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ROBUSTNESS ANALYSIS OF LINEAR SYSTEMS WITH TIME-VARYING PARAMETERS

Minyue Fu[1], *Carlos E. de Souza*[2] *and Soura Dasgupta*[3]

[1] Dept. Electrical and Computer Eng., Univ. Newcastle, Australia (eemf@ee.newcastle.edu.au)
[2] Dept. Systems and Control, Laboratório Nacional de Computação Científica – LNCC, Brazil
[3] Dept. Electrical and Computer Eng., Univ. Iowa, U.S.A.

1. INTRODUCTION

The objective of this paper is to provide a useful tool for robustness analysis of linear systems which are subject to time-varying parameters. More specifically, we consider the following uncertain system

$$\dot{x} = A(q(t))x = (A_0 - BD^{-1}(q(t))C(q(t)))x, \quad (1.1)$$

where $q(t) \in R^p$ represents uncertain parameters, belonging to a bounding set $Q = [-1,\ 1]^p$; $q(\cdot)$ is assumed to be differentiable and belongs to a set $\mathbf{Q}$; $A_0 \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ is a constant full column rank matrix, $m \le n$, $C(q) \in \mathcal{R}^{m\times n}$ and $D(q) \in \mathcal{R}^{m\times m}$ are affine in q:

$$C(q) = \sum_{i=1}^{p} q_i C_i, \quad D(q) = I + \sum_{i=1}^{p} q_i D_i \quad (1.2)$$

The inverse of $D(q)$ exists and is uniformly bounded in Q.

Further, the uncertain parameter vector $q(t)$ is assumed to be differentiable with its derivative constrained to be specified later. The set of admissible $q(\cdot)$ is denoted by $\mathbf{Q}$.

Our problem is of two-fold. First, we want to determine if the uncertain system (1.1) is robustly asymptotically stable when the parameters are time-invariant. We will call this property *frozen-time robust stability.* Second, if the system is frozen-time robustly stable, we want to determine how much time variation, in some appropriate sense, can the parameters have without losing robust stability. That is, we want to characterise the set $\mathbf{Q}$ that preserves the robustness.

A general framework for studying the uncertain system (1.1) is to transform it into a structure depicted in Figure 1. This is done by rewriting the system as follows:

$$\begin{aligned} \dot{x} &= A_0 x + \bar{B}u \\ y &= \bar{C}x + \bar{D}u \\ u &= \Delta y \end{aligned} \quad (1.3)$$

where

$$\bar{C} = \begin{bmatrix} C_1 \\ \cdots \\ C_p \end{bmatrix} \in \mathcal{R}^{mp\times n};\ \bar{D} = \begin{bmatrix} D_1 \\ \cdots \\ D_p \end{bmatrix} \bar{I} \in \mathcal{R}^{mp\times mp}$$

$$\bar{B} = B\bar{I} \in \mathcal{R}^{n\times mp};\ \bar{I} = [I \ \cdots \ I] \in \mathcal{R}^{m\times mp} \quad (1.4)$$

$$\Delta = \text{diag}\{-q_1(t)I, \cdots, -q_p(t)I\} \in \mathcal{R}^{mp\times mp} \quad (1.5)$$

with the corresponding bounding set $\mathbf{\Delta}$ defined implicitly. The transfer matrix $\bar{G}(s)$ in Figure 1 is given by

$$\bar{G}(s) = \bar{C}(sI - A_0)^{-1}\bar{B} + \bar{D} \quad (1.6)$$

Using the representation above, a general technique called *integral quadratic constraint approach* can be applied. In this regard, results can be found in many references; see, e.g., [7]. However, the essence of this paper is to present a new approach which utilizes the particular structure of the system, i.e., the structure of $\bar{B}$, $\bar{D}$ and Δ.

