xi_2 \end{bmatrix} \geq 0. \tag{4.21}$$

Hence, we have the result. ■

The next Lemma gives some key properties of g_p.

Lemma 4.3: Suppose that $\mathcal{G}_n(z/\rho)$ is stable invariant with $\rho > 1$. Then, under (3.10)–(3.11), for all $1 \leq p \leq n$

$$g_p - \rho g_{p-1} = |\alpha_p|(\rho g_p g_{p-1} - 1) \tag{4.22}$$

and

$$\rho g_{p-1} - |\alpha_p| = \frac{(1-\alpha_p^2) g_p}{1 + |\alpha_p| g_p}. \tag{4.23}$$

Proof: From (3.10), (3.11), and Lemma 3.2

$$\begin{aligned} g_p - \rho g_{p-1} &= \frac{\rho g_{p-1} - |\alpha_p|}{1 - \rho g_{p-1}|\alpha_p|} - \rho g_{p-1} \\ &= \frac{-|\alpha_p| + \rho^2 g_{p-1}^2 |\alpha_p|}{1 - \rho g_{p-1}|\alpha_p|} \\ &= |\alpha_p| \frac{\rho^2 g_{p-1}^2 - 1}{1 - \rho g_{p-1}|\alpha_p|}. \end{aligned} \tag{4.24}$$

Further

$$\begin{aligned} \rho g_p g_{p-1} - 1 &= \rho \frac{\rho g_{p-1} - |\alpha_p|}{1 - \rho g_{p-1}|\alpha_p|} g_{p-1} - 1 \\ &= \frac{\rho^2 g_{p-1} - 1}{1 - \rho g_{p-1}|\alpha_p|}. \end{aligned} \tag{4.25}$$

Then, (4.22) follows easily. Now, from (3.11)

$$\begin{aligned} g_p - \rho g_{p-1} g_p |\alpha_p| &= \rho g_{p-1} - |\alpha_p| \\ \Leftrightarrow \rho g_{p-1} &= \frac{g_p + |\alpha_p|}{1 + g_p |\alpha_p|}. \end{aligned} \tag{4.26}$$

Thus

$$\begin{aligned} \rho g_{p-1} - |\alpha_p| &= \frac{g_p + |\alpha_p|}{1 + g_p|\alpha_p|} - |\alpha_p| \\ &= \frac{g_p + |\alpha_p| - |\alpha_p| - g_p|\alpha_p|^2}{1 + g_p|\alpha_p|} \end{aligned} \tag{4.27}$$

hence, we have the result. ■

We are now in a position to proceed with the inductive step of relating Π_p to Π_{p+1}.

Lemma 4.4: Suppose $\mathcal{G}_n(z/\rho)$ is stable invariant with $\rho > 1$. Consider Π_p as in Lemma 4.1. Define

$$\gamma_p = \rho(1 - \alpha_p^2) \frac{g_p}{g_{p-1}} \tag{4.28}$$

and

$$\Pi_{p+1} = \begin{bmatrix} \gamma_p \Pi_p & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.29}$$

Then

$$S_p' \Pi_{p+1} S_p - \Pi_{p+1} \leq 0. \tag{4.30}$$

Proof: We will treat the $\alpha_p \neq 0$ case separately from the $\alpha_p = 0$ case.

Case I: $\alpha_p = 0$. In this case, from (3.11)

$$g_p = \rho g_{p-1}. \tag{4.31}$$

Further, from Theorem 2.2 and (4.11)

$$S_p = \begin{bmatrix} \rho A_{p-1} & 0 & b_{p-1}/\rho g_{p-1} \\ \rho c_{p-1} & 0 & d_{p-1}/\rho g_{p-1} \\ 0 & \rho & 0 \end{bmatrix}. \tag{4.32}$$

From (4.28) and (4.31)

$$\gamma_p = \rho^2. \tag{4.33}$$

Thus

$$\Pi_{p+1} = \begin{bmatrix} \rho^2 \Pi_p & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.34}$$

Define the $(p+1) \times (p+1)$ matrix

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1/\rho \\ 0 & \rho & 0 \end{bmatrix}.$$

Note that $E = E^{-1}$. Because of (4.11)

$$S_p E = \begin{bmatrix} \rho A_{p-1} & b_{p-1}/g_{p-1} & 0 \\ \rho c_{p-1} & d_{p-1}/g_{p-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{p-1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.35}$$

Moreover, because of (4.14)

$$E' \Pi_{p+1} E = \Pi_{p+1}. \tag{4.36}$$

Thus, from (4.34)–(4.36)

$$\begin{aligned} &E' S_p' \Pi_{p+1} S_p E - E' \Pi_{p+1} E \\ &\quad = \begin{bmatrix} \rho^2 S_{p-1}' \Pi_p S_{p-1} & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \rho^2 \Pi_p & 0 \\ 0 & 1 \end{bmatrix} \leq 0 \end{aligned}$$

the last inequality following from (4.15). Hence, the result holds.

