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NEW EXTREME POINT RESULTS ON ROBUST STRICT POSITIVE REALNESS

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ABSTRACT

This paper considers the robust strict positive real (SPR) problem for a family of plants of the form $G(s,q) = N(s,q_n)D^{-1}(s,q_d)-\alpha$, where $N(s,q_n)$ and $D(s,q_d)$ are multiaffine in uncertain parameters q_n and q_d, respectively, and $\alpha > 0$. In the discrete-time setting, this problem plays an important role in digital quantization. Several results are presented. First, we prove that this plant family is robustly SPR if and only if all "corner plants" in the family are SPR. Secondly, we show that, if this plant family is robustly SPR, it admits a multiaffine Lyapunov matrix for the Kalman-Yakubovic-Popov (KYP) inequality, i.e, the Lyapunov matrix is multiaffine in the uncertain parameters. This result is useful in robustness analysis of time-varying systems. Thirdly, we relate the robust SPRness of this plant family to the robust strict bounded realness (SBRness) of a plant family involving the inverse of $N(s,q_n)D^{-1}(s,q_d)$. We show that multiaffine Lyapunov matrix for the KYP inequality of the first plant family yields a multiaffine Lyapunov matrix for the bounded real inequality of the second plant family. Finally, the robust SPR problem is considered for a more general plant family with applications in circuits and communication systems.

1. INTRODUCTION

Consider a family of linear time-invariant plants

$$\mathcal{G} = \{G(s,q) : \ q \in Q\} \tag{1}$$

where Q is a hyperrectangle representing the set of uncertainty. We are interested in necessary and sufficient conditions for the family $\mathcal{G}$ to be robustly strictly positive real (SPR), i.e., every member of $\mathcal{G}$ being SPR.

For an interval plant family, i.e., $G(s,q)$ is a ratio of two interval polynomials, Dasgupta *et. al.* [3] show that this family is robustly SPR if and only if 16 special "corner plants" called Kharitonov plants are SPR. This result is recently generalized by Chapellat *et. al.* [2] to the case

$$G(s,q) = G_I(s,q) - \alpha \tag{2}$$

where $G_I(s,q)$ is an interval plant and α is any scalar number. Again, they show that it is necessary and sufficient to check the 16 Kharitonov plants.

Since interval plant families are often very restrictive in applications and that the results above do not generalize to discrete-time plants, many other results have been reported. It is shown in Dasgupta *et. al.* [3] that, when the numerator and denominator of the plant are multiaffine functions of two separate sets of uncertain parameters, the robust SPRness of the whole family is equivalent to the SPRness of all corner plants. For the case where the numerator and denominator are affine functions of the same set of uncertain parameters. It is shown in Fu [5] that the family of plants is robustly SPR if and only if all the "edge plants" (the plants corresponding to the edges of Q) are all SPR. Also analyzed in [5] is a more general case where the numerator and denominator of the plant are multiaffine functions of two separate sets of uncertain parameters, and they both are affine in a third set of parameters. The results above are applicable to both continuous- and discrete-time plants.

This paper considers the robust SPR problem associated with the following uncertain plant family:

$$\begin{aligned}\mathcal{G} \;\; &= \;\; \{G(s,q) = N(s,q_n)D^{-1}(s,q_d) - \alpha : \\ &\qquad q_n \in Q_n, \;\; q_d \in Q_d\}, \;\; \alpha \geq 0,\end{aligned} \tag{3}$$

where $N(s,q_n)$ and $D(s,q_d)$ are square polynomial matrices in s and are multiaffine in q_n and q_d, respectively, Q_n and Q_d are hyperrectangles, s denotes the Laplace operator for continuous-time plants or the z-operator for discrete-time plants.

The robust SPR problem above in the discrete-time setting is motivated from applications in digital communications. For example, in decision feedback equalization, finite time recovery from errors is guaranteed if the channel transfer function $G(z)$ obeys

$$G(z) - \frac{1}{2} \text{ is SPR} \tag{4}$$

See, e.g., Kennedy, Anderson and Bitmead [7]. In the continuous-time setting, this problem arises in robustness analysis for electrical circuits and tuned filters in communication systems; see Example 2 in Section 3 for details.

