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Consensus for high-order multi-agent systems with communication delay

Zhenhua WANG[1,2*], Huanshui ZHANG[3], Minyue FU[4,5] & Huaxiang ZHANG[1,2]

[1] *School of Information Science and Engineering, Shandong Normal University, Jinan* 250014*, China;*
[2] *Institute of Data Science and Technology, Shandong Normal University, Jinan* 250014*, China;*
[3] *School of Control Science and Engineering, Shandong University, Jinan* 250061*, China;*
[4] *School of Electrical Engineering and Computer Science, University of Newcastle,* NSW 2308*, Australia;*
[5] *School of Automation, Guangdong Key Laboratory of IoT Information Technology, Guangdong University of Technology, Guangzhou* 510006*, China*

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Abstract In this study, consensus problem for general high-order multi-agent systems with communication delay is investigated. Given the unstable agent dynamics and a known communication delay, two consensus protocols are designed to guarantee consensus over undirected network. By jointly researching the effects of agent dynamics and network topology, allowable delay bounds depending on the maxima of concave functions are easy to calculate. Especially, the maximum delay bound is derived when the network topology is completely connected. The main approach for the same involves designing the control gains on the basis of the solution of a parametric algebraic Riccati equation. Finally, the theoretical results are demonstrated via numerical simulations.

Keywords consensus, communication delay, historical input information, parametric algebraic Riccati equation, eigenratio

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1 Introduction

In recent years, increasing attention has been given to the problem of consensus because of its broad ranges of applications, such as sensor fusion [1], multiple vehicle control [2], distributed filtering [3]. Consensus aims at analyzing the way in which an agreement can be reached through local interactions among agent individuals. A number of results on consensus have recently been established. Consensus for single-integrator multi-agent systems has been previously researched in [4]. By investigating the joint impact of the network topology and agent dynamics, a necessary and sufficient condition was provided in [5] for linear multi-agent systems to reach consensus. Furthermore, consensus conditions for a class of multi-agent systems with noise disturbances were reported in [6].

Time delays related to information transmission will inevitably occur in practical applications. This delay usually significantly degrades closed-loop performance and stability [7]; therefore, conducting research on the effects of delay on consensus is highly important. Many studies have been conducted till

* Corresponding author (email: wzhua111@126.com)

now to address this issue. For single-integrator multi-agent systems, a necessary and sufficient condition on the upper bound of communication delay was established in [8] to guarantee consensus. In addition, the consensus problem with delayed noisy measurements was considered in [9,10]. Furthermore, ref. [11] researched systems with communication delays and limited data rates. By introducing delayed state information into the protocol design, consensus conditions were obtained in [12] for second-order multi-agent systems with communication delay. A new consensus methodology was introduced in [13] for high-order integrator multi-agent systems with input delay. For high-order multi-agent systems that are at most critically unstable (all the eigenvalues of the system matrix are in the closed left-half plane), refs. [14,15] considered a consensus problem with constant communication delay. Furthermore, if the agent dynamics is exponentially unstable (the system matrix has eigenvalues in the open right-half plane), the maximum input delay margin was determined in [16] for multi-agent systems with only one unstable pole to achieve consensus. For information transition affected by diverse communication delays, consensus for a class of high-order multi-agent systems with single input was investigated in [17]. For general linear multi-agent systems with communication delay, allowable delay bounds related to linear matrix inequalities (LMIs) were reported in [18,19].

This study considers the consensus problem for general high-order multi-agent systems in the presence of communication delay. Thus, the agent dynamics studied in this paper is more general than those studied in [13–17]. The consensus problem in this study is converted into a simultaneous stabilization of a number of delayed systems. The agent dynamics may be exponentially unstable; therefore, the low-gain feedback method for dealing with the consensus of critically unstable systems, reported in [14,15], is unavailable for this study. Moreover, the tools for analyzing the roots of the characteristic equations of delayed systems reported in [13,16,17] become more complicated or even incapable of dealing with the stability of general high-order delayed systems. Thus, the generality of state and input matrix for high-order agent dynamics and the existence of a time delay result in difficulties in the research about consensus. In addition, the delay bounds in [18,19] are considered to be conservative to a certain extent because several matrices are required to be defined and they must satisfy a set of LMIs. To overcome the above difficulties and derive consensus, the truncated predictor feedback method reported in [20] can be employed, i.e., the solution of a parametric algebraic Riccati equation (ARE) and delay information are used to design the control gains. However, compared with the parameters of the control gains designed in [20], new parameters need to be designed to simultaneously stabilize a number of delayed systems, rendering the problem in this study more challenging.

Two consensus protocols are proposed in this study to guarantee consensus under the assumption that the undirected network topology is affected by communication delay. The Razumikhin Stability Theorem in [21] is adopted to deal with the communication delay. The main contribution is three-fold. First, for general multi-agent systems that are exponentially unstable, allowable delay bounds in terms of the maxima of concave functions are provided on the basis of two protocols proposed in this study. In particular, so long as the network topology and agent dynamics are provided, maxima of the concave functions, i.e., the delay bounds for consensus are easy to calculate. Second, based on the above method, any large but bounded delay is tolerant for consensus if the agent dynamics is at most critically unstable, which is consistent with the previous results obtained in [15]. Third, if the network topology is complete, any large yet bounded delay is allowed for consensus under the new designed protocol, which comprises the delayed relative information and a part of agents's own historical input information.

The remainder of the paper is organized as follows. In Section 2, some preliminary results on graph theory are reviewed. Section 3 delineates the problem formulation. In Section 4, allowable delay bounds guaranteeing consensus are shown based on two designed protocols. In Section 5, simulation examples are delivered. Finally, the conclusion is stated in Section 6.

Before ending this section, some notations used in this study are listed as follows. The set of real numbers is denoted by $\mathcal{R}$. For any integers p and q, we define $I[p,q] \triangleq \{p, p+1, \dots, q\}$. We use $\mathbf{1}_N = [1, 1, \dots, 1]^{\mathrm{T}}$ to denote a column vector with all entries as one. Let $\|\cdot\|$ represent the 2-norm of a vector. For a matrix A, $\lambda(A)$ and $\mathrm{Re}(\lambda(A))$ represent the eigenvalues of A and the real part of $\lambda(A)$, respectively. In addition, the Kronecker product [22] between matrices A and B is denoted by $A \otimes B$.