Case II: $\alpha_p \neq 0$. Because of (2.25), (4.11), and Theorem 2.2

$$S_p = \begin{bmatrix} \rho A_p & (A_p e_p/\alpha_p g_p) \\ \rho(1-\alpha_p^2)e_p' & -(\alpha_p/g_p) \end{bmatrix} = \rho \begin{bmatrix} A_p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & (e_p/\rho\alpha_p g_p) \\ (1-\alpha_p^2)e_p' & -(\alpha_p/\rho g_p) \end{bmatrix}. \quad (4.37)$$

Observe through direct verification that

$$\begin{bmatrix} I & e_p/\alpha_p \\ (1-\alpha_p^2)e_p' & -\alpha_p \end{bmatrix}^{-1} = \begin{bmatrix} I-(1-\alpha_p^2)e_p e_p' & e_p \\ \alpha_p(1-\alpha_p^2)e_p' & -\alpha_p \end{bmatrix}. \quad (4.38)$$

Observe from (4.14) that

$$\Pi_p e_p = e_p. \quad (4.39)$$

Because of (4.37) and (4.39), (4.30) is, in turn, equivalent to

$$\begin{bmatrix} \rho^2\gamma_p A_p'\Pi_p A_p & 0 \\ 0 & \rho^2 \end{bmatrix} - \begin{bmatrix} I-(1-\alpha_p^2)e_p e_p' & \alpha_p(1-\alpha_p^2)e_p \\ e_p' & -\alpha_p \end{bmatrix} \cdot \begin{bmatrix} \gamma_p\Pi_p & 0 \\ 0 & \rho^2 g_p^2 \end{bmatrix} \begin{bmatrix} I-(1-\alpha_p^2)e_p e_p' & e_p \\ \alpha_p(1-\alpha_p^2)e_p' & -\alpha_p \end{bmatrix} \leq 0$$

and (4.40), shown at the bottom of the page. Because of (4.16) and (4.39), (4.30) is guaranteed, provided we have (4.41), shown at the bottom of the page. Observe from Lemma 4.2 that this is equivalent to (4.42), shown at the bottom of the page. Now, select γ_p as in (4.28). Then, using (4.2), the (1, 1) block of (4.42) equals

$$F(1,1) = \rho\alpha_p^2(1-\alpha_p^2)\frac{g_p}{g_{p-1}}|\alpha_p|(\rho g_p g_{p-1}-1) \cdot (\rho g_{p-1} - |\alpha_p|) \quad (4.43)$$

where the last equality is from Lemma 4.2. Likewise

$$F(2,1) = F(1,2) = \rho\alpha_p^2\frac{g_p}{g_{p-1}}(1-\alpha_p^2)(1-\rho g_p g_{p-1}). \quad (4.44)$$

Finally, from Lemmas 4.3 and 3.2

$$F(2,2) = \frac{\rho}{g_{p-1}}|\alpha_p|(\rho g_p g_{p-1}-1)(1+g_p|\alpha_p|) > 0. \quad (4.45)$$

Then, it is readily verified using Lemma 4.3 that

$$\det(F) = 0. \quad (4.46)$$

Consequently, (4.4) together with (4.45) implies that (4.42) holds. Hence, (4.30) holds. ■

Then, the proof of Theorem 4.1 follows by noting that

$$S_0 = 1$$

and hence, $\Pi_1 = 1$. Then, with the induction so initiated, we can repeatedly use Lemmas 4.1 and 4.4 to obtain the desired result that

$$\rho^2 A_n'\Pi_n A_n - \Pi_n \leq -(1-\rho g_{p-1}\alpha_p)^2 e_p e_p'. \quad (4.47)$$

V. Relative Stability of the LTV Lattice

This section addresses the relative stability of LTV lattice filters. We will assume that there exists $\epsilon_i > 0$ arbitrarily small such that for all $1 \leq i \leq n$

$$|\alpha_i(k)| \leq \delta_i - \epsilon_i. \quad (5.1)$$

We will further assume that the f_i in (3.3)–(3.4) obey (3.5) for all $1 \leq i \leq n$, i.e., all frozen systems are stable with degree of stability $\ln\rho$. The question is, given

$$1 < \rho_1 < \rho \quad (5.2)$$

what rates of time variations can be sustained to ensure that the LTV Lattice has degree of stability $\ln\rho_1$?