In addition to finding simple conditions for determining the robust SPRness of a plant family, we are interested in a perhaps more important problem: Find a simply parametrized Lyapunov matrix $P(q)$ which establishes the robust SPRness in the state-space via the well-known Kalman-Yakubovic-Popov (KYP) Lemma [1]. It is known that such Lyapunov matrices play a critical role in robust stability analysis of systems which involve time-varying uncertain parameters; see, e.g., Dasgupta *et. al.* [4].

It is known that the robustness analysis for systems with time-varying parameters requires a slightly stronger notion of SPR called ρ-SPR. In this paper, a continuous-time plant $G(s)$ is ρ-SPR for $\rho \geq 0$ if $G(s-\rho)$ is SPR. Similarly, a discrete-time plant $G(z)$ is called ρ-SPR for $\rho > 0$ if $G(z/(1+\rho))$ is SPR. Most of our results will be in terms of ρ-SPR.

The third problem we are concerned with is to relate the robust SPR condition of $\mathcal{G}$ to the robust strict bounded realness (SBR) of a corresponding plant family. We also want to know how the Lyapunov matrix for the robust SPR condition of the first plant family is related to the Lyapunov matrix for the robust SBR condition for the second plant family. The notion of ρ-SBR will be used. Its definition is analogous to ρ-SPR.

Our main results are as follows: First we prove that the family of plants $\mathcal{G}$ in (3) is robustly ρ-SPR if and only if all corner plants are ρ-SPR. Secondly, we show that when $\mathcal{G}$ is robustly ρ-SPR, there is a multiaffine Lyapunov matrix $P(q)$ for establishing the robust ρ-SPRness via the KYP Lemma. Thirdly, we show that $\mathcal{G}$ is robustly ρ-SPR if and only if the following family of plants

$$\begin{aligned} \tilde{\mathcal{G}} &= \{\tilde{G}(s,q) = 2\alpha D(s,q_d)N^{-1}(s,q_n) - I : \\ &\qquad q_n \in Q_n, \quad q_d \in Q_d\} \end{aligned} \tag{5}$$

is robust ρ-SBR, and the latter occurs if and only if the corner plants of $\tilde{\mathcal{G}}$ are all ρ-SBR. Note in particular that $\tilde{G}(s,q)$ is affine in q_d while $G(s,q)$ is affine in q_n. Further, the Lyapunov matrix $\tilde{P}(q)$ for establishing the ρ-SBRness of $\tilde{\mathcal{G}}$ is also multiaffine and simply related to $P(q)$ by $\tilde{P}(q) = 2\alpha P(q)$. Finally, the corner result for the plant family in (3) is extended to a more general plant family which finds applications in robust SPR analysis of circuits and tuned filters for communication systems.

2. MAIN RESULTS

Consider the plant family $\mathcal{G}$ in (3), the first main result of our paper is given as follows:

Theorem 1 *Given $\rho > 0$, $\alpha > 0$ and the plant family $\mathcal{G}$ in (3). The following are equivalent:*

(i) *$\mathcal{G}$ is robustly ρ-SPR;*

(ii) *All the corner plants of $\mathcal{G}$ are ρ-SPR;*

(iii) *$N(s,q)$ is invertible for all $q_n \in Q_n$ and the plant family $\tilde{\mathcal{G}}$ in (3) is robust ρ-SBR;*

(iv) *$N(s,q)$ is invertible at all corners of Q_n and the corner plants of $\tilde{\mathcal{G}}$ are ρ-SBR.*

Further, the equivalence between (i) and (ii) above hold also when $\alpha = 0$.

Proof. We treat the continuous-time case only because the discrete-time case is almost identical. Also, we only need to consider the case when $\alpha > 0$ because the equivalence between (i) and (ii) when $\alpha = 0$ is known; see [3].

Note that $\mathcal{G}$ is ρ-SPR if and only if there exists $\epsilon > 0$ such that, for every complex s with $\mathrm{Re}[s] \geq -\rho - \epsilon$, the following holds:

$$\begin{aligned} &D(s,q_d) \neq 0, \\ &N(s,q_n)D^{-1}(s,q_d) + (D^*(s,q_d))^{-1}N^*(s,q_n) \\ &-2\alpha I > 0, \\ &\qquad \forall q_n \in Q_n, q_d \in Q_d \end{aligned}$$

The above is equivalent to

$$\begin{aligned} &D^*(s,q_d)N(s,q_n) + N^*(s,q_n)D(s,q_d) \\ -\ &2\alpha D^*(s,q_d)D(s,q_d) > 0, \\ &\qquad \forall q_n \in Q_n, q_d \in Q_d \end{aligned} \tag{6}$$

which is in turn equivalent to

$$\begin{bmatrix} D^*N + N^*D & D^* \\ D & (2\alpha)^{-1}I \end{bmatrix} > 0, \quad \forall q_n \in Q_n, q_d \in Q_d,$$

which is multiaffine in (q_n, q_d). Hence, (i)<=> (ii).