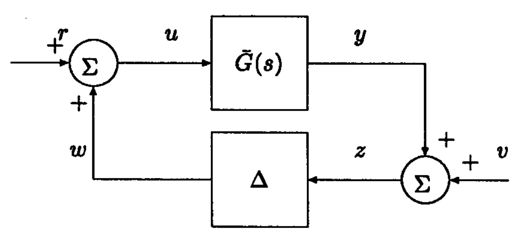

Figure 1: Pictorial Representation of System (1.3)

The approach we propose in this paper is based on the classical multiplier theory. Instead of dealing with the robust stability using the representation in (1.3) (or Figure 1), we deal with an equivalent problem of a much lower dimension by exploring the special structure of the system (1.1). Subsequetly, our results are easier to use in comparison with those derived from the IQC approach which relies on the structure in Figure 1.

The main result of the paper can be viewed as a generalization of the classical result of Freedman and Zames [6] and the result of Dasgupta et. al. [2]. More specifically, we give an upper bound on the time-varying rate of the parameters for preserving robust stability. This bound is an average logarithmic variation rate (ALVR), a term used by Freedman and Zames. A unique feature of the ALVR is that a clear tradeoff is presented between the time-varying rate of the parameters and the degree of the stability (characterized in some sense) of the "frozen-time" system.

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Due to page limit, thoerems and lemmas are presented without proofs.

Notation. We will use $\mathcal{L}_2^n$ to denote a set of $\mathcal{L}_2^n[0,\infty)$ time functions, but the superscript n will usually be suppressed. For $u \in \mathcal{L}_2$ (also denoted by $u(t)$), its Fourier transform is $u(j\omega)$. For $u, v \in \mathcal{L}_2$, the inner product is given by

$$\langle u, v\rangle = \int_{-\infty}^{\infty} \mathrm{He}(u^*(j\omega)v(j\omega))d\omega \qquad (1.7)$$

where $\mathrm{He}(x) = (x + x^*)/2$ and $\|u\| = \sqrt{\langle u, u\rangle}$.

2. PASSIVITY CONDITION

Based on the classical multiplier approach, a simple yet fundamental result, called *passivitiy condition*, is derived. This condition utilizes the special structure of the system (1.1) and avoids the use of the high dimensional representation in Figure 1. It will be shown in the next section that this passivity condition is not conservative under some mild condition when compared with the IQC approach.

To understand the passivity condition, we first consider the following system:

$$\begin{aligned} \dot{x} &= A_0 x + Bu \\ y_i &= C_i x + D_i u \\ y &= u + \sum_{i=1}^{p} q_i(t) y_i \\ &= C(q(t))x + D(q(t))u \end{aligned} \qquad (2.8)$$

It is trivial to verify that this is the inverse system of

$$\begin{aligned} \dot{x} &= \left(A_0 - BD^{-1}(q(t))C(q(t))\right)x + BD^{-1}(q(t))\tilde{u} \\ \tilde{y} &= -D^{-1}(q(t))C(q(t))x + D^{-1}(q(t))\tilde{u} \end{aligned} \qquad (2.9)$$

Denoting the transfer function from u to y_i in (2.8) by

$$G_i(s) = C_i(sI - A_0)^{-1}B + D_i \qquad (2.10)$$

and the corresponding input-output mapping by G_i, then the input-output mapping of (2.8) is given by

$$I + G(q) = I + \sum_{i=1}^{p} q_i(t) \circ G_i \qquad (2.11)$$

Next, we consider the system depicted in Figure 2, where M_1 and M_2, called *multipliers*, are *bistable*, i.e., stable with a stable inverse.

The multiplier approach used in this paper is based on the following result:

Theorem 2.1 *Consider the uncertain system (1.1) and the corresponding inverse system (2.8). Suppose there exist multipliers M_1 and M_2 such that the mapping from $\hat{u}$ to $\hat{y}$ in Figure 2 is strictly passive, or equivalently,*

$$\langle M_1 \circ u, M_2 \circ (I + G(q)) \circ u\rangle - \epsilon\langle u, u\rangle \geq 0 \qquad (2.12)$$

holds for all $u \in \mathcal{L}_2$, $q(\cdot) \in \mathbf{Q}$, and some constant $\epsilon > 0$. Then, the system (1.1) is robustly stable.

