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Finite Test of Robust Strict Positive Realness†

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Abstract
This paper is concerned with the problem of testing the robust strict positive realness (SPRness) of a family of rational functions with both numerator and denominator dependent on the same set of parameters. We show that this problem can be solved by using a series of Routh tables. In other words, the robust SPRness of the whole family can be tested by performing only a finite number of elementary operations (arithmetic operations, logical operations and sign tests).

1 Introduction

It is well known that the problem of strict positive realness (SPRness) plays an important role in many system analysis and design problems. Examples range from absolute stability analysis for linear systems with uncertain/nonlinear perturbations[1] to the convergence study of adaptive controllers[2]. There are already several papers on the SPRness of rational functions with uncertain parameters. A family of rational functions called *interval plant* was considered in [5]and it is shown that the robust SPR of this family of rational functions is equivalent to the SPR of 16 special members. This result is extended in [3]to the SPR problem of a real shifted family of interval transfer functions. In [7], the condition in [5] is strengthened such that the the number of functions need to be checked is reduced to 8. In [6] a family of rational functions with the denominator and the numerator multilinearly or linearly dependent on two *independent* sets of parameters is considered and it is shown that the whole family is robustly SPR if and only if the rational functions associated with the extreme values of the parameters are SPR. A more general case is considered in [4] where the transfer function is allowed to have both independent multilinear parameters and dependent linear parameters in both numerator and denominator. It is shown that only certain extreme points and edges of the parameter set need to be tested for the SPRness of the whole family.

This paper considers the same problem as in [4], i.e., the SPR problem of a family of rational functions with dependent linear parameters in both numerator and denominator. Following some recent papers on finite decidability of stability and other related problems, we show that the SPRness of the whole family can be determined by using a series of Routh tables which involve only a finite number of elementary operations (i.e. arithmetic operations, logical operations and sign tests). The structure of this paper is as follows: in Section 2 we formulate the robust SPR problem and recapture the main result in [4] and results for testing the positivity of polynomials. The main result is given in Section 3 and its computational aspects are discussed in Section 4 .

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2 Problem Formulation and Preliminaries

Definition: [9] A rational function

$$G(s) = \frac{N(s)}{D(s)} \qquad (1)$$

is called positive real (PR) if

i) $G(s)$ is real for real s, and

ii) $\mathrm{Re}\, G(s) \geq 0, \ \forall Re[s] > 0$.

Suppose $G(s)$ is not identically zero. Then $G(s)$ is called strictly positive real (SPR) if $G(s-\varepsilon)$ is PR for some $\varepsilon > 0$.

Properties: $G(s)$ *is* PR iff $1/G(s)$ is PR; $G(s)$ is SPR iff $1/G(s)$ is SPR. Further, a family of rational functions is said to be robustly SPR if every member of the family is SPR.

Consider the following parameterized rational function:

$$G(s, q_n, q_d, q_b) = \frac{N(s, q_n, q_b)}{D(s, q_d, q_b)} = \frac{\sum_{i=0}^{m} b_i(q_n, q_b)s^i}{\sum_{i=0}^{n} a_i(q_d, q_b)s^i}$$

$$= \frac{\sum_{i=0}^{m} [b_{i0} + b_{i1}(q_n, q_b)]s^i}{\sum_{i=0}^{n} [a_{i0} + a_{i1}(q_d, q_b)]s^i} \qquad (2)$$

where a_{i0} and b_{i0} are the coefficients of the nominal parts of the denominator and numerator respectively; $a_{i1}(.)$ and $b_{i1}(.)$ represent uncertainties in the coefficients;
$q_n \in Q_n \subset R^{p_n}, q_d \in Q_d \subset R^{p_d}$, $q_b \in Q_b \subset R^{p_b}$;
Q_n, Q_d, Q_b
are given bounding sets. It is assumed that
A1). $a_{i1}(q_d, q_b)$ are multilinear functions of q_d and linear functions of q_b;
A2). $b_{i1}(q_n, q_b)$ are multilinear functions of q_n and linear functions of q_b;
A3). Q_n, Q_d, Q_b are hyperrectangles, all containing the origin;
A4).The leading coefficients
$a_{n0} + a_{n1}(q_d, q_b)$ and $b_{m0} + b_{m1}(q_n, q_b)$ do not vanish for any $q_n \in Q_n$, $q_d \in Q_d$, $q_b \in Q_b$.