To see (i)<=>(iii) and (ii)<=>(iv), we simply note that

$$
\begin{aligned}
& G(s,q) \text{ is } \rho-\text{SPR} \\
<=> \;& \alpha^{-1}G(s,q) \text{ is } \rho-\text{SPR} \\
<=> \;& \alpha^{-1}(G(s,q)+G^*(s,q))>0, \\
& \qquad \forall \text{Re}[s] \geq -\rho-\epsilon \text{ for some } \epsilon>0 \\
<=> \;& (I-\alpha^{-1}G)^*(I-\alpha^{-1}G) \\
& \quad < (I+\alpha^{-1}G)^*(I+\alpha^{-1}G), \\
& \qquad \forall \text{Re}[s] \geq -\rho-\epsilon \text{ for some } \epsilon>0 \\
<=> \;& (I-\alpha^{-1}G)(I+\alpha^{-1}G)^{-1} \text{ is } \rho-\text{SBR} \\
<=> \;& \tilde{G}(s,q) \text{ is } \rho-\text{SBR}
\end{aligned}
$$

$\nabla\nabla\nabla$

Remark 1 *We require $\alpha \geq 0$ in the extreme-point result above. It is shown in Chapellat et. al. [2] that, when $N(s,q_n)$ and $D(s,q_d)$ are interval polynomials and that the continuous-time robust SPR (rather than robust ρ-SPR) problem is concerned, the assumption of $\alpha \geq 0$ can be removed. More specifically, they show that the plant family $\mathcal{G}$ in this case is robustly SPR if and only if 16 corner plants are SPR. However, we show in the next example, the extreme-point result can no longer be guaranteed in the general case when $\alpha < 0$.*

Example 1: Consider the following family of plants:

$$\mathcal{G} = \{G(s,q) : q \in [0,\ 1]\} \tag{7}$$

where

$$G(s,q) = \frac{0.01s^3}{s^3+(1.5+q)s^2+(1.5+q)s+2.06+4q}+1 \tag{8}$$

It is verified that both $G(s,0)$ and $G(s,1)$ are SPR. But $G(s,0.5)$ is not SPR because it is unstable.

The next result deals with parametric Lyapunov matrices:

Theorem 2 *Given $\rho > 0$, suppose the continuous-time plant family $\mathcal{G}$ in (3) is robustly ρ-SPR. Let $a(s)$ be any Hurwitz-stable polynomial with its degree equal to the larger of the McMillian degrees of the $N(s,q_n)$ and $D(s,q_d)$. Let*

$$\{A,B,C_n(q_n),D_n(q_n)\} \text{ and } \{A,B,C_d(q_d),D_d(q_d)\}$$

be state-space realizations of

$$\frac{N(s,q_n)}{a(s+\rho)} \text{ and } \frac{D(s,q_d)}{a(s+\rho)},$$

respectively. Then we have the following properties:

(i) *$G(s,q)$ has the following realization:*

$$
\begin{aligned}
& G(s,q) = \mathcal{C}_n(q)(sI-\mathcal{A}_n(q))^{-1}\mathcal{B}_n(q)+\mathcal{D}_n(q) \\
=\; & (C_n - D_nD_d^{-1}C_d)\left(sI-A+BD_d^{-1}C_d\right)^{-1} \\
& \cdot BD_d^{-1} + (D_nD_d^{-1}-\alpha I)
\end{aligned} \tag{9}
$$

(ii) *When $\alpha > 0$, $\tilde{G}(s,q)$ exists and has the following realization:*

$$
\begin{aligned}
& \tilde{G}(s,q) = \mathcal{C}_d(q)(sI-\mathcal{A}_d(q))^{-1}\mathcal{B}_d(q)+\mathcal{D}_d(q) \\
=\; & 2\alpha(C_d - D_dD_n^{-1}C_n)\left(sI-A+BD_n^{-1}C_n\right)^{-1} \\
& \cdot BD_n^{-1} + (2\alpha D_dD_n^{-1}-I)
\end{aligned} \tag{10}
$$