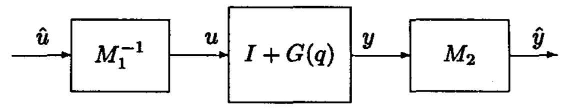

Figure 2: The Multiplier Approach

For the case where M_1 and M_2 are linear time-invariant systems of finite dimensions and the uncertain parameters are time-invariant, the inequality (2.12) is simplified to

$$\mathrm{He}\left(M_1^*(j\omega)M_2(j\omega)(I + G(j\omega, q)) - \epsilon I\right) \geq 0 \qquad (2.13)$$

for all $\omega \in \mathcal{R}$ and $q \in Q$ and some constant $\epsilon > 0$.

3. MULTIPLIERS VS. INTEGRAL QUADRATIC CONSTRAINTS

The simplicity of the multiplier approach in the previous section may lead one to wonder whether it would be very conservative. It turns out that this is not the case. Indeed, we compare it with the IQC approach and show that the multiplier approach is equivalent to the IQC approach under a mild *convexity condition*, but with the advantage of lower computational complexity.

Before we carry out the comparative study, a brief introduction to the IQC theory is in order. Consider the interconnected feedback system in Figure 2, where $\bar{G}(s) = \bar{C}(sI - A_0)^{-1}\bar{B} + \bar{D}$ with an asymptotically stable A_0, and $\Delta \in \mathbf{\Delta}$ is, for simplicity, assumed to be an $\mathcal{L}_2$ operator. Further, $G_f(s) = C_f(sI - A_f)^{-1}B_f + D_f$ is a stable *filter* to be designed for testing the robust stability of the feedback loop.

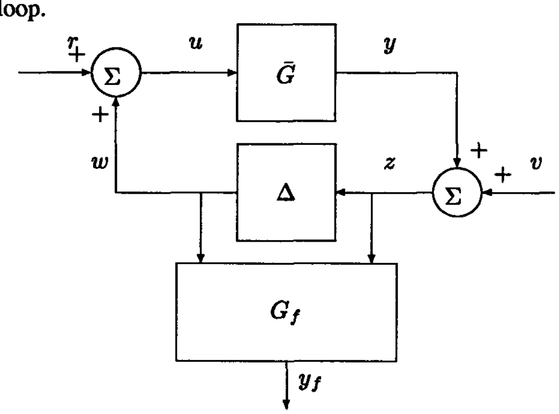

Figure 3: Interconnected Feedback System

The essence of the IQC approach is to construct a filter $G_f(s)$ and a constant kernal matrix $\bar{\Phi}$ such that the following two conditions are simultaneously satisfied:

$$\int_{-\infty}^{+\infty} [z^*(j\omega)\ \ w^*(j\omega)]\Phi(j\omega)\begin{bmatrix} z(j\omega) \\ w(j\omega)\end{bmatrix} d\omega \geq 0,$$

$$\forall\, \Delta \in \mathbf{\Delta}, w = \Delta \circ z, z \in \mathcal{L}_2 \quad (3.14)$$

$$[\bar{G}^*(j\omega)\ \ I]\Phi(j\omega)\begin{bmatrix} \bar{G}(j\omega) \\ I \end{bmatrix} + \epsilon I \leq 0, \ \ \forall\, \omega \in (-\infty, \infty) \quad (3.15)$$

for some $\epsilon > 0$, where

$$\Phi(s) = G_f^T(-s)\tilde{\Phi}G_f(s) \quad (3.16)$$

The fundamental result of the IQC approach is as follows (see, e.g., Megretski and Rantzer [7]):

Theorem 3.1 (The IQC Theorem) *The interconnected system in Figure 3 is robustly stable if there exists some $\Phi(s)$ of the form (3.16) (or equivalently, $G_f(s)$ and $\tilde{\Phi}$) and a constant $\epsilon > 0$ such that both (3.14) and (3.15) are satisfied.*

The IQC approach can be viewed as a generalization of the classical multiplier approach depicted in Figure 4, where $\bar{M}_1(s)$ and $\bar{M}_2(s)$ are bistable linear time-invariant transfer matrices. Typically, these multipliers are constructed such that either of the following (sufficient) robust stabiity conditions can be satisfied:

- **Bounded Realness Condition**:

  $||\bar{M}_2(s)\bar{G}(s)\bar{M}_1^{-1}(s)|| < 1; \ ||\bar{M}_1 \circ \Delta \circ \bar{M}_2^{-1}|| \leq 1$

  for all $\Delta \in \mathbf{\Delta}$, where $||\cdot||$ is the induced $\mathcal{L}_2$ norm;

- **Passivity Condition**:

  $\bar{M}_2(s)\bar{G}(s)\bar{M}_1^{-1}(s)$ is strictly passive and $-\bar{M}_1 \circ \Delta \circ \bar{M}_2^{-1}$ is passive for all $\Delta \in \mathbf{\Delta}$.

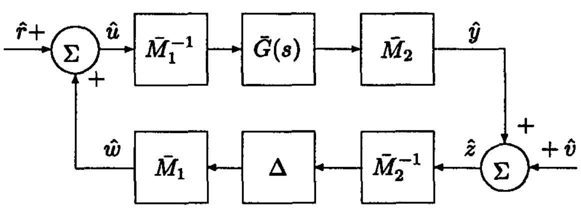

Figure 4: Transformed Sysmtem Using Multipliers

Not only the bounded realness condition and passivity condition are examples of the IQC apporach, many other IQCs have been proposed, as summarized in Megretski and Rantzer [7]. Partitioning

$$\Phi(j\omega) = \begin{bmatrix} Q(j\omega) & F(j\omega) \\ F^*(j\omega) & R(j\omega) \end{bmatrix} \quad (3.17)$$

the most relevant ones corresponding to the structure of Δ in (1.5) are listed below:

- **Time-invariant Parameters**:

$$\Phi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y^*(j\omega) & -X(j\omega) \end{bmatrix} \quad (3.18)$$

  with $X(j\omega) = X^*(j\omega) \geq 0, Y(j\omega) = -Y^*(j\omega)$.

- **Fast Time-varying Paramters**: Same as above except that X and Y must be real constant matrices.

- **Slow Time-Varying Parameters**: For time-varying parameters with derivative bounded by d,

$$\Phi(j\omega) = \begin{bmatrix} \Phi_{11}(j\omega) & 0 \\ 0 & -H^*(j\omega)H(j\omega) \end{bmatrix} \quad (3.19)$$

  where

$$\Phi_{11}(j\omega) = (1+\rho)(H^*(j\omega)H(j\omega) + \phi(H,d)/\rho),$$

  ρ is any positive scalar, $H(j\omega)$ is any linear time-invariant stable operator, and $\phi(H,d)/\rho)$ is any bound of the operator $H \circ \Delta - \Delta \circ H$.

Now we are ready to compare the multiplier approach in Theorem 2.1 with the IQC approach. The comparison relies on the following key observation which arises not only from the examples above but also from systems involving many other types of uncertainties: **In almost all applications of the IQC approach, the matrix $R(j\omega)$ in $\Phi(j\omega)$, is always non-positive definite.** We will call this the *convexity condition for IQC* for the reason below.

Rewrite the condition (3.14) using the partition in (3.19):

$$\int_{-\infty}^{\infty} (z^*Qz + z^*Fw + w^*F^*z + w^*Rw)d\omega \geq 0 \quad (3.20)$$

Then, it is clear that the above condition is convex in Δ. Subsequently, if $\mathbf{\Delta}$ is a compact (bounded and closed) convex set, then only the boundary of $\mathbf{\Delta}$ needs to be considered for robust stability analysis. In particular, if $\mathbf{\Delta}$ is a polytope, only its vertices need attention.