To this end, we present two results. The first is a simple consequence of the comments made at the end of the previous section. The second constitutes the main result of this section.

$$\left[\begin{array}{c|c} \begin{matrix}\rho^2\gamma_p A_p'\Pi_p A_p - \Pi_p\gamma_p \\ -\{[-2(1-\alpha_p^2)+(1-\alpha_p^2)^2]\gamma_p \\ +\rho^2 g_p^2(1-\alpha_p^2)^2\alpha_p^2\}e_p e_p'\end{matrix} & \alpha_p^2[\rho^2 g_p^2(1-\alpha_p^2)-\gamma_p]e_p \\ \hline \alpha_p^2[\rho^2 g_p^2(1-\alpha_p^2)-\gamma_p]e_p' & \rho^2(1-g_p^2\alpha_p^2)-\gamma_p \end{array}\right] \leq 0. \quad (4.40)$$

$$\left[\begin{array}{c|c} \begin{matrix}\{\gamma_p[-(1-\rho^2\alpha_p^2 g_{p-1}^2) \\ +(1-\alpha_p^4)] - \rho^2 g_p^2(1-\alpha_p^2)^2\alpha_p^2]\}e_p e_p'\end{matrix} & \alpha_p^2[\rho^2 g_p^2(1-\alpha_p^2)-\gamma_p]e_p \\ \hline \alpha_p^2[\rho^2 g_p^2(1-\alpha_p^2)-\gamma_p]e_p' & \rho^2(1-g_p^2\alpha_p^2)-\gamma_p \end{array}\right] \leq 0. \quad (4.41)$$

$$F = \begin{bmatrix} \alpha_p^2\rho^2 g_p^2(1-\alpha_p^2)^2 - \gamma_p\alpha_p^2(\rho^2 g_{p-1}^2 - \alpha_p^2) & \alpha_p^2[\gamma_p - \rho^2 g_p^2(1-\alpha_p^2)] \\ \alpha_p^2[\gamma_p - \rho^2 g_p^2(1-\alpha_p^2)] & \gamma_p - \rho^2(1-g_p^2\alpha_p^2) \end{bmatrix} \geq 0. \quad (4.42)$$

Theorem 5.1: Consider the lattice filter depicted in Fig. 1. Suppose that (5.1) holds and that $\mathcal{G}_n(z/\rho)$ is stable invariant for some $\rho > 1$. Suppose also that there exist a_i such that for all $1 \leq i \leq n-1$ and all k

$$|\alpha_i(k)| = |a_i|. \tag{5.3}$$

Then, the LTV lattice filter is stable with degree of stability $\ln \rho$.

Proof: Suppose $\{A_n(k), b_n(k), c_n(k), d_n(k)\}$ is the SVR of the lattice filter as exemplified in Theorem 2.2. Define

$$\begin{aligned}\Pi_n(k) = \text{diag}\,\{&\rho^{n-1}(1-\alpha_{n-1}^2(k))\cdots(1-\alpha_1^2(k))\\ &g_{n-1}(k)/g_0(k), \cdots, \rho(1-\alpha_{n-1}^2(k))\\ &\cdot g_{n-1}(k)/g_{n-2}(k), 1\}.\end{aligned} \tag{5.4}$$

Observe that because of Lemma 3.2, (3.10), (3.11), and (5.1), there exists $\mu_1, \mu_2 > 0$ such that for all k

$$\mu_1 I \leq \Pi_n(k) \leq \mu_2 I. \tag{5.5}$$

Further, because of (5.3) and (3.10)–(3.11)

$$\Pi_n(k+1) = \Pi_n(k). \tag{5.6}$$

Then, because of Theorem 4.1

$$\begin{aligned}&\rho^2 A_n'(k)\Pi_n(k+1)A_n(k) - \Pi_n(k)\\ &\quad= \rho^2 A_n'(k)\Pi_n(k)A_n(k) - \Pi_n(k)\\ &\quad\leq -(1-\rho^2 g_{n-1}^2(k)\alpha_n^2(k))e_n e_n'.\end{aligned} \tag{5.7}$$

Then, we show in the Appendix that

$$Q_n(k) = \sqrt{1-\rho^2 g_{n-1}^2(k)\alpha_n^2(k)}e_n' \tag{5.8}$$

obeys (2.1). Note that (5.8) is real for all $|\alpha_i(k)| \leq \delta_i$. Hence, we have the result. ■

Observe that this theorem states that as long as the frozen LTI systems have degree of stability $\ln \rho$, the LTV filter sustains the same degree of stability for arbitrary rates of variation in $\alpha_n(k)$ as long as the $\alpha_i(k), 1 \leq i \leq n-1$ sustain only changes in sign, and (5.1) holds for all $1 \leq i \leq n$.