(iii) *There exists a parametric matrix $P(q) = P(q_n,q_d)$, which is multiaffine in (q_n,q_d), symmetric and positive-definite such that the following KYP inequality for $G(s,q)$ holds for all $q_n \in Q_n, q_d \in Q_d$:*

$$\begin{bmatrix} \Omega_n(q) & P(q)\mathcal{B}_n(q)-\mathcal{C}_n^T(q) \\ \mathcal{B}_n(q)^TP(q)-\mathcal{C}_n(q) & -\mathcal{D}_n^T(q)-\mathcal{D}_n(q) \end{bmatrix} < 0 \tag{11}$$

where

$$\Omega_n(q) = (\mathcal{A}_n(q)+\rho I)^T P(q) + P(q)(\mathcal{A}_n(q)+\rho I)$$

(iv) *Let $P(q)$ be given as above and $\alpha > 0$. Define $\tilde{P}(q) = 2\alpha P(q)$. Then, (11) holds for all $q_n \in Q_n, q_d \in Q_d$ if and only if the following SBR inequality for $\tilde{G}(s,q)$ holds for all $q_n \in Q_n, q_d \in Q_d$:*

$$\begin{bmatrix} \Omega_d(q) & \Gamma_d(q) \\ \Gamma_d^T(q) & -I+\mathcal{D}_d^T(q)\mathcal{D}_d(q) \end{bmatrix} < 0 \tag{12}$$

where

$$
\begin{aligned}
\Omega_d(q) \;=\;& (\mathcal{A}_d(q)+\rho I)^T\tilde{P}(q) \\
+\;& \tilde{P}(q)(\mathcal{A}_d(q)+\rho I) + \mathcal{C}_d^T(q)\mathcal{C}_d(q)
\end{aligned}
$$

and

$$\Gamma_d(q) = \tilde{P}(q)\mathcal{B}_d(q) + \mathcal{C}_d^T(q)\mathcal{D}_d(q)$$

(v) *The inequality (11) (resp. (12)) holds for all $q_n \in Q_n, q_d \in Q_d$ if and only if it holds at all corners of Q_n and Q_d.*

Proof. See Appendix A. $\nabla\nabla\nabla$

Remark 2 *The significance of (v) above lies in the fact that both (11) and (12) are affine in $P(q)$. Denoting*

$$q_n = (q_{n,1}, q_{n,2}, \cdots), \quad q_d = (q_{nd,1}, q_{d,2}, \cdots)$$

$P(q)$ can be written as

$$P(q) = P_0 + \sum_i P_{n,i}q_{n,i} + \sum_i P_{d,i}q_{d,i} + \sum_{i\neq j} P_{n,ij}q_{n,i}q_{n,j} + \sum_{i\neq j} P_{d,ij}q_{d,i}q_{d,j} + \cdots \quad (13)$$

Therefore, (11) (or (12)) represents a finite set of linear matrix inequalities (LMIs) for P_0, $P_{n,i}$, $P_{d,i}$, $P_{n,ij}$, $P_{d,ij}, \cdots$. That is, $P(q)$ can be solved using standard LMI algorithms. Alternatively, $P(q)$ can be directly constructed from the solutions of $P(q)$ at all corners of Q_n and Q_d; as demonstrated in Dasgupta et. al. [3] for the case when the plant denominator is fixed.

The discrete-time version of the above result is given below:

Theorem 3 *Given $\rho > 0$, suppose the discrete-time plant family $\mathcal{G}$ in (3) is robustly ρ-SPR. Let $a(z)$ be any Schur stable polynomial with its degree equal to the larger of the McMillian degrees of the $N(z,q_n)$ and $D(z,q_d)$. Let*

$$\{A, B, C_n(q_n), D_n(q_n)\} \text{ and } \{A, B, C_d(q_n), D_d(q_d)\}$$

be state-space realizations of

$$\frac{N(z,q_n)}{a(z(1+\rho)^{-1})} \text{ and } \frac{D(z,q_d)}{a(z(1+\rho)^{-1})},$$

respectively. Then the properties (i) -(v) in Theorem 2 hold when (11) and (12) are replaced by the following two inequalities:

$$\begin{bmatrix} \mathcal{A}_n^T P \mathcal{A}_n - \frac{P}{(1+\rho)^2} & \mathcal{A}_n^T P \mathcal{B}_n - \mathcal{C}_n^T \\ \mathcal{B}_n^T P \mathcal{A}_n - \mathcal{C}_n & \mathcal{B}_n^T P \mathcal{B}_n - \mathcal{D}_n^T - \mathcal{D}_n \end{bmatrix} < 0 \quad (14)$$

$$\begin{bmatrix} \mathcal{A}_d^T \tilde{P} \mathcal{A}_d - \frac{\tilde{P}}{(1+\rho)^2} + \mathcal{C}_d^T \mathcal{C}_d & \mathcal{A}_d^T \tilde{P} \mathcal{B}_d + \mathcal{C}_d^T \mathcal{D}_d \\ \mathcal{B}_d^T \tilde{P} \mathcal{A}_d + \mathcal{D}_d^T \mathcal{C}_d & \mathcal{B}_d^T \tilde{P} \mathcal{B}_d - I + \mathcal{D}_d^T \mathcal{D}_d \end{bmatrix} < 0 \quad (15)$$

Proof. See Appendix B. $\nabla\nabla\nabla$

For single-input-single-output (SISO) plants, we generalize $\mathcal{G}$ in (3) to the following plant family:

$$\mathcal{G} = \{\sum_{i=1}^{k} \frac{n_i(s,q_{n_i})}{d_i(s,q_{d_i})} - \frac{n_0(s,q_{n_0})}{d_0(s)} : q_{n_i} \in Q_{n_i},\ \ q_{d_i} \in Q_{d_i}\} \quad (16)$$

where Q_{n_0}, Q_{n_i} and Q_{d_i}, $i = 1, \cdots, k$, are all hyperrectangles. When $k = 1$ and $G_0(s, q_0) = \alpha$, we obviously return to the plant family in (3).

The motivation for this problem stems from robustness analysis of circuits and tuned filters for communication systems. This will be demonstrated through an example (Example 2) in the next section.

Define

$$\mathcal{G}_i = \{\frac{n_i(s,q_{n_i})}{d_i(s,q_{d_i})} : q_{n_i} \in Q_{n_i},\ \ q_{d_i} \in Q_{d_i}\}, i = 1, \cdots, k \quad (17)$$

and

$$\mathcal{G}_0 = \{\frac{n_0(s,q_{n_0})}{d_0(s)} : q_{n_0} \in Q_{n_0}\} \quad (18)$$

We then have the following result, which holds for both continuous-time and discrete-time plants.

Theorem 4 *Suppose all the corner plants of $\mathcal{G}_i, i = 0, 1, \cdots, k$, are ρ-SPR. Then, $\mathcal{G}$ is robustly ρ-SPR if and only if all the corner plants of $\mathcal{G}$ are ρ-SPR.*

Proof. As before, only the continuous-time case is considered. The necessity is obvious. To see the sufficiency, we assume that all corner plants of $\mathcal{G}$ and $\mathcal{G}_i$ are ρ-SPR. Let $\epsilon > 0$ be such that

$$\text{Re}[G_i(s)] > 0, \forall \text{Re}[s] \geq -\rho - \epsilon \quad (19)$$

holds for every corner plant of $\mathcal{G}$ and $\mathcal{G}_i$, $i = 0, 1, \cdots, k$. Suppose there exist some $G(s,q) \in \mathcal{G}$ and some s with $\text{Re}[s] \geq -\rho - \epsilon$ such that

$$\text{Re}[G(s,q)] \leq 0 \quad (20)$$

Since (20) is multiaffine in q_{n_i}, it must also hold for some (different) q for which every q_{n_i} takes a corner vector. Suppose some q_{d_j} component, $1 \leq j \leq k$, takes an interior vector, we claim that (20) holds when this q_{d_j} is replaced by a corner vector. To see this, we rewrite (20) as follows:

$$\text{Re}[\frac{n_j(s,q_{n_j})}{d_j(s,q_{d_j})}] - \alpha \leq 0 \quad (21)$$

where

$$\alpha = \text{Re}\left[-\sum_{i\neq j} \frac{n_i(s,q_{n_i})}{d_i(s,q_{d_i})} + \frac{n_0(s,q_{n_0})}{d_0(s)}\right]$$