In the sequel, let

$$q = (q_n, q_d, q_b), \ Q = Q_n \oplus Q_d \oplus Q_b \qquad (3)$$

and denote the set of vertices and the set of edges of a hyperrectangle **H** by V(**H**) and E(**H**), respectively. We also define two subsets of Q as follows:

$$Q_{edge} = V(Q_n) \oplus V(Q_d) \oplus E(Q_b) \qquad (4)$$

$$Q_v = V(Q_n) \oplus V(Q_d) \oplus V(Q_b) \qquad (5)$$

and define the set of admissible rational functions as

$$\mathcal{G} := \{ G(s, q_n, q_d, q_b) : q_n \in Q_n, q_d \in Q_d, q_b \in Q_b \} \qquad (6)$$

or equivalently

$$\mathcal{G} := \{ G(s, q) : q \in Q \}. \qquad (7)$$

Then we have the following result.

Lemma 1 [4]: Given the family of rational functions $\mathcal{G}$ in (6) satisfying assumptions A1)–A4), $\mathcal{G}$ is robustly SPR if and only if $G(s,q)$ is SPR for every $q \in Q_{edge}$. □

This result shows that it is sufficient to test certain number of edges of Q. To check the SPRness of an individual rational function, we have the following result.

Lemma 2 [8] Assume $G(s)$ is a real rational function, not identically zero for all s. Then $G(s)$ is SPR if and only if i) $G(s)$ is analytic in Re[s] ≥ 0, i.e. $D(s)$ is strictly Hurwitz,
ii) Re[$G(j\omega)$] > 0, $\forall \omega \in [0, \infty)$, and
iii) a) $\lim_{|\omega| \to \infty} \mathrm{Re}[G(j\omega)] > 0$ when r * = 0 ,
b) $\lim_{|\omega| \to \infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0$ when r * = 1
, or
c) $\lim_{|\omega| \to \infty} \mathrm{Re}[G(j\omega)] > 0$, $\lim_{|\omega| \to \infty} \frac{G(j\omega)}{j\omega} > 0$ when r * = − 1 ,
where r * is the relative degree of $G(s)$. □

Remark: Condition iii) a) above is actually implied by ii). We list it here simply for convenience.

3 Main Results

First, we need a robust version of Lemma 2. For the robust SPRness of a family of rational functions, we give the following result.

Theorem 1 Consider the family of rational functions (7) then $\mathcal{G}$ is robustly SPR if and only if the following conditions hold:
i) $D(s, q^0)$ is strictly Hurwitz, for some $q^0 \in Q_{edge}$,
ii) $\mathrm{Re}[N(j\omega, q)]\ \mathrm{Re}[D(j\omega, q)] + \mathrm{Im}[N(j\omega, q)]\ \mathrm{Im}[D(j\omega, q)] > 0$
$\forall \omega \in [0, \infty)$, $\forall q \in Q_{edge}$, and
iii) a) $\lim_{|\omega| \to \infty} \mathrm{Re}[G(j\omega, q)] > 0$, $\forall q \in Q_{edge}$ if $r^* = 0$, or
b) $\lim_{|\omega| \to \infty} \omega^2 \mathrm{Re}[G(j\omega, q)] > 0$, $\forall q \in Q_{edge}$ if r * = 1 , or
c) $\lim_{|\omega| \to \infty} \mathrm{Re}[G(j\omega, q)] > 0$ &
$\lim_{|\omega| \to \infty} \frac{G(j\omega, q)}{j\omega} > 0$, $\forall q \in Q_{edge}$,
if $r^* = -1$,
where r^* is the relative degree of $G(j\omega, q)$.
□

Proof: (*Necessity*) Follows directly from Lemma 2 , noting that

$$\mathrm{Re}[G(j\omega, q)] = \frac{\mathrm{Re}[N(j\omega, q)]\ \mathrm{Re}[D(j\omega, q)] + \mathrm{Im}[N(j\omega, q)]\ \mathrm{Im}[D(j\omega, q)]}{(\mathrm{Re}[D(j\omega, q)])^2 + (\mathrm{Im}[D(j\omega, q)])^2}. \qquad (8)$$