The observation above leads to an interesting conclusion that the IQC approach is indeed equivalent to the multiplier approach under the convexity condition for IQC:

Theorem 3.2 *Consider the interconnected feedback system in Figure 3. Suppose a stable $G_f(s)$ and a kernal matrix $\tilde{\Phi}$ exist such that the IQC conditions (3.14)-(3.15) are satisfied and that the convexity condition $R(j\omega) \leq 0$ holds. Then, the multiplier condition (2.12) is satisfied for some bistable multipliers $M_1(s)$ and $M_2(s)$. A special choice of them is given by the following steps:*

1.
$$\begin{aligned} X(s) &= -2G_f(s)\begin{bmatrix} \bar{G}(s) \\ I \end{bmatrix}; \\ Y(s) &= \tilde{\Phi}G_f(s)\begin{bmatrix} 0 \\ I \end{bmatrix} \end{aligned} \quad (3.21)$$

2. *Let U be any unitary matrix which converts $X(s)$ and $Y(s)$ into*

$$\begin{aligned} UX(s) &= \begin{bmatrix} X_1(s) \\ 0 \end{bmatrix}; \\ UY(s) &= \begin{bmatrix} Y_1(s) \\ Z_1(s) \end{bmatrix} \end{aligned} \quad (3.22)$$

3. *Set*

$$X_2(s) = \bar{I}'X_1^{-1}(s); \quad Y_2(s) = \bar{I}'Y_1^{-1}(s) \qquad (3.23)$$

4. *Let V be any unitary matrix which converts $X_2(s)$ and $Y_2(s)$ into*

$$X_2(s)V = \left[\begin{array}{cc} X_3(s) & 0 \end{array}\right]; \quad Y_2(s)V = \left[\begin{array}{c} Y_3(s) \\ Z_3(s) \end{array}\right] \qquad (3.24)$$

5. *Set*

$$M_1(s) = X_3^{-1}(s); \quad M_2(s) = Y_3^{-1}(s) \qquad (3.25)$$

4. TIME-INVARIANT PARAMETERS

In this section, we consider several types of uncertain systems with time-invariant parameters and discuss the structures of the associated multipliers. We will use $G(s,q)$ to denote the transfer function of $G(q)$.

The relevant stability notion used here will be the so-called σ-stability [2]. A system $G(s)$ is called σ-stable for some scalar $\sigma > 0$ if $G(s-\sigma)$ is stable. Also, an uncertain system $G(s,q), q \in Q$ is called robustly σ-stable if every $G(s,q)$ is σ-stable. A system $G(s)$ is called σ-bistable if $G(s)$ is invertible and both $G(s)$ and $G^{-1}(s)$ are σ-stable. Similarly, $G(s)$ is called σ-SPR if $G(s-\sigma)$ is SPR. Robust σ-SPR can be defined accordingly.

The value of σ can be viewed as a sort of stability margin. As shown in [2], this margin plays a key role in determining the time-variation rate that the uncertain parameters can have without losing robust stability.

It is worth to know that the passivity condition in Theorem 2.1 now becomes the following:

Theorem 4.1 *The system in (1.1) with time-invariant parameters is robustly σ-stable for a given σ if there exist σ-bistable multipliers $M_1(s,q)$ and $M_2(s,q)$, possibly parameterized in q, such that the transfer function*

$$H(s,q) = M_2(s,q)(I + G(s,q))M_1^{-1}(s,q) \qquad (4.26)$$

is robustly σ-SPR.

Single-input and Single-output (SISO) Systems

Consider the case where $I + G(s,q)$ is SISO, i.e.,

$$I + G(s,q) = 1 + g(s,q) = 1 + \sum_{i=1}^{p} q_i g_i(s), \qquad (4.27)$$

where $g_i(s) = c_i(sI - A_0)^{-1}b$ are SISO transfer functions.

In this case, the multiplier approach given in Theorem 4.1 is non-conservative. It is known (Anderson, *et. al.* [1]) that the inverse of $1 + g(s,q)$ is robustly σ-stable iff there exists a SISO σ-bistable multiplier $m(s)$ such that $m^{-1}(s)(1 + g(s,q))$ is robustly σ-SPR. This corresponds to $M_1(s) = m(s)$, $M_2(s) = 1$ in Theorem 4.1. A procedure for constructing $m(s)$ is given in [1]. It is worth to note that the order of the dynamics in $m(s)$ may exceed that of $g(s,q)$.