From (19), we know that $\text{Re}[\frac{n_j(s,q_{n_j})}{d_j(s,q_{d_j})}] > 0$ even when q_{d_i} doesn't take a corner vector (see, [3]). Subsequently, $\alpha > 0$. Since (21) is equivalent to

$$\begin{aligned} f(q_{d_j}) &= \text{Re}[n_j(s,q_{n_j})d_j^*(s,q_{d_j})] \\ &- \alpha d_j^*(s,q_{d_j})d_j(s,q_{d_j}) \leq 0 \end{aligned}$$

The minimum of $f(\cdot)$ over q_{d_j} must occur at a corner because $\alpha > 0$. Therefore, the above inequality (and hence (21)) must also hold when q_{d_j} takes one of the corner vectors. Since the analysis above holds for all $j = 1, \cdots, k$, (20) must hold at a corner q, which contradicts our assumption. This completes our proof. $\nabla\nabla\nabla$

3. APPLICATION EXAMPLE

In this section, we demonstrate via an example how Theorem 3 is applied in the so-called tuned filters for communication systems.

Example 2: Consider the tuned filter in Figure 1. The impedence of the circuit is given by

$$Z(s) = R_1 + \frac{s}{s + \Gamma G_2} + \frac{1}{Cs + G_3} \quad (22)$$

where $G_2 = R_2^{-1}, G_3 = R_3^{-1}, \Gamma = L^{-1}$. Suppose R_1, G_2, G_3, Γ and C are,say, all normalized to be 1 with $\pm 10\%$ tolerance value. We are interested in determining the largest $\alpha > 0$ such that $Z(s) - \alpha$ remains to be SPR. Obviously, each individual term in $Z(s)$ is SPR. Using Theorem 4, we know $Z(s) - \alpha$ is SPR for all admissible parameters if and only if it is so at all 32 corners. So $\alpha_{\max}$ can be simply determined, e.g., from the 32 Nyquist plots of $G(s)$.

Alternatively, for a given $0 < \alpha < \alpha_{\max}$, we can seek for the maximum ρ such that ρ-SPRness of $Z(s) - \alpha$ is preserved for parameter values. Again, we only need to seek at all the corner plants only.

Collectively, ρ and α provide two different measures of the "degree" of passivity for the filter.

4. CONCLUSIONS

In this paper, we have established simple necessary and sufficient conditions to determine the robust strict positive realness of a plant family in (3). We have also shown how to derive simple parametric Lyapunov matrices for the plant family. This result is of particular use in robustness analysis of time-varying systems. Due to the nature of SPRness, we are able to treat both the continuous-time setting and the discrete-time setting by using the same framework. It is also interesting to note that the results in Theorems 1-3 allow the plant family to be multi-input-multi-output. In the single-input-single-outut case, the robust SPR result for the plant family (3) is generalized to the plant family (16) which finds applications in robustness analysis for electrical circuits and communication systems.

A. PROOF OF THEOREM 2

The following lemma, called Parametric KYP Lamma, will play an vital role in the proof of Theorem 1. This result is quoted from Fu and Dasgupta [6].

Lemma 1 (Parametric KYP Lemma) *Given matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$, $m \le n$, a hyperrectangular set $Q \subset \mathbf{R}^p$, a parametric matrix described by*

$$\Omega(q) = \Omega^T(q) = \Omega_M(q) + \sum_{i=1}^{p} q_i^2 \Omega_{ii} \in \mathbf{R}^{(n+m)\times(n+m)} \quad (23)$$

where $\Omega_M(q)$ is multiaffine in q and

$$\Omega_{ii} \ge 0, \quad i = 1, \cdots, p \quad (24)$$

Then the following two conditions are equivalent:

i) *There exists $\epsilon > 0$ such that*

$$[B^T((sI-A)^{-1})^* \;\; I]\Omega(q)\begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \mathrm{Re}[s] \ge -\epsilon \quad (25)$$

for all $q \in Q$.

ii) *There exists a multiaffine matrix*

$$P(q) = P^T(q) \in \mathbf{R}^{n\times n}, \quad q \in Q \quad (26)$$

such that

$$\Pi(q) = \begin{bmatrix} A^T P(q) + P(q)A & P(q)B \\ B^T P(q) & 0 \end{bmatrix} + \Omega(q) < 0 \quad (27)$$

for all $q \in Q$.

iii) *The inequality (25) holds at all vertices of Q.*

iv) *The inequality (27) holds at all vertices of Q.*

Proof of Theorem 2: The conditions (i) and (ii) of Theorem 2 are straightforward to verify, so the details are omitted. We set out to prove (iii)-(v).