2 Preliminaries

This section reviews some notations and results about the algebraic graph theory. We use an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ to describe the network topology among multiple agents, where $\mathcal{V}=\{1,2,\ldots,N\}$ and $\mathcal{E}=\{(i,j):i,j\in\mathcal{V}\}\subseteq\mathcal{V}\times\mathcal{V}$ represent the agent set and edge set, respectively. Matrix $\mathcal{A}=[a_{ij}]\in\mathcal{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ with symmetric nonnegative elements. Here, $a_{ij}>0$ means that there exists an information flow between agents i and j, i.e., $(i,j)\in\mathcal{E}$. In addition, assume $a_{ii}=0$ for all $i\in\mathcal{V}$. The degree of agent i is represented by $d_i\triangleq\sum_{j=1}^{N}a_{ij}$. If we denote the degree matrix as $D\triangleq\mathrm{diag}\{d_1,d_2,\ldots,d_N\}$, then the Laplacian matrix of $\mathcal{G}$ is defined as $L_{\mathcal{G}}\triangleq D-\mathcal{A}$, and this matrix $L_{\mathcal{G}}$ is clearly symmetric. Undirected graph $\mathcal{G}$ is connected if any two distinct agents can be connected via a path that follows the edges of $\mathcal{G}$. If $\mathcal{G}$ is connected, it follows that $L_{\mathcal{G}}$ has only one zero eigenvalue and all other eigenvalues are positive [23]. In this case, all the eigenvalues of $L_{\mathcal{G}}$ are ordered as $0=\lambda_1<\lambda_2\leqslant\cdots\leqslant\lambda_N$. In addition, eigenratio, $\frac{\lambda_2}{\lambda_N}$ known as synchronizability, is an important index of a network topology [24]. In particular, $\lambda_2=\lambda_N$ holds if $\mathcal{G}$ is complete, i.e., every agent can directly communicate with the others.

3 Problem statement

In this study, the dynamics of the i-th agent is described as

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t),\quad i\in I[1,N], \tag{1}$$

where $x_i(t)\in\mathcal{R}^n,u_i(t)\in\mathcal{R}^m$ are the state and input of the i-th agent, respectively; $A\in\mathcal{R}^{n\times n}$ and $B\in\mathcal{R}^{n\times m}$ are general constant matrices.

In network control, communication delay is non-negligible because of data packet losses during information transmission and asynchronous agent clocks [25]. Assume that every agent receives a message sent by its neighbors after a delay of τ. The protocol employing the delayed state information is given as

$$u_i(t)=K\sum_{j=1}^{N}a_{ij}[x_j(t-\tau)-x_i(t-\tau)], \tag{2}$$

where $K\in\mathcal{R}^{m\times n}$ is a constant matrix to be designed.

Similar to the assumption in [15], the communication delay in this study is also known for every agent. To further eliminate the effect of delay on consensus, we design the following new protocol:

$$u_i(t)=K_1\sum_{j=1}^{N}a_{ij}[x_j(t-\tau)-x_i(t-\tau)]+K_2\int_0^{\tau}\mathrm{e}^{As}Bu_i(t-s)\mathrm{d}s, \tag{3}$$

where $K_1\in\mathcal{R}^{m\times n}$, $K_2\in\mathcal{R}^{m\times n}$ and $K_2\neq\mathbf{0}$ are the control gains to be designed.

Remark 1. The idea that protocol (3) contains a part of agent's own historical input information is inspired by previous studies [26,27], in which historical input information is used to design the stabilizing controller. To make this protocol practical, we further assume that all agents have memory for storing data packets.

To make systems (1) operational under protocols (2) or (3), we additionally take the initial values $x_i(\theta)=\mathbf{0}$, $u_i(\theta)=\mathbf{0}$ for any $\theta<0$ and $i\in I[1,N]$.

Definition 1. The multi-agent systems defined in (1) under protocols (2) or (3) are said to reach consensus if for any initial values, there holds $\lim_{t\to\infty}\|x_j(t)-x_i(t)\|=0,\ \forall i,j\in I[1,N]$.

If all the eigenvalues of A lie in the open left-half plane, i.e., $\mathrm{Re}(\lambda(A))<0$, it is easy to observe that consensus can be achieved by designing $u_i(t)\equiv 0$ using the above definition. Thus, for the sake of making the problem meaningful, it is sensible to assume that matrix A in (1) has eigenvalues in the closed right-half plane.

Problem statement. To establish conditions such that the multi-agent systems described by (1) reach consensus under protocols (2) and (3), respectively.

4 Main results

To present the main results, the following technical lemmas need to be provided first.

4.1 Lemmas

Lemma 1 ([28])**.** Suppose that (A, B) is controllable, and $\gamma > 0$ is a scalar such that $\gamma > -2\min\{\mathrm{Re}(\lambda(A))\}$. Then, the parametric ARE

$$A^{\mathrm{T}}P(\gamma) + P(\gamma)A - P(\gamma)BB^{\mathrm{T}}P(\gamma) = -\gamma P(\gamma) \tag{4}$$

has a unique positive-definite solution $P(\gamma)$ with $P^{-1}(\gamma) = \int_0^{\infty} \mathrm{e}^{-(A+\frac{\gamma}{2}I)t}BB^{\mathrm{T}}\mathrm{e}^{-(A+\frac{\gamma}{2}I)^{\mathrm{T}}t}\mathrm{d}t$.

For the sake of convenience, we use $P \triangleq P(\gamma)$ in the following. For given matrices A and B, it is easy to see that the solution of (4) is only related to parameter γ from Lemma 1.

Lemma 2 ([20])**.** Suppose that all the eigenvalues of A are in the closed right-half plane and (A, B) is controllable. Then, for any $t > 0$,

$$\mathrm{tr}(B^{\mathrm{T}}PB) = 2\mathrm{tr}\left(A + \frac{\gamma}{2}I\right), \quad PBB^{\mathrm{T}}P \leqslant 2\mathrm{tr}\left(A + \frac{\gamma}{2}I\right)P, \quad \text{and} \quad \mathrm{e}^{A^{\mathrm{T}}t}P\mathrm{e}^{At} \leqslant \mathrm{e}^{w\gamma t}P$$

hold, where $w = \frac{2tr(A)}{\gamma} + n > 0$, and γ is shown in Lemma 1.

Lemma 3 ([29])**.** For any scalars γ_1 and γ_2 with $\gamma_1 < \gamma_2$, vector function $\omega : [\gamma_1, \gamma_2] \to \mathcal{R}^n$ such that the following integrations are well-defined, then

$$\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\mathrm{d}\beta P \int_{\gamma_1}^{\gamma_2} \omega(\beta)\mathrm{d}\beta \leqslant (\gamma_2 - \gamma_1)\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)P\omega(\beta)\mathrm{d}\beta.$$

Lemma 4. Assume that $0 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_N$ are $N-1$ non-zero constants, and $\delta > 0$ is a parameter to be determined. Then, the maximum value of h satisfying the following inequality:

$$1 - 2(\delta\lambda_i) + (\delta\lambda_i)^2 h \leqslant 0 \tag{5}$$

for $i \in I[2, N]$ is $h^* = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$. Specifically, $h = h^*$ if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$.