The next theorem addresses relative stability under simultaneous magnitude variations in multiple reflection coefficients.

Theorem 5.2: Consider the LTV lattice in Fig. 1, which obeys (5.1). Suppose that $\mathcal{G}_n(z/\rho)$ is stable invariant and that $\rho > 1$. Then, the LTV lattice is stable with degree of stability ρ_1 obeying (5.2), if the following holds: There exists an integer $N > 0$ and $0 < \beta < 1$ such that

$$\sup_{k\geq 0} \frac{1}{N} \sum_{l=k}^{k+N-1} \nu(l) \leq 2\ln\left[\frac{\rho\beta}{\rho_1}\right] \tag{5.9}$$

where

$$\begin{aligned}\nu(k) = \sup_{i\in\{1,\cdots,n-1\}} \Bigg\{&\Bigg(\ln\left[\frac{g_{n-1}(k+1)}{g_{n-1}(k)}\right]\\ &+ \ln\left[\frac{g_{i-1}(k)}{g_{i-1}(k+1)}\right]\\ &+ \sum_{l=i}^{n-1}\ln\left[\frac{1-\alpha_l^2(k+1)}{1-\alpha_l^2(k)}\right]\Bigg)^+\Bigg\}.\end{aligned} \tag{5.10}$$

Proof: Consider the Lyapunov matrix $\Pi_n(k)$ in (5.4), and observe that (5.5) prevails. Note that two $n \times n$ diagonal matrices Λ_1 and Λ_2 obey

$$\Lambda_1 \leq \Lambda_2 \tag{5.11}$$

iff for all $1 \leq i \leq n$, the ith diagonal of Λ_1 is less than or equal to the ith diagonal of Λ_2. Then, from (5.4), it is readily observed that $e^{\nu(k)}$ is the *smallest* scalar for which

$$\Pi_n(k+1) \leq e^{\nu(k)}\Pi_n(k). \tag{5.12}$$

The "+" notation in (5.10) accounts for the fact that the $n \times n$ elements of both $\Pi_n(k+1)$ and $\Pi_n(k)$ equal 1.

With $x_n(k)$ the state vector of the lattice (see Theorem 2.2), consider the zero input state solution of

$$x_n(k+1) = A_n(k)x_n(k). \tag{5.13}$$

Consider

$$V_n(x_n(k), k) = x_n'(k)\Pi_n(k)x_n(k). \tag{5.14}$$

Then, because of (5.5), it suffices to show that under (5.10), there exists $\beta_2 > 0, 0 \leq \beta_1 < 1$ such that along (5.13), for all k, k_0

$$\rho^{(k-k_0)}V_n(x_n(k), k) \leq \beta_2 V_n(x_n(k_0), k_0)\beta_1^{(k-k_0)}. \tag{5.15}$$

Now, observe that under our assumptions, $A_n(k)$ is a bounded matrix. Hence, (5.13) cannot have finite escape time. In fact, because of (5.5) along (5.13), there exists β_3 such that for all k, k_0

$$V_n(x_n(k+1), k+1) \leq \beta_3 V_n(x_n(k), k). \tag{5.16}$$

Now, consider

$$\begin{aligned}&V_n(x_n(k+1), k+1)\\ &\quad\leq x_n'(k+1)\Pi_n(k+1)x_n(k+1)\\ &\quad\leq e^{\nu(k)}x_n'(k+1)\Pi_n(k)x_n(k+1)\\ &\quad= e^{\nu(k)}x_n'(k)A_n'(k)\Pi_n(k)A_n(k)x_n(k).\end{aligned} \tag{5.17}$$

Then, because of Theorem 4.1

$$\rho_1^2 V_n(x_n(k+1), k+1) \leq e^{\nu(k)}\frac{V_n(x_n(k), k)}{\rho^2}\rho_1^2. \tag{5.18}$$