We also note that if $m(s)$ is given, a (possibly non-minimal) state-space representation of $m(s)$ and $g_i(s)$ can be given in the following way:

$$m(s) = \tilde{c}_k(sI - \tilde{A}_0)^{-1}\tilde{b} + \tilde{d}_k \qquad (4.28)$$

$$g_i(s) = \tilde{c}_i(sI - \tilde{A}_0)^{-1}\tilde{b} + \tilde{d}_i, \; i = 1, \cdots, p \qquad (4.29)$$

That is, they share the same $\tilde{A}_0$ and $\tilde{b}$.

Multi-input and Multi-output (MIMO) Systems

When $G(s,q)$ is MIMO, finding non-conservative multipliers is in general difficult. In Fu and Dasgupta [3], we have proposed to use parameter-dependent multipliers. More specifically, we require

$$M_1^{-1}(s,q) = C_k(q)(sI - A_0)^{-1}B + D_k(q); \quad M_2(s) = I \qquad (4.30)$$

where

$$C_k(q) = C_{k0} + \sum_{i=1}^{p} q_i C_{ki}; \quad D_k(q) = D_{k0} + \sum_{i=1}^{p} q_i D_{ki} \qquad (4.31)$$

That is, M_1 is square and affine in q. The matrices C_{ki} and D_{ki} are variables to be found. Using the Parametric KYP Lemma (to be explained later), these matrices can be searched using linear matrix inequalities.

Applying the multipliers as above, the robust σ-stability condition given in Theorem 4.1 becomes that the transfer function $H(s,q)$ is robustly σ-SPR, where

$$H(s,q) = (I + G(s,q))M_1^{-1}(s,q) \qquad (4.32)$$

A more general version of the above can be easily given as follows: First, we choose $M_2(s,q) = M_2(s)$ to be any fixed square, σ-bistable system which is also monic, i.e. $M_2(\infty) = I$. Since the parameters are time-invariant, $M_2 \circ q_i = q_i \circ M_2$, for all i. Using these properties, we get

$$M_2(s)(I + G(s,q) = I + \tilde{G}(s,q)) = I + \tilde{G}_0(s) + \sum_{i=1}^{q} \tilde{G}_i(s) \qquad (4.33)$$

for some strictly proper $\tilde{G}_i(s), i = 0, \cdots, p$. We further take a common-state representation of $\tilde{G}_i(s)$ and $M_1^{-1}(s,q)$, i.e.,

$$\tilde{G}_i(s) = \tilde{C}_i(sI - \tilde{A}_0)^{-1}\tilde{B} \qquad (4.34)$$

and

$$M_1^{-1}(s,q) = \tilde{C}_k(q)(sI - \tilde{A}_0)^{-1}\tilde{B} + \tilde{D}_k(q) \qquad (4.35)$$

where $\tilde{C}_k(q)$ and $\tilde{D}_k(q)$ are similarly defined as in (4.31) and are to be searched for. We note that $\tilde{G}_i(s)$ and $M_1^{-1}(s,q)$ share the same $\tilde{A}_0$ and $\tilde{B}$.

5. PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

We show in this section how to transform the frequency domain conditions in the previous sections into a state-space form to allow construction of parameter-dependent Lyapunov functions. We will start by developing a generalized KYP Lemma, called *Parametric KYP Lemma*. One of the main contributions of this result is that it allows us to establish multi-affine Lyapunov matrices of the following form:

$$P(q) = P_0 + \sum_{i=1}^{p} q_i P_i + \sum_{i \neq j} q_i q_j P_{ij} + \cdots \quad (5.36)$$

This multi-affine dependence will be crucial to analysis of time-varying parameters. Another main contribution of the Parametric KYP Lemma is that it permits us to search for multipliers using linear matrix inequalities.

Parametric KYP Lemma

We first introduce a known generalized KYP lemma by Willems [8].