Proof. Inequality (5), holding for all $i \in I[2, N]$, yields $h \leqslant -\frac{1}{(\delta\lambda_i)^2} + \frac{2}{(\delta\lambda_i)}$ for $i \in I[2, N]$. For any fixed $\delta > 0$, define function $f(\delta\lambda_i) \triangleq -\frac{1}{(\delta\lambda_i)^2} + \frac{2}{(\delta\lambda_i)}$. When $0 < \delta \leqslant \frac{1}{2\lambda_2}$, $f(\delta\lambda_2) = \frac{2(\delta\lambda_2 - \frac{1}{2})}{(\delta\lambda_2)^2} \leqslant 0$ holds. Thus, in this case, the maximum h satisfying inequality (5) for $i \in I[2, N]$ is not more than 0. However, when $\delta > \frac{1}{2\lambda_2}$, there holds $f(\delta\lambda_i) = \frac{2(\delta\lambda_i - \frac{1}{2})}{(\delta\lambda_i)^2} > 0$ for $i \in I[2, N]$, which means that the maximum value of h satisfying inequality (5) for $i \in I[2, N]$ is more than 0.

Therefore, given the above analysis, the problem in this lemma is equivalent to asserting that

$$h^* = \max_{\delta > \frac{1}{2\lambda_2}} \min\{f(\delta\lambda_i), i \in I[2, N]\} = \frac{4\lambda_N\lambda_2}{[\lambda_N + \lambda_2]^2}$$

holds if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$. When $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$, it is easy to obtain $f(\delta^*\lambda_2) = f(\delta^*\lambda_N) = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$ and

$$f(\delta^*\lambda_N) - f(\delta^*\lambda_i) = -\frac{4\lambda_N\lambda_2}{[\lambda_N + \lambda_2]^2}(\lambda_N - \lambda_i)(\lambda_i - \lambda_2) \leqslant 0.$$

Thus, it yields $f(\delta^*\lambda_N) \leqslant f(\delta^*\lambda_i)$ for $i \in I[2, N]$.

The following shows that, for any $\delta \neq \delta^*$, $\min\{f(\delta\lambda_i),\ i \in I[2, N]\} < h^*$ holds. The proof is divided into two cases.

(1) $\frac{1}{2\lambda_2} < \delta < \delta^*$. In this case, $\frac{1}{\delta\lambda_2} > \frac{1}{\delta^*\lambda_2} = \frac{2\lambda_N}{\lambda_N+\lambda_2} \geqslant 1$ holds. Thus,

$$\min\{f(\delta\lambda_i),\ i \in I[2, N]\} \leqslant f(\delta\lambda_2) = -\frac{1}{(\delta\lambda_2)^2} + \frac{2}{(\delta\lambda_2)} < -\frac{1}{(\delta^*\lambda_2)^2} + \frac{2}{\delta^*\lambda_2} = h^*.$$

(2) $\delta > \delta^*$. In this case, we have $\frac{1}{\delta\lambda_N} < \frac{1}{\delta^*\lambda_N} = \frac{2\lambda_2}{\lambda_N+\lambda_2} \leqslant 1$. Then,

$$\min\{f(\delta\lambda_i),\ i \in I[2,N]\} \leqslant f(\delta\lambda_N) = -\frac{1}{(\delta\lambda_N)^2} + \frac{2}{\delta\lambda_N} < -\frac{1}{(\delta^*\lambda_N)^2} + \frac{2}{\delta^*\lambda_N} = h^*.$$

Combining the above discussion, it can be seen that h takes the maximum value $h^* = \frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$ if and only if $\delta = \delta^* = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$. The proof is complete.

Lemma 5. Assume $x \in \mathcal{R}^n, y \in \mathcal{R}^n, r \in \mathcal{R}$. Then, for any $l > 0$,

$$rx^{\mathrm{T}}y + ry^{\mathrm{T}}x \leqslant lx^{\mathrm{T}}x + \frac{|r|^2}{l}y^{\mathrm{T}}y.$$

Proof. For any $l > 0$, it follows that

$$[rx^{\mathrm{T}}y + ry^{\mathrm{T}}x] - \left[lx^{\mathrm{T}}x + \frac{|r|^2}{l}y^{\mathrm{T}}y\right] = -\frac{1}{l}[ry - lx]^{\mathrm{T}}[ry - lx] \leqslant 0.$$

Thus, the proof is complete.

Lemma 6. Suppose that $0 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$ are $N-1$ nonzero constants. Then, the minimum value of function $\frac{1}{l} \times (\max_{i\in I[2,N]} |h_1\lambda_i - h_2|^2)$ subject to constraints $l > 0$, $h_1 \in \mathcal{R}$, $h_2 \in \mathcal{R}$ and $1 - 2h_2 + l \leqslant 0$ is $(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2})^2$. In this case, $l = 1$, $h_1 = \frac{2}{\lambda_N+\lambda_2}$ and $h_2 = 1$.

Proof. For any fixed $l > 0$, relation $h_2 \geqslant \frac{1+l}{2} > \frac{1}{2}$ holds from the constraint $1 - 2h_2 + l \leqslant 0$. Take $h_1 = \frac{2h_2}{\lambda_N+\lambda_2}$. Then

$$|h_1\lambda_2 - h_2|^2 = |h_1\lambda_N - h_2|^2 = h_2^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2.$$

Consider that

$$|h_1\lambda_i - h_2|^2 - |h_1\lambda_N - h_2|^2 = \frac{4h_2^2}{[\lambda_N + \lambda_2]^2}(\lambda_i - \lambda_2)(\lambda_i - \lambda_N) \leqslant 0.$$

Thus, when $h_1 = \frac{2h_2}{\lambda_N+\lambda_2}$, $|h_1\lambda_i - h_2|^2 \leqslant |h_1\lambda_N - h_2|^2$ holds, i.e.,

$$|h_1\lambda_2 - h_2|^2 = |h_1\lambda_N - h_2|^2 = \max_{i\in I[2,N]} |h_1\lambda_i - h_2|^2.$$

Next, we prove that $\max_{i\in I[2,N]} |\overline{h}_1\lambda_i - h_2|^2 > h_2^2(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2})^2$ holds for any $\overline{h}_1 \neq \frac{2h_2}{\lambda_N+\lambda_2}$. On the one hand, if $\overline{h}_1 < \frac{2h_2}{\lambda_N+\lambda_2}$, it follows from $h_2 - \overline{h}_1\lambda_2 > h_2 - \frac{2h_2\lambda_2}{\lambda_N+\lambda_2} \geqslant 0$ that

$$|\overline{h}_1\lambda_2 - h_2|^2 > \left|h_2 - \frac{2h_2\lambda_2}{\lambda_N + \lambda_2}\right|^2 = h_2^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2.$$