Following the proof of Theorem 1, $\mathcal{G}$ is robustly ρ-SPR if and only if (6) holds. Using the state-space realization of $N(s, q_n)/a(s+\rho)$ and $D(s, q_d)/a(s+\rho)$, (6) is equivalent to the following:

$$[\; ((sI-A)^{-1}B)^* \;\; I \;]\,\Omega(q)\begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \mathrm{Re}[s] \ge -\rho - \epsilon, q_n \in Q_n, q_d \in Q_d \quad (28)$$

for some $\epsilon > 0$, where $\Omega(q)$ is given by

$$\begin{aligned} \Omega(q) &= -\begin{bmatrix} C_d^T \\ D_d^T \end{bmatrix} [\; C_n \;\; D_n \;] \\ &- \begin{bmatrix} C_n^T \\ D_n^T \end{bmatrix} [\; C_d \;\; D_d \;] \\ &+ 2\alpha \begin{bmatrix} C_d^T \\ D_d^T \end{bmatrix} [\; C_d \;\; D_d \;] \end{aligned} \quad (29)$$

Shifting s by ρ and denoting $A - \rho I$ by A_ρ, (6) becomes

$$\left[\; ((sI - A_\rho)^{-1}B)^* \quad I \; \right] \Omega(q) \left[\begin{array}{c} (sI - A_\rho)^{-1}B \\ I \end{array} \right] < 0,$$

$$\forall \mathrm{Re}[s] \geq -\epsilon, q_n \in Q_n, q_d \in Q_d \qquad (30)$$

Now applying the Parametric KYP Lemma, we know that (30) holds if and only if there exists a multiaffine $P(q) = P(q)^T$ such that the following holds

$$\left[\begin{array}{cc} (A^T + \rho I)P(q) + P(q)(A + \rho I) & P(q)B \\ B^T P(q) & 0 \end{array} \right] + \Omega(q) < 0 \qquad (31)$$

for all $q_n \in Q_n$ and $q_d \in Q_d$, or equivalently, at all corners of Q_n and Q_d. Now, (11) is obtained by right- and left-multiplying (31) with

$$\left[\begin{array}{cc} I & 0 \\ -D_d^{-1}C_d & D_d^{-1} \end{array} \right]$$

and its transpose. Similarly, (12) is obtained by right- and left-multiplying (31) with

$$\sqrt{2\alpha} \left[\begin{array}{cc} I & 0 \\ -D_n^{-1}C_n & D_n^{-1} \end{array} \right]$$

and its transpose. The details are straightforward (but a bit tedious) and thus omitted.

B. PROOF OF THEOREM 3

Similar to the proof of Theorem 1, we need to have a discrete time version of the Parametric KYP Lamma. This result is also quoted from Fu and Dasgupta [6].

Lemma 2 (Discrete-time Parametric KYP Lemma) *Given matrices $A, B, \Omega(q), Q$ as in Lemma 1. Let $0 < \mu < 1$. Then the following two conditions are equivalent:*

i)

$$[B^T((zI - A)^{-1})^* \quad I]\Omega(q) \left[\begin{array}{c} (zI - A)^{-1}B \\ I \end{array} \right] < 0,$$

$$\forall |z| \geq \mu, \; q \in Q \qquad (32)$$

for all $q \in Q$.

ii) *There exists a multilinear matrix*

$$P(q) = P^T(q) \in \mathbf{R}^{n \times n}, \quad q \in Q \qquad (33)$$

such that

$$\Pi(q) = \left[\begin{array}{cc} A^T PA - \mu^2 P & A^T PB \\ B^T PA & B^T PB \end{array} \right] + \Omega(q) < 0 \qquad (34)$$

for all $q \in Q$.

iii) *The inequality (32) holds at all vertices of Q;*

iv) *The inequality (33) holds at all vertices of Q.*

Proof of Theorem 2: The proof is almost identical and thus omitted.

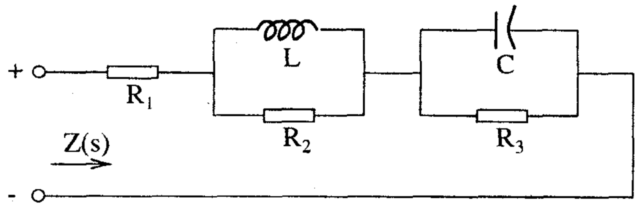

Figure 1: Tuned Filter

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