Then, the recursive application of (5.18) reveals that for all k

$$\begin{aligned}&\rho_1^{2N}V_n(x_n(k+N-1), k+N-1)\\ &\quad\leq \left[\left(\frac{\rho_1}{\rho}\right)^{2N}\exp\left(\sum_{i=k}^{k+N-1}\nu(k)\right)\right]\\ &\qquad\cdot V_n(x_n(k-1), k-1)\\ &\quad\leq V_n(x_n(k-1), k-1)\beta^{2N}\end{aligned} \tag{5.19}$$

where the last inequality is because of (5.9). Then, because of (5.16), for all k, k_0

$$\begin{aligned}&\rho_1^{2(k-k_0)}V_n(x_n(k), k)\\ &\quad\leq \left(\frac{\beta_3}{\beta^2}\right)^N \rho_1^{2N}V_n(x_n(k_0), k_0)\beta^{2(k-k_0)}.\end{aligned} \tag{5.20}$$

Hence, we have the result. ■

A few comments concerning (5.9)–(5.10) are now appropriate. Essentially, this condition represents a tradeoff between frozen systems and LTV system degree of stability with the rate of variations in the *magnitude* of the α_i. Sign changes are inconsequential.

Observe that with

$$\gamma_p(k) = \rho(1-\alpha_p^2(k))\frac{g_p(k)}{g_{p-1}(k)} \tag{5.21}$$

$$\nu(k) = \sup_{i\in\{1,\cdots,n-1\}}\left\{\left[\sum_{p=i}^{n-1}\ln\frac{\gamma_p(k+1)}{\gamma_p(k)}\right]^+\right\}. \tag{5.22}$$

Thus, (5.9)–(5.10) essentially quantify the potentially destabilizing time variations as those that increase $\gamma_p(k)$ and limit the average increase in these $\gamma_p(k)$. Declining values of $\gamma_p(k)$ are found not to be destabilizing.

VI. Conclusion

We have studied the relative stability of both the LTI and the LTV lattice. We have shown that when the LTI set of lattice filters is defined by bounds on the reflection coefficients, then there is a simple necessary and sufficient condition for all such LTI lattices to have degree of stability $\ln\rho$. We also show that verification of stable invariance can be effected by checking a single corner of $\mathcal{G}_n(z/\rho)$.

We provide a Lyapunov matrix for checking this degree of stability requirement and show that it specializes to the matrix of [7]. Finally, we give a logarithmic rate of variation result that suffices for the relative stability of LTV unnormalized lattices.

Although the LTI results of Section III apply to the normalized lattice as well, the LTV and Lyapunov results do not. An important issue is a generalization that captures this normalized case.

Appendix

This appendix shows that $(A_n(k), Q_n(k))$ is UCO for $A_n(k)$ and $Q_n(k)$ defined in Theorem 2.2 and (5.8), respectively. When d_i are LTI, the UCO property of $(A_n(k), Q_n(k))$ implies that $W_n'W_n$ in (4.8) is positive definite. In (2.1), choose $N = n$. Call

$$\sigma(k) = \sqrt{1-\rho^2 g_{n-1}^2(k)\alpha_n^2(k)}. \tag{A.1}$$

Observe from Lemma 3.4 that $\sigma(k)$ is real for all k and, because of the stable invariance of $\mathcal{G}_n(z/\rho)$, obeys for some m_1, M_1

$$0 < m_1 \le \sigma(k) \le M_1 \quad \forall k. \tag{A.2}$$

Then, with

$$\sum(k) = \text{diag}\,\{\sigma(k), \sigma(k+1), \cdots, \sigma(k+n-1)\} \tag{A.3}$$

$$\Psi_n(k) = \begin{bmatrix} e_n' \\ \rho e_n' A_n(k) \\ \rho^2 e_n' A_n(k+1)A_n(k) \\ \vdots \\ \rho^{n-1}e_n' A_n(k+n-2)\cdots A_n(k)\end{bmatrix} \tag{A.4}$$

and

$$W_n(k) = \Sigma(k)\Psi_n(k) \tag{A.5}$$

the matrix in (2.1) is $W_n'(k)W_n(k)$. Observe from (2.14) and (2.16) that $A_n(k)$ has the form

$$A_n(k) = \begin{bmatrix} a_1(k) & a_2(k) \\ D(k) & \Psi(k)\end{bmatrix} \tag{A.6}$$

where $D(k)$ is $(n-1)\times(n-1)$ upper triangular with the ith element

$$(1-\alpha_i^2(k)).$$

Thus, $\Psi_n(k)$ is triangular with the property that its (i,j)th element is zero whenever $i+j \le n$. Further, for all $1 \le i \le n$, its $(i, n-i+1)$th element is

$$\prod_{l=1}^{i-1}\rho(1-\alpha_{n-l+1}^2(k+l)). \tag{A.7}$$

Clearly, since elements in (A.7) are bounded away from zero and all elements of $\Psi_n(k)$ and, hence, $W_n(k)$ are bounded, (2.1) holds.

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