On the other hand, in case of $\overline{h}_1 > \frac{2h_2}{\lambda_N+\lambda_2}$, it yields from $\overline{h}_1\lambda_N - h_2 > \frac{2h_2\lambda_N}{\lambda_N+\lambda_2} - h_2 \geqslant 0$ that

$$|\overline{h}_1\lambda_N - h_2|^2 > \left|h_2 - \frac{2h_2\lambda_N}{\lambda_N + \lambda_2}\right|^2 = h_2^2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2.$$

Thus, for any $l > 0$, $h_1 \in \mathcal{R}$ and $h_2 \in \mathcal{R}$, the following holds

$$\frac{1}{l} \times \left(\max_{i\in I[2,N]} |h_1\lambda_i - h_2|^2\right) \geqslant \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 \times \min_{h_2\in\mathcal{R}} \frac{h_2^2}{l}.$$

In light of $h_2 \geqslant \frac{1+l}{2}$, it follows $\min_{h_2\in\mathcal{R}} \frac{h_2^2}{l} = \frac{(1+l)^2}{4l}$ when $h_2 = \frac{1+l}{2}$, and then

$$\frac{1}{l} \times \left(\max_{i\in I[2,N]} |h_1\lambda_i - h_2|^2\right) \geqslant \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 \times \frac{(1+l)^2}{4l} \geqslant \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2.$$

The minimum $(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2})^2$ is taken if and only if $l = 1$, and in this case $h_1 = \frac{2}{\lambda_N+\lambda_2}, h_2 = 1$. The proof is complete.

4.2 Consensus results

Using the parametric ARE given in (4), the control gains for (2) and (3) are designed in what follows. It is known from Lemma 1 that the key to deriving consensus is finding appropriate parameter γ. We first give the consensus result for the multi-agent systems (1) under protocol (2). In this case, the analysis in [30] shows that the stable eigenvalues of the system matrix have no effect on consensus. Hence, the following theorem is obtained.

Theorem 1. Consider an undirected graph $\mathcal{G}$ and assume it is connected. Suppose that agent dynamics (1) is exponentially unstable and matrix A has no stable eigenvalue. If (A, B) is controllable, design $K = \delta B^{\mathrm{T}} P \mathrm{e}^{A\tau}$ in (2) using $\delta = \frac{1}{2\lambda_2} + \frac{1}{2\lambda_N}$, where P is the unique positive-definite solution of the parametric ARE (4). Then, the allowable delay bound for the multi-agent systems (1) to reach consensus under protocol (2) is

$$\tau^* = \max_{q>0} \frac{q\left[1 - n \times \frac{(\lambda_N+\lambda_2)^2}{4\lambda_N\lambda_2} q^2 \mathrm{e}^{2q}\right]}{2\mathrm{tr}(A)}.$$

Moreover, for $0 < \tau_1 < \tau^*$, there exists a set $\Omega = [\gamma^*, \gamma_1]$ such that the consensus is reached if $\gamma \in \Omega$.

Proof. According to Lemma 1 and the conditions in the theorem, we know that Eq. (4) has a unique positive-definite solution P. Combining protocol (2) and system (1) leads to

$$\dot{x}_i(t) = Ax_i(t) + \delta BB^{\mathrm{T}} P \mathrm{e}^{A\tau} \sum_{j=1}^{N} a_{ij}[x_j(t-\tau) - x_i(t-\tau)].$$

Then, $\dot{x}(t) = I_N \otimes Ax(t) - \delta L_{\mathcal{G}} \otimes BB^{\mathrm{T}} P \mathrm{e}^{A\tau} x(t-\tau)$ with $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Represent the average state of all agents at time t by $\overline{X}(t) \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}[\mathbf{1}_N^{\mathrm{T}} \otimes I_n]x(t)$. Hence, it follows from $\mathbf{1}_N^{\mathrm{T}} L_{\mathcal{G}} = \mathbf{0}^{\mathrm{T}}$ that $\dot{\overline{X}}(t) = A\overline{X}(t)$. Further, we denote the deviation of every agent from the average state by $\xi_i(t) \triangleq x_i(t) - \overline{X}(t)$ and stack $\xi_i(t)$ to acquire a new vector $\xi(t) = [\xi_1^{\mathrm{T}}(t), \xi_2^{\mathrm{T}}(t), \ldots, \xi_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then

$$\dot{\xi}(t) = I_N \otimes A\xi(t) - \delta L_{\mathcal{G}} \otimes BB^{\mathrm{T}} P \mathrm{e}^{A\tau} \xi(t-\tau).$$

The undirected graph $\mathcal{G}$ is connected; therefore, it is easy to know that matrix $L_{\mathcal{G}}$ has a simple eigenvalue of 0. Thus, we can construct a unitary matrix $\Phi = [\frac{\mathbf{1}_N}{\sqrt{N}}, \phi_2, \ldots, \phi_N]$ to transform $L_{\mathcal{G}}$ into a diagonal form, i.e., $\Phi^{\mathrm{T}} L_{\mathcal{G}} \Phi = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$. Defining $\tilde{\xi}(t) \triangleq [\Phi \otimes I_n]^{\mathrm{T}} \xi(t)$ and partitioning $\tilde{\xi}(t) \in \mathcal{R}^{nN}$ into N parts $\tilde{\xi}(t) = [\tilde{\xi}_1^{\mathrm{T}}(t), \tilde{\xi}_2^{\mathrm{T}}(t), \ldots, \tilde{\xi}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, it follows that $\tilde{\xi}_1(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \xi_i(t) \equiv 0$ and

$$\dot{\tilde{\xi}}_i(t) = A\tilde{\xi}_i(t) - \sigma_i BB^{\mathrm{T}} P \mathrm{e}^{A\tau} \tilde{\xi}_i(t-\tau) \tag{6}$$

for $i \in I[2, N]$, where $\sigma_i = \delta\lambda_i$. Obviously, the consensus can be achieved if $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ holds simultaneously for $i \in I[2, N]$.