Lemma 5.1 (Generalized KYP Lemma) *Given $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times k}$ and symmetric $\Omega \in \mathbf{R}^{(n+k)\times(n+k)}$, there exists a symmetric matrix $P \in \mathbf{R}^{n\times n}$ such that*

$$\begin{bmatrix} A'P + PA & PB \\ B'P & 0 \end{bmatrix} + \Omega < 0 \quad (5.37)$$

if and only if there exists some $\epsilon > 0$ such that

$$[B'((sI-A)^{-1})^* \; I]\Omega \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \, \mathrm{Re}[s] \geq \epsilon \quad (5.38)$$

Further, if A is Hurwitz stable and the upper left $n \times n$ block of Ω is positive semidefinite, then P as above, when it exists, is positive definite.

Our desired result is given as follows:

Lemma 5.2 (Parametric KYP Lemma) *Given matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$, $m \leq n$, a hyper-rectangular set $Q \subset \mathbf{R}^p$, a parametric matrix $\Omega(q) \in \mathbf{R}^{(n+m)\times(n+m)}$ described by*

$$\Omega(q) = \Omega'(q) = \Omega_M(q) + \sum_{i=1}^{p} q_i^2 \Omega_{ii} \quad (5.39)$$

where $\Omega_M(q)$ is multi-affine in q and

$$\Omega_{ii} \geq 0, \; i = 1, \cdots, p \quad (5.40)$$

Then the following conditions are equivalent:

i) *There exists $\epsilon > 0$ such that*

$$[B'((sI-A)^{-1})^* \; I]\Omega(q) \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \, \mathrm{Re}[s] \geq -\epsilon, q \in Q \quad (5.41)$$

ii) *There exists a multi-affine matrix of the form (5.36) such that*

$$\Pi(q) = \begin{bmatrix} A'P(q) + P(q)A & P(q)B \\ B'P(q) & 0 \end{bmatrix} + \Omega(q) < 0 \quad (5.42)$$

for all $q \in Q$.

iii) *The inequality (5.41) holds at all vertices of Q.*

iv) *The inequality (5.42) holds at all vertices of Q.*

Parameter-Dependent Lyapunov Functions

Under a "convexity condition", our main result below provides a necessary and sufficient condition for the existence of parameter-dependent multipliers studied in the previous section. This condition automatically renders a multi-affine Lyapunov matrix.

Theorem 5.1 *Given the uncertain system in (1.1) and $\sigma > 0$, suppose there exists an affine multiplier $M_1(s,q)$ of the form (4.30) such that the transfer matrix $H(s,q)$ in (4.32) is σ-SPR at all vertices of Q. In addition, the* convexity condition *below is satisfied:*

$$\mathbf{He}\left[\begin{bmatrix} C_i' \\ D_i' \end{bmatrix} [C_{ki} \; D_{ki}]\right] \leq 0, \; i = 1, \cdots, p \quad (5.43)$$

Then, the following properties hold:

i) *$H(s,q)$ is σ-SPR for all $q \in Q$.*

ii) *$H(s,q)$ has the following n-th order realization*

$$\begin{aligned} H(s,q) &= (C_k(q) - D_k(q)D^{-1}(q)C(q)) \\ &\quad \cdot (sI - A_0 + BD^{-1}(q)C(q))^{-1}BD^{-1}(q) \\ &\quad + D_k(q)D^{-1}(q) \end{aligned} \quad (5.44)$$

iii) *There exists a multi-affine $P(q) = P'(q)$ of (5.36) to establish the robust σ-SPR property of $H(s,q)$, i.e.,*

$$\Pi(q) = \begin{bmatrix} \Pi_{11}(q) & \Pi_{12}(q) \\ \Pi_{12}'(q) & \Pi_{22}(q) \end{bmatrix} < 0 \quad (5.45)$$

holds for all $q \in Q$, where

$$\begin{aligned} \Pi_{11}(q) &= (\sigma I + A(q))'P(q) + P(q)(\sigma I + A(q)) \\ \Pi_{12}(q) &= P(q)BD^{-1}(q) - C_k'(q) \\ &\quad + C'(q)(D'(q))^{-1}D_k'(q) \\ \Pi_{22}(q) &= -(D_k(q)D^{-1}(q) - (D'(q))^{-1}D_k'(q)) \end{aligned}$$

iv) *(5.45) holds $\forall \, q \in Q$ iff it holds at all corners of Q.*

v) *The same $P(q)$ above is a Lyapunov matrix for establishing the robust σ-stability of (1.1).*

Conversely, if there exists $P(q)$ of the form (5.36) and a multiplier $M_1(s,q)$ of the form (5.37) such that the convexity condition (5.36) is satisfied and that the LMI (5.45) holds at all vertices of Q. Then, $(I + G(s,q))M_1^{-1}(s)$ is robustly σ-SPR.