(*Sufficiency*) Assuming that conditions i)– iii) hold, we need to show that $\mathcal{G}$ is robustly SPR. From Lemma 1, we only need to show that $G(s, q)$ is robustly SPR for every $q \in Q_{edge}$. Suppose there exists some $\tilde{q} \in Q_{edge}$ such that $G(s,\tilde{q})$ is not SPR. Then, from (8) and Lemma 2, the only possibility is that $G(s,\tilde{q})$ is not analytic in $Re[s] \geq 0$, i.e. $D(s,\tilde{q})$ has roots in the closed right half plane (CRHP). Note that the set Q_{edge} is a connected set. Then from the continuity of the roots of $D(s,q)$ with respect to q, there must be another $\hat{q} \in Q_{edge}$ and $\hat{\omega} \in \mathbb{R}$, such that $D(j\hat{\omega},\hat{q}) = 0$, which means

$$\mathrm{Re}[N(j\hat{\omega},\hat{q})]\ \mathrm{Re}[D(j\hat{\omega},\hat{q})] + \mathrm{Im}[N(j\hat{\omega},\hat{q})]\ \mathrm{Im}[D(j\hat{\omega},\hat{q})] = 0.$$

Clearly this contradicts ii). So $D(s,q)$ is analytic in $\mathrm{Re}[s] \geq 0$, $\forall q \in Q_{edge}$.
Consequently, $G(s,q)$ is SPR for all $q \in Q_{edge}$ and, by Lemma 1, this is equivalent to that $G(s,q)$ is SPR for all $q \in Q$. ▼▼▼

Now we concentrate on checking condition ii) in Theorem 1, i.e., whether the image of an edge stays in the open right half plane for all ω. For each edge in Q_{edge}, the corresponding transfer function (2) is typically expressed as

$$g(s,\lambda) := \frac{N(s,\lambda)}{D(s,\lambda)} = \frac{N_0(s) + \lambda N_1(s)}{D_0(s) + \lambda D_1(s)}$$

$$= \frac{\sum_{i=0}^{m} c_i s^i}{\sum_{i=0}^{n} d_i s^i} = \frac{\sum_{i=0}^{m} [c_{i0} + \lambda c_{i1}] s^i}{\sum_{i=0}^{n} [d_{i0} + \lambda d_{i1}] s^i}, \quad \underline{\lambda} \leq \lambda \leq \bar{\lambda} \tag{9}$$

Where λ represent the free parameter in Q_b. From (9) it follows that

$$\mathrm{Re}[g(j\omega,\lambda)] > 0$$

if and only if

$$P(\lambda\ ,\ \omega) := \lambda^2 \mathrm{Re}(N_1 D_1^*) + \lambda \mathrm{Re}(N_0 D_1^* + N_1 D_0^*) + \mathrm{Re}(N_0 D_0^*) > 0,$$
$$\forall \omega \in [0, \infty),\ \lambda \in [\underline{\lambda},\ \bar{\lambda}] \quad . \tag{10}$$

In the sequel, we will derive a finite algorithm for checking (10).

Lemma 3 (Key Lemma): Denote

$$P(\lambda,\omega) = a_1(\omega)\lambda^2 + a_2(\omega)\lambda + a_3(\omega) \tag{11}$$

where $a_i(\omega)$, $i = 1, 2, 3$, are real polynomials in ω. Then
$P(\lambda,\omega) > 0$, $\forall \lambda \in [\underline{\lambda},\bar{\lambda}]$ *and* $\forall \omega \in [0, +\infty)$
if and only if the following conditions hold:

i) $a_1(+\infty)\lambda^2 + a_2(+\infty)\lambda + a_3(+\infty) > 0$, $\forall \lambda \in [\underline{\lambda},\bar{\lambda}]$;

ii) $a_1(\omega)\bar{\lambda}^2 + a_2(\omega)\bar{\lambda} + a_3(\omega) > 0$,
$a_1(\omega)\underline{\lambda}^2 + a_2(\omega)\underline{\lambda} + a_3(\omega) > 0$, $\forall \omega \in [0, +\infty)$;
and

iii) $2a_3(\omega) + a_2(\omega)(\bar{\lambda} + \underline{\lambda}) + 2a_1(\omega)\underline{\lambda}\bar{\lambda} > 0$
whenever $a_2^2(\omega) - 4a_1(\omega)a_3(\omega) = 0$. ❑