According to (6) and the variation of constants formula, we have

$$\tilde{\xi}_i(t) = \mathrm{e}^{A\tau}\tilde{\xi}_i(t-\tau) - \sigma_i \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BB^{\mathrm{T}} P \mathrm{e}^{A\tau} \tilde{\xi}_i(s-\tau)\mathrm{d}s,$$

which, together with (6), implies

$$\begin{aligned}\dot{\tilde{\xi}}_i(t) &= [A - \sigma_i BB^{\mathrm{T}} P]\tilde{\xi}_i(t) - \sigma_i^2 BB^{\mathrm{T}} P \int_{t-\tau}^{t} \mathrm{e}^{A(t-s)} BB^{\mathrm{T}} P \mathrm{e}^{A\tau} \tilde{\xi}_i(s-\tau)\mathrm{d}s \\ &\triangleq [A - \sigma_i BB^{\mathrm{T}} P]\tilde{\xi}_i(t) - \sigma_i^2 BB^{\mathrm{T}} P \Pi_i(t).\end{aligned} \tag{7}$$

Consider the Lyapunov function $V(\tilde{\xi}_i(t)) = \tilde{\xi}_i^{\mathrm{T}}(t) P \tilde{\xi}_i(t)$. Using (4), the derivative of $V(\tilde{\xi}_i(t))$ along the trajectories (7) satisfies

$$\dot{V}(\tilde{\xi}_i(t)) = \dot{\tilde{\xi}}_i^{\mathrm{T}}(t) P \tilde{\xi}_i(t) + \tilde{\xi}_i^{\mathrm{T}}(t) P \dot{\tilde{\xi}}_i(t)$$

$$
\begin{aligned}
&= \tilde{\xi}_i^{\mathrm{T}}(t)[A^{\mathrm{T}}P + PA - 2\sigma_i PBB^{\mathrm{T}}P]\tilde{\xi}_i(t) - \sigma_i^2\Pi_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{\xi}_i(t) \\
&\quad -\sigma_i^2\tilde{\xi}_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\Pi_i(t) \\
&\leqslant -\gamma\tilde{\xi}_i^{\mathrm{T}}(t)P\tilde{\xi}_i(t) + [1-2\sigma_i+\sigma_i^2 h]\tilde{\xi}_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{\xi}_i(t) + \frac{1}{h}\Pi_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\Pi_i(t),
\end{aligned}
$$

where $h>0$ is a constant that is given below. Applying Lemmas 2 and 3 yields

$$
\begin{aligned}
\Pi_i^{\mathrm{T}}(t)P\Pi_i(t) &= \left[\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(s-\tau)\mathrm{d}s\right]^{\mathrm{T}} P \left[\int_{t-\tau}^{t} \mathrm{e}^{A(t-s)}BB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(s-\tau)\mathrm{d}s\right] \\
&\leqslant \tau\int_{t-\tau}^{t} [\tilde{\xi}_i^{\mathrm{T}}(s-\tau)\mathrm{e}^{A^{\mathrm{T}}\tau}PBB^{\mathrm{T}}(\mathrm{e}^{A^{\mathrm{T}}(t-s)}P\mathrm{e}^{A(t-s)})BB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(s-\tau)]\mathrm{d}s \\
&\leqslant \tau\int_{t-\tau}^{t} \mathrm{e}^{w\gamma(t-s)}[\tilde{\xi}_i^{\mathrm{T}}(s-\tau)\mathrm{e}^{A^{\mathrm{T}}\tau}(PBB^{\mathrm{T}}PBB^{\mathrm{T}}P)\mathrm{e}^{A\tau}\tilde{\xi}(s-\tau)]\mathrm{d}s \\
&\leqslant 4\mathrm{tr}\left(A+\frac{\gamma}{2}I\right)^2\tau\int_{t-\tau}^{t} \mathrm{e}^{w\gamma(t-s+\tau)}\tilde{\xi}_i^{\mathrm{T}}(s-\tau)P\tilde{\xi}_i(s-\tau)\mathrm{d}s,
\end{aligned}
$$

where $w=\frac{2\mathrm{tr}(A)}{\gamma}+n$, and $PBB^{\mathrm{T}}PBB^{\mathrm{T}}P \leqslant PB\mathrm{tr}(B^{\mathrm{T}}PB)B^{\mathrm{T}}P \leqslant 4\mathrm{tr}(A+\frac{\gamma}{2}I)^2P$ is utilized in the second inequality.

Let $V(\tilde{\xi}_i(t+\theta)) < \phi V(\tilde{\xi}_i(t))$ for $\forall\theta\in[-\tau,0]$, where $\phi>1$ is to be specified. Then,

$$
\begin{aligned}
\Pi_i^{\mathrm{T}}(t)P\Pi_i(t) &\leqslant 4\phi\mathrm{tr}\left(A+\frac{\gamma}{2}I\right)^2\tau\int_{t-\tau}^{t}\mathrm{e}^{w\gamma(t-s+\tau)}\mathrm{d}sV(\tilde{\xi}_i(t)) \\
&= \frac{4}{w\gamma}\phi\mathrm{tr}\left(A+\frac{\gamma}{2}I\right)^2\tau\mathrm{e}^{w\gamma\tau}[\mathrm{e}^{w\gamma\tau}-1]V(\tilde{\xi}_i(t)).
\end{aligned}
$$

It is easy to obtain $\Pi_i^{\mathrm{T}}(t)P\Pi_i(t) \leqslant 4\phi\mathrm{tr}(A+\frac{\gamma}{2}I)^2\tau^2\mathrm{e}^{2w\gamma\tau}V(\tilde{\xi}_i(t))$ from the fact that $\mathrm{e}^{w\gamma\tau}-1\leqslant w\gamma\tau\mathrm{e}^{w\gamma\tau}$, which yields

$$
\begin{aligned}
\dot{V}(\tilde{\xi}_i(t)) &\leqslant -\gamma V(\tilde{\xi}_i(t)) + [1-2\sigma_i+\sigma_i^2h]\tilde{\xi}_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{\xi}_i(t) \\
&\quad + \frac{8\phi\mathrm{tr}(A+\frac{\gamma}{2}I)^3\tau^2}{h}\mathrm{e}^{2w\gamma\tau}V(\tilde{\xi}_i(t)). \qquad (8)
\end{aligned}
$$

To recede the influence of the latter two parts of (8) on the asymptotic stability of $\tilde{\xi}_i(t)$, we take $h=\frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$. It is known from Lemma 4 that $h=\frac{4\lambda_N\lambda_2}{[\lambda_N+\lambda_2]^2}$ is the maximum value satisfying $1-2\sigma_i+\sigma_i^2h\leqslant 0$ for $i\in I[2,N]$. As a result, Eq. (8) simplifies to

$$
\dot{V}(\tilde{\xi}_i(t)) \leqslant -\left(\gamma - 8\phi\times\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}\mathrm{tr}\left(A+\frac{\gamma}{2}I\right)^3\tau^2\mathrm{e}^{2w\gamma\tau}\right)V(\tilde{\xi}_i(t)).
$$