6. TIME-VARYING PARAMETERS: AVERAGE LOGARITHMIC VARIATION RATES

The key to analysis of time-varying parameters is the following result modified from Dasgupta *et. al.* [2].

Lemma 6.1 *Consider the time-varying system:*

$$\dot{x} = A(q(t))x(t) \tag{6.46}$$

where $A(q)$ is any continuous function of q, $q(t) \in Q = [-1, 1]^p$ for all $t \geq 0$. Suppose there exists a multi-affine Lyapunov matrix $P(q) = P'(q) > 0$ for frozen system of (5.36) and $\sigma > 0$ such that

$$(\sigma I + A(q))P(q) + P(q)(\sigma I + A(q)) \leq 0,\ \forall\, q \in Q \tag{6.47}$$

Then, the time-varying system (6.46) is robustly stable if the following conditions are satisfied:

1. *q are restricted to a slightly smaller region in Q, i.e.,*

$$q_i(t) \in [-1 + \epsilon_0, 1 - \epsilon_0],\ \forall\, i = 1, \cdots, p \tag{6.48}$$

for some (sufficiently small) $\epsilon_0 \in (0, 1)$.

2. *For some (sufficiently large) $T > 0$,*

$$\sup_{t \geq 0} \frac{1}{T} \int_t^{t+T} \sum_{i=1}^{p} \left[\frac{d}{d\tau} \ln \frac{q_i(\tau) + 1}{1 - q_i(\tau)} \right]^+ d\tau < 2\sigma \tag{6.49}$$

where $[x]^+ = \max\{x, 0\}$.

Remark 6.1 *The condition 2 above is known as the average logarithmic variation rate. This is actually generalized from Freedman and Zames [6]. The condition 1 is needed to assure that the second condition is properly defined. It should be noted that this condition is used only for convenience and can be removed [3].*

Applying the lemma above and Theorem 5.1 to the system (1.1), we obtain the following result:

Theorem 6.1 *Consider the uncertain system in (1.1) and an affine multiplier $M_1(s, q)$ of the form (4.30). Let $P(q)$, $C_k(q)$, $D_k(q)$ and $\sigma > 0$ be any solution to (5.45) and $P(q) > 0$ at all vertices of Q and (5.43). Also, let the time-variations of $q(\cdot)$ satisfy (6.49) for some $T > 0$ and $\delta \in (0, \sigma)$. Then, the system (1.1) is robustly stable.*

Remark 6.2 *It is worth to know that the maximum σ (supremum in fact) can be approximated using the semidefinite programming techniques because (5.43) and (5.45) are jointly linear in C_{ki}, D_{ki}, P_i, P_{ij}, ...*

7. CONCLUSIONS

In this paper, we have studied the robust stability problem for linear systems with uncertain time-varying parameters. This is done using a multiplier approach in conjunction with parameter-dependent Lyapunov functions. The main result of the paper is an average logarithmic variation rate for the uncertain parameters for robust stability. In the process of doing so, we have derived an extended version of the KYP lemma, parametric KYP lemma, as a general tool to study the robust stability with parameter uncertainty. Using this lemma, we have provided conditions under which an affinely parameterized multiplier exists to establish the robust stability of the uncertain system. This type of parametric multiplier then naturally leads to a multi-affine Lyapunov function for robust stability analysis in the state space domain.

The multiplier approach used in this paper is very general. In particular, many previous results in the literature on parametric Lyapunov functions and time-varying parameters lead to special multipliers.

Another advantage of the proposed multiplier approach is that discrete-time systems can be treated in a similar way.

8. REFERENCES

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