Proof: For each $\omega \in \mathbb{R}$, consider two cases according to the value of $a_1(\omega)$.
Case One: $a_1(\omega) \neq 0$. In this case $P(\lambda,\omega)$ is part of a parabola with two end points $\mathcal{E} = a_1\bar{\lambda}^2 + a_2\bar{\lambda} + a_3$ and $\mathcal{S} = a_1\underline{\lambda}^2 + a_2\underline{\lambda} + a_3$. This parabola attains its minimum or maximum value at $\lambda^0 = -\frac{a_2}{2a_1}$ with the corresponding

$$P(\lambda^0,\omega) = \frac{4a_1a_3 - a_2^2}{4a_1}.$$

So, we need to show that, under conditions i) and ii), $P(\lambda,\omega) > 0$, $\forall \lambda \in [\underline{\lambda},\ \bar{\lambda}\ ]$ *and* $\forall \omega \in [\ 0, +\infty)$,
if and only if iii) is true. For this purpose we consider two sub-cases.

Sub-case one: $a_1(\omega) < 0$. In this case, λ^0 corresponds to the maximum of $P(\lambda,\omega)$. Therefore $\min_{\lambda} P(\lambda,\omega) = \min(\mathcal{S},\mathcal{E}) > 0$ will guarantee that $4a_1a_3 - a_2^2 \neq 0$. That is, the condition iii) is void.

Sub-case two: $a_1(\omega) > 0$. Now

$$\min_{\lambda \in \mathbb{R}} P(\lambda,\omega) = P(\lambda^0,\omega) = \frac{4a_1a_3 - a_2^2}{4a_1}.$$

From i) we know that
$P(\lambda, +\infty) > 0$, $\forall \lambda \in [\underline{\lambda},\ \bar{\lambda}\ ]$. Since $P(\lambda,\omega)$ is a continuous function of λ & ω, if there were $\tilde{\lambda}$ & $\tilde{\omega}$ such that $P(\tilde{\lambda},\tilde{\omega}) < 0$, there would exist an $\hat{\omega} \in [0, +\infty)$ such that $P(\lambda^0,\hat{\omega}) = 0$. So it is enough to check that

$$\lambda^0 \notin [\underline{\lambda},\ \bar{\lambda}], \qquad \text{whenever}$$
$$P(\lambda^0,\omega) = \frac{4a_1a_3 - a_2^2}{4a_1} = 0 \tag{12}$$

To check (12) we observe that

$$\lambda^0 \notin [\underline{\lambda},\ \bar{\lambda}] \Leftrightarrow (\lambda^0 - \bar{\lambda})(\lambda^0 - \underline{\lambda}) > 0$$
$$\Leftrightarrow a_2^2 + 2a_1a_2(\underline{\lambda} + \bar{\lambda}) + 4a_1^2\underline{\lambda}\bar{\lambda} > 0 \tag{13}$$

which is equivalent to the condition iii) after substituting in $a_2^2 - 4a_1a_3 = 0$.
Case Two: $a_1(\omega) = 0$. Obviously, ii) is both necessary and sufficient for

$$P(\lambda,\omega) > 0,\ \forall \lambda \in [\underline{\lambda},\ \bar{\lambda}].$$

We need to confirm that iii) hold automatically. Indeed, if $a_2^2 - 4a_1a_3 = 0$, then $a_2 = 0$. Consequently, $2a_3 + a_2(\bar{\lambda} + \underline{\lambda}) + 2a_1\underline{\lambda}\bar{\lambda} = 2a_3 > 0$, due to ii).

▼▼▼

From Lemma 3 and the following result, it become clear that the SPRness of an edge can be tested with a finite number of elementary operations.