To derive consensus, we need to find proper values for $\tau>0,\gamma>0$, and $\phi>1$ to ensure that

$$
\gamma - 8\phi\times\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}\mathrm{tr}\left(A+\frac{\gamma}{2}I\right)^3\tau^2\mathrm{e}^{2w\gamma\tau} > 0. \qquad (9)
$$

Let $R(\lambda_2,\lambda_N)\triangleq\frac{[\lambda_N+\lambda_2]^2}{4\lambda_N\lambda_2}$ and $q\triangleq w\gamma\tau = 2\mathrm{tr}(A+\frac{\gamma}{2}I)\tau$. Recalling that matrix A has no stable eigenvalues, it is easy to see that $q>0$ for any $\tau>0$. Then, employing $\gamma=\frac{2\mathrm{tr}(A+\frac{\gamma}{2}I)}{n}-\frac{2\mathrm{tr}(A)}{n}$, inequality (9) is converted into $2\mathrm{tr}(A)\tau < q[1-\phi nR(\lambda_2,\lambda_N)q^2\mathrm{e}^{2q}]$. Denote function $F(q)\triangleq q[1-nR(\lambda_2,\lambda_N)q^2\mathrm{e}^{2q}]$. If $2\mathrm{tr}(A)\tau<F(q)$, take constant $\phi=\frac{nR(\lambda_2,\lambda_N)q^3\mathrm{e}^{2q}+q-2\mathrm{tr}(A)\tau}{2R(\lambda_2,\lambda_N)q^3\mathrm{e}^{2q}}>1$. By calculation, we know that

$$
2\mathrm{tr}(A)\tau < \mathrm{tr}(A)\tau + \frac{1}{2}q[1-nR(\lambda_2,\lambda_N)q^2\mathrm{e}^{2q}] = q[1-\phi nR(\lambda_2,\lambda_N)q^2\mathrm{e}^{2q}].
$$

Therefore, based on the Razumikhin Stability Theorem in [21], consensus is reached if $2\mathrm{tr}(A)\tau<F(q)$, i.e., $\tau<\frac{F(w\gamma\tau)}{2\mathrm{tr}(A)}$.

We now give the maximum value of function $F(q)$ on $q > 0$. Obviously, $F(0) = 0$ holds, and $F(q) < 0$ for any $q \geqslant 1$. Furthermore, it is easy to obtain $F'(q) = 1 - nR(\lambda_2, \lambda_N)q^2\mathrm{e}^{2q}[3 + 2q]$ and $F''(q) = -2nR(\lambda_2, \lambda_N)q\mathrm{e}^{2q}[2q^2 + 6q + 3]$. It follows $F''(q) < 0$ for any $q > 0$, thus $F'(q)$ decreases monotonically on $q > 0$. $F'(0) = 1 > 0$ and $F'(q) < 0$ for $q \geqslant 1$; therefore, there exists $0 < q^* < 1$ such that $F'(q^*) = 0$. That is, $F(q)$ increases monotonically in interval $[0, q^*]$ and decreases monotonically in interval $[q^*, +\infty)$. Thus, $\max_{q>0} F(q) = F(q^*)$.

For the following equation, let τ^* satisfy $F(q^*) = 2\mathrm{tr}(A)\tau^*$. The allowable delay for consensus is shown to be $\tau_1 \in (0, \tau^*)$. First, we demonstrate that there exists γ^* such that $q^* = 2\mathrm{tr}(A + \frac{\gamma^*}{2})\tau^*$. In fact, if $q^* < 2\mathrm{tr}(A + \frac{\gamma}{2})\tau^*$ for all $\gamma > 0$, it is easy to conclude $q^* \leqslant 2\mathrm{tr}(A)\tau^*$. Accordingly,

$$q^* \leqslant 2\mathrm{tr}(A)\tau^* = F(q^*) = q^*[1 - nR(\lambda_2, \lambda_N)(q^*)^2\mathrm{e}^{2q^*}] < q^*,$$

which is a contradiction. Thus, there exists γ^* such that $q^* = 2\mathrm{tr}(A + \frac{\gamma^*}{2})\tau^*$.

Next, for $0 < \tau_1 < \tau^*$, we demonstrate that there exists a $\gamma_1 > \gamma^*$ such that the consensus is guaranteed for $\gamma \in \Omega \triangleq [\gamma^*, \gamma_1]$. In fact, we select a γ_1 that satisfies $w_1\gamma_1\tau_1 = 2\mathrm{tr}(A + \frac{\gamma_1}{2}I)\tau_1 = q^*$. $\tau_1 < \tau^*$ and $q^* = 2\mathrm{tr}(A + \frac{\gamma^*}{2}I)\tau^* = w^*\gamma^*\tau^*$; therefore it is easy to see that $\gamma^* < \gamma_1$ and $F(w^*\gamma^*\tau_1) < F(q^*)$. Because

$$2\mathrm{tr}(A)\tau^* = F(q^*) = (w^*\gamma^*\tau^*)[1 - nR(\lambda_2, \lambda_N)(w^*\gamma^*\tau^*)^2\mathrm{e}^{2w^*\gamma^*\tau^*}],$$

we obtain $2\mathrm{tr}(A) = (w^*\gamma^*)[1 - nR(\lambda_2, \lambda_N)(w^*\gamma^*\tau^*)^2\mathrm{e}^{2w^*\gamma^*\tau^*}]$, and

$$\begin{aligned} 2\mathrm{tr}(A)\tau_1 &= (w^*\gamma^*\tau_1)[1 - nR(\lambda_2, \lambda_N)(w^*\gamma^*\tau^*)^2\mathrm{e}^{2w^*\gamma^*\tau^*}] \\ &< (w^*\gamma^*\tau_1)[1 - nR(\lambda_2, \lambda_N)(w^*\gamma^*\tau_1)^2\mathrm{e}^{2w^*\gamma^*\tau_1}] \\ &= F(w^*\gamma^*\tau_1). \end{aligned}$$

In addition, due to $F(w^*\gamma^*\tau_1) \leqslant F(w\gamma\tau_1)$ for $\gamma \in [\gamma^*, \gamma_1]$, it follows that $2\mathrm{tr}(A)\tau_1 < F(w\gamma\tau_1)$, which implies that consensus is guaranteed. Therefore, the proof is complete.

Remark 2. The allowable delay bound in Theorem 1 is related to the eigenratio $\frac{\lambda_2}{\lambda_N}$ of $\mathcal{G}$. In particular, when the network topology $\mathcal{G}$ is complete, i.e., $\lambda_2 = \lambda_N$, the delay bound takes the maximum value.

For the special case, when all the eigenvalues of the system matrix are on the imaginary axis, the consensus result corresponds with the case proposed in [15].

Corollary 1. Assume that all the eigenvalues of matrix A are on the imaginary axis and (A, B) is controllable. Then, for any connected undirected graph $\mathcal{G}$, the multi-agent systems (1) under protocol (2) achieve consensus for any large yet bounded communication delay.

Proof. The assumption that all the eigenvalues of A are on the imaginary axis and the fact that $A \in \mathcal{R}^{n\times n}$ lead to $\mathrm{tr}(A) = 0$. From Theorem 1, we know that consensus is achieved if $2\mathrm{tr}(A)\tau < F(q) = q[1 - nR(\lambda_2, \lambda_N)q^2\mathrm{e}^{2q}]$. For any large yet bounded delay $\tau > 0$, considering $\mathrm{tr}(A) = 0$ and $q = 2\mathrm{tr}(A + \frac{\gamma}{2}I)\tau = n\gamma\tau > 0$, the condition becomes $nR(\lambda_2, \lambda_N)(n\gamma\tau)^2\mathrm{e}^{2n\gamma\tau} < 1$. Thus, taking $0 < \gamma < \min\{\frac{1}{n\tau}, \frac{1}{n\tau\mathrm{e}\sqrt{nR(\lambda_2,\lambda_N)}}\}$, it yields $nR(\lambda_2, \lambda_N)(n\gamma\tau)^2\mathrm{e}^{2n\gamma\tau} < nR(\lambda_2, \lambda_N)(n\gamma\tau)^2\mathrm{e}^2 < 1$. Then, the proof is complete.

In the following, we study the consensus of the multi-agent systems (1) under protocol (3). If the synchronizability of the network topology satisfies $(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2})^2 < \frac{1}{n}$, we design gains $K_1 = h_1B^{\mathrm{T}}P\mathrm{e}^{A\tau}$ and $K_2 = -h_2B^{\mathrm{T}}P$, where h_1, h_2 are constants to be determined. Then, the following theorem can be given.

Theorem 2. Under the assumptions in Theorem 1, the control gains in protocol (3) are designed as $K_1 = h_1B^{\mathrm{T}}P\mathrm{e}^{A\tau}$, $K_2 = -h_2B^{\mathrm{T}}P$ with $h_1 = \frac{2}{\lambda_N+\lambda_2}$ and $h_2 = 1$. Then, the allowable delay bound for the multi-agent systems (1) under protocol (3) to achieve consensus is

$$\overline{\tau}^* = \max_{q>0} \frac{q\left[1 - n \times \left(\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2}\right)^2\mathrm{e}^q\right]}{2\mathrm{tr}(A)}.$$

Furthermore, for $0 < \overline{\tau}_1 < \overline{\tau}^*$, there exists a set $\overline{\Omega} = [\overline{\gamma}^*, \overline{\gamma}_1]$ such that the consensus is reached for $\gamma \in \overline{\Omega}$. In particular, if the undirected graph $\mathcal{G}$ is complete, any large yet bounded communication delay is allowed for consensus.

Proof. Employing the variation of constants formula, it follows from (1) that $x_i(t) = \mathrm{e}^{A\tau}x_i(t-\tau) + \int_o^\tau \mathrm{e}^{As}Bu_i(t-s)\mathrm{d}s$. On the basis of this equation and protocol (3), the agent dynamics (1) becomes

$$\dot{x}_i(t) = [A + BK_2]x_i(t) + BK_1 \sum_{j=1}^{N} a_{ij}[x_j(t-\tau) - x_i(t-\tau)] - BK_2\mathrm{e}^{A\tau}x_i(t-\tau).$$

Then, similar to the discussion in Theorem 1, the consensus problem is equivalent to the simultaneous stabilization of error systems

$$\dot{\tilde{\xi}}_i(t) = [A + BK_2]\tilde{\xi}_i(t) - [\lambda_i BK_1 + BK_2\mathrm{e}^{A\tau}]\tilde{\xi}_i(t-\tau) \tag{10}$$

for $i \in I[2,N]$. Substituting gains $K_1 = h_1 B^{\mathrm{T}}P\mathrm{e}^{A\tau}$ and $K_2 = -h_2 B^{\mathrm{T}}P$ into (10) gives

$$\dot{\tilde{\xi}}_i(t) = [A - h_2 BB^{\mathrm{T}}P]\tilde{\xi}_i(t) - [h_1\lambda_i - h_2]BB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(t-\tau).$$

Take $V(\tilde{\xi}_i(t)) = \tilde{\xi}_i^{\mathrm{T}}(t)P\tilde{\xi}_i(t)$. Then, applying (4) and Lemma 5, we obtain

$$\begin{aligned}
\dot{V}(\tilde{\xi}_i(t)) &= \dot{\tilde{\xi}}_i^{\mathrm{T}}(t)P\tilde{\xi}_i(t) + \tilde{\xi}_i^{\mathrm{T}}(t)P\dot{\tilde{\xi}}_i(t)\\
&= \tilde{\xi}_i^{\mathrm{T}}(t)[A^{\mathrm{T}} - h_2 PBB^{\mathrm{T}}]P\tilde{\xi}_i(t) - [h_1\lambda_i - h_2]\tilde{\xi}_i^{\mathrm{T}}(t-\tau)\mathrm{e}^{A^{\mathrm{T}}\tau}PBB^{\mathrm{T}}P\tilde{\xi}_i(t)\\
&\quad + \tilde{\xi}_i^{\mathrm{T}}(t)P[A - h_2BB^{\mathrm{T}}P]\tilde{\xi}_i(t) - [h_1\lambda_i - h_2]\tilde{\xi}_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(t-\tau)\\
&\leqslant -\gamma\tilde{\xi}_i^{\mathrm{T}}(t)P\tilde{\xi}_i(t) + [1 - 2h_2 + l]\tilde{\xi}_i^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{\xi}_i(t)\\
&\quad + \frac{1}{l}|h_1\lambda_i - h_2|^2\tilde{\xi}_i^{\mathrm{T}}(t-\tau)\mathrm{e}^{A^{\mathrm{T}}\tau}PBB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(t-\tau),
\end{aligned} \tag{11}$$

where $l > 0$ is a constant to be determined.