Lemma 4 [15] Given two real polynomials $r_1(\omega)$ and $r_2(\omega)$, the following two statements are equivalent:

i) $r_2(\omega) > 0$, whenever $r_1(\omega) = 0$, for every $\omega \in (-\infty, +\infty)$;

ii) $I_{-\infty}^{+\infty}\frac{\dot{r}_1(\omega)}{r_1(\omega)} = I_{-\infty}^{+\infty}\frac{\dot{r}_1(\omega)r_2(\omega)}{r_1(\omega)}$. □

Now, it is straightforward to obtain the following theorems.

Theorem 2 Define

$$f_1 = a_2^2 - 4a_1a_3, \quad f_2 = 2a_3 + a_2(\bar{\lambda} + \underline{\lambda}) + 2a_1\underline{\lambda}\bar{\lambda} \tag{14}$$

Then, the transfer function in (9) is SPR $\forall\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and $\forall\omega \in \mathbb{R}$ if and only if the following conditions hold:

i) $D_0(s)$ is strictly Hurwitz,

ii) Condition ii) in Lemma 3 are satisfied,

iii) $I_{-\infty}^{+\infty}\frac{\dot{f}_1(\omega)}{f_1(\omega)} = I_{-\infty}^{+\infty}\frac{\dot{f}_1(\omega)f_2(\omega)}{f_1(\omega)}$, and

iv) a) $\frac{c_m}{d_n} > 0$, if $r^* = 0$ (i.e., m=n);

b) $c_{n-1}d_{n-1} - c_{n-2}d_n > 0$, if $r^* = 1$ (i.e., m = n−1);

c) $\begin{Bmatrix} c_nd_n - c_{n+1}d_{n-1} > 0 \\ \frac{c_{n+1}}{d_n} > 0 \end{Bmatrix}$, if $r^* = -1$ (i.e., m=n+1). □

Proof: The proof of Theorem 2 follows from Theorem 1, Lemma 3 & Lemma 4. It is obvious that condition (i) is the same in both Theorem 1 & Theorem 2. From the definition of $P(\lambda, \omega)$ in (10), it is clear that condition ii) in Theorem 1 holds if and only if conditions i)−iii) in Lemma 3 do. Note that condition iii) in Lemma 3 is the same as condition iii) in Theorem 2 (due to Lemma 4). Also, condition i) in Lemma 3 is implied by condition iii) in Theorem 1. Therefore, it suffices to show that condition iii) in Theorem 1 is equivalent to condition iv) in Theorem 2. To this end, we obtain from (9) that

$$g(j\omega,\lambda) = \frac{(-1)^nc_md_n(j\omega)^{m+n} + (-1)^{n-1}c_md_{n-1}(j\omega)^{m+n-1} + (-1)^nc_{m-1}d_n(j\omega)^{m+n-1} + \dots}{(-1)^nd_n^2(j\omega)^{2n} + \dots} \tag{15}$$

and analyze three cases:

a) $r^* = 0$. In this case, $\lim_{|\omega|\to\infty} Re[g(j\omega,\lambda)] = \frac{c_n}{d_n}$.

b) $r^* = 1$. This implies that

$$\lim_{|\omega|\to\infty} \omega^2Re[g(j\omega,\lambda)] = \frac{c_{n-1}d_{n-1} - c_{n-2}d_n}{d_n^2} > 0.$$

c) $r^* = -1$. This gives

$$\lim_{|\omega|\to\infty} Re[g(j\omega,\lambda)] = \frac{c_nd_n - c_{n+1}d_{n-1}}{d_n^2} > 0,$$

$$\lim_{|\omega|\to\infty} \frac{g(j\omega,\lambda)}{j\omega} = \frac{c_{n+1}}{d_n}.$$

It is therefore clear that condition iii) in Theorem 1 is equivalent to condition iv) in Theorem 2. ▼▼▼

In view of Lemma 1 and Theorem 2, we have the following general result (the proof is omitted due to its triviality).

Theorem 3 Similar to that of (10) − (14), we define

$$f_{1j} = a_{2j}^2 - 4a_{1j}a_{3j},$$

$$f_{2j} = 2a_{3j} + a_{2j}(\bar{\lambda} + \underline{\lambda}) + 2a_{1j}\underline{\lambda}\bar{\lambda} \tag{16}$$

where j denote the jth edge of Q_{edge}. Then, $\mathcal{G}$ in (7) is Robustly SPR if and only if the following conditions hold:

i) $\sum_{i=0}^{n} a_i(q^0)s^i$ is strictly Hurwitz for some $q^0 \in V(Q_d) \oplus E(Q_b)$;

ii) $\mathrm{Re}\, G(j\omega, q) > 0$, $\forall q \in Q_v$, $\omega \in [0, +\infty)$; and

iii) $I_{-\infty}^{+\infty}\frac{\dot{f}_{1j}(\omega)}{f_{1j}(\omega)} = I_{-\infty}^{+\infty}\frac{\dot{f}_{1j}(\omega)f_{2j}(\omega)}{f_{1j}(\omega)}$, $j = 1, \dots, 2^{p_n}\,2^{p_d}\,p_b\,2^{p_b-1}$; and

iv) a) $\frac{b_n(q)}{a_n(q)} > 0$, $\forall q \in Q_{edge}$, if $r^* = 0$; or

b) $b_{n-1}(q)a_{n-1}(q) - b_{n-2}(q)a_n(q) > 0$, $\forall q \in Q_{edge}$, if $r^* = 1$; or

c) $\begin{Bmatrix} b_n(q)a_n(q) - b_{n+1}(q)a_{n-1}(q) > 0 \\ \frac{b_{n+1}(q)}{a_n(q)} > 0 \end{Bmatrix}$, $\forall q \in Q_{edge}$, if $r^* = -1$. □

4 Computational aspects

We now briefly exam the computational requirement of the algorithm in Theorem 3. Note that there are all together $2^{p_n}\,2^{p_d}\,2^{p_b}$ vertices and $2^{p_n}\,2^{p_d}\,p_b\,2^{p_b-1}$ edges in $\mathcal{G}$. To check

Theorem 3 i), one Routh table of degree n is needed. In Theorem 3 ii), a polynomial is strictly positive if and only if it has not real zero, i.e. its corresponding Cauchy index is zero. This requires one Routh table of degree 2n for each vertex. As for Theorem 3 iii), if we express

$$\frac{\dot{f}_{1j} f_{2j}}{f_{1j}} = R_{1j} + \frac{\tilde{f}_{2j}}{f_{1j}}, \tag{17}$$

where R_{1j} is a polynomial and $\deg \tilde{f}_{2j} < \deg f_{1j}$, then

$$I_{-\infty}^{+\infty} \frac{\dot{f}_{1j} f_{2j}}{f_{1j}} = I_{-\infty}^{+\infty} \frac{\tilde{f}_{2j}}{f_{1j}}. \tag{18}$$

Hence, we can check Theorem 3 iii) by two Routh tables of degree 2n for each edge.

In addition, some simple computation is needed for Theorem 3 iv), i.e., we have to determine whether for each edge of Q_{edge} a portion of a parabola stays positive. We can summarize the above in the table at the end of this paper. Note that a Routh table of degree n requires $O(n^2)$ number of calculations, so the complexity of the algorithm is $O(n^2)$.

5 Conclusion

The results above address the SPR problem of a family of rational functions with respect to the open left half plane. We have provided a finite algorithm which can test the robust SPRness by using only $O(n^2)$ elementary operations, where n is the degree of the rational functions. Thus, the commonly used value set approach, or frequency sweeping, is obviated. Although our results are for the Hurwitz stability region, an extension of the results can be obtained for more general stability regions, such as the unit circle or other circular regions which are of importance to filter designs[16]. This can be done by using the bilinear transformation which converts the circular regions to the open left plane. For more general stability regions, similar results can also be developed provided that the region can be converted into the open left half plane by using the so-called strongly admissible rational function(see[18] for definition).

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Number of Routh tables	Deg. *	Required by	Note
one	n	Hurwitzness of $\sum_{i=0}^{n} a_{i0} s^i$	Theorem 3 i)
$2^{p_n}\ 2^{p_d}\ 2^{p_b}$	2n	$\mathrm{Re}\, G(j\omega, q) > 0$	Theorem 3 ii)
$2^{p_n}\ 2^{p_d}\ p_b\ 2^{p_b}$	2n	Cauchy indices	Theorem 3 iii)

* Denotes the degree of the polynomial.