To weaken the effect of the latter two parts on the asymptotic stability of $\tilde{\xi}_i(t)$, take $0 < l = 2h_2 - 1$, $h_1 = \frac{2h_2}{\lambda_N + \lambda_2}$ and $h_2 = 1$ in (11). From Lemma 6, we know $(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2})^2$ is the minimum value of $\max\{\frac{1}{l}|h_1\lambda_i - h_2|^2,\ i \in I[2,N]\}$ under the condition that $1 - 2h_2 + l \leqslant 0$. Based on above analysis, Eq. (11) becomes

$$\dot{V}(\tilde{\xi}_i(t)) \leqslant -\gamma\tilde{\xi}_i^{\mathrm{T}}(t)P\tilde{\xi}_i(t) + \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 \tilde{\xi}_i^{\mathrm{T}}(t-\tau)\mathrm{e}^{A^{\mathrm{T}}\tau}PBB^{\mathrm{T}}P\mathrm{e}^{A\tau}\tilde{\xi}_i(t-\tau).$$

From Lemma 2, we conclude

$$\mathrm{e}^{A^{\mathrm{T}}\tau}PBB^{\mathrm{T}}P\mathrm{e}^{A\tau} \leqslant 2\mathrm{tr}\left(A + \frac{\gamma}{2}I\right)\mathrm{e}^{A^{\mathrm{T}}\tau}P\mathrm{e}^{A\tau} \leqslant 2\mathrm{tr}\left(A + \frac{\gamma}{2}I\right)\mathrm{e}^{w\gamma\tau}P,$$

and thus,

$$\dot{V}(\tilde{\xi}_i(t)) \leqslant -\gamma V(\tilde{\xi}_i(t)) + 2\left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 \mathrm{tr}\left(A + \frac{\gamma}{2}I\right)\mathrm{e}^{w\gamma\tau}V(\tilde{\xi}_i(t-\tau)).$$

If the undirected graph $\mathcal{G}$ is completely connected, it follows $\lambda_2 = \lambda_N$ and $\dot{V}(\tilde{\xi}_i(t)) \leqslant -\gamma V(\tilde{\xi}_i(t))$. Thus, for any parameter $\gamma > 0$, there holds $\lim_{t\to\infty}\tilde{\xi}_i(t) = 0$ for $i \in I[2,N]$, which means that the consensus is guaranteed for any large yet bounded communication delay.

Next, we consider the general case of $\lambda_2 \neq \lambda_N$. Assume $V(\tilde{\xi}_i(t-\tau)) < \eta V(\tilde{\xi}_i(t))$, where $\eta > 1$ is a constant to be determined. Let $q \triangleq w\gamma\tau = 2\mathrm{tr}(A + \frac{\gamma}{2}I)\tau$. Then, making use of $\gamma = \frac{2\mathrm{tr}(A+\frac{\gamma}{2}I)}{n} - \frac{2\mathrm{tr}(A)}{n}$, the consensus problem is transformed into one for finding proper $\tau > 0, \gamma > 0$ and $\eta > 1$ such that

$$2\mathrm{tr}(A)\tau < q\left[1 - \eta n \times \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2 \mathrm{e}^q\right].$$

Toward this objective, denote function $G(q) \triangleq q[1 - nS(\lambda_2,\lambda_N)\mathrm{e}^q]$ with $S(\lambda_2,\lambda_N) = \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^2$. If $2\mathrm{tr}(A)\tau < G(q)$, we design constant $\eta = \frac{q - 2\mathrm{tr}(A)\tau + qnS(\lambda_2,\lambda_N)\mathrm{e}^q}{2qnS(\lambda_2,\lambda_N)\mathrm{e}^q} > 1$. Then, $2\mathrm{tr}(A)\tau < q[1 - \eta nS(\lambda_2,\lambda_N)\mathrm{e}^q]$ holds. Thus, the consensus is guaranteed if $2\mathrm{tr}(A)\tau < G(q)$ according to the Razumikhin Stability Theorem in [21]. The remaining part is similar to the proof in Theorem 1 and hence omitted. Therefore, the proof is complete.

Remark 3. The method in Theorem 2 is also available if matrix A has stable eigenvalues. In fact, assume A and B have the following forms by applying appropriate transformations

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_- \\ B_+ \end{bmatrix},$$

where $\mathrm{Re}(\lambda(A_-)) < 0$ and $\mathrm{Re}(\lambda(A_+)) \geqslant 0$. Let $\tilde{\xi}_i(t) = \left[\begin{smallmatrix} \tilde{\xi}_i^-(t) \\ \tilde{\xi}_i^+(t) \end{smallmatrix}\right]$ and $K_1 = [0, K_1^+], K_2 = [0, K_2^+]$. Then, system (10) in Theorem 2 can be rewritten as

$$\begin{aligned} \dot{\tilde{\xi}}_i^-(t) &= A_- \tilde{\xi}_i^-(t) + B_- K_2^+ \tilde{\xi}_i^+(t) - [\lambda_i B_- K_1^+ + B_- K_2^+ \mathrm{e}^{A_+ \tau}] \tilde{\xi}_i^+(t-\tau), \\ \dot{\tilde{\xi}}_i^+(t) &= [A_+ + B_+ K_2^+] \tilde{\xi}_i^+(t) - [\lambda_i B_+ K_1^+ + B_+ K_2^+ \mathrm{e}^{A_+ \tau}] \tilde{\xi}_i^+(t-\tau). \end{aligned}$$

If $\lim_{t\to\infty} \tilde{\xi}_i^+(t) = 0$, it follows from $\mathrm{Re}(\lambda(A_-)) < 0$ that $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$, which means that consensus cannot be affected by the stable eigenvalues of A. Thus, the technique in Theorem 2 also works when matrix A has stable eigenvalues.

Remark 4. The allowable delay bounds in Theorem 1 and 2 can be easily calculated if the network topology and agent dynamics are provided. The two delay bounds can be enlarged if we improve the synchronizability (i.e., increase $\frac{\lambda_2}{\lambda_N}$) of the network. However, comparing Theorem 2 with Theorem 1 shows that the allowable delay bound under protocol (3) is larger than that under protocol (2) if the network topology has better synchronizability. An example to demonstrate this conclusion is provided in the next section.

5 Simulation

In this section, simulations are shown to demonstrate the effectiveness of the previous results. Assume that there are four agents in the network and the dynamics of the i-th agent is

$$\begin{bmatrix} \dot{x}_{i1}(t) \\ \dot{x}_{i2}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \quad i = 1, 2, 3, 4.$$

It is obvious that the system given above is controllable and has an exponentially unstable eigenvalue $\lambda = \frac{1}{2}$. The initial values are arbitrarily selected as $x_1(0) = [2; -3], x_2(0) = [-6; 18], x_3(0) = [16; 9]$ and $x_4(0) = [10; 4]$. In addition, let $x_i(s) = [0; 0], u_i(s) = 0$ for any $s < 0$. From Lemma 1, the solution of (4) is computed as $P(\gamma) = \left[\begin{smallmatrix} \frac{(\gamma+1)(2\gamma+1)^2}{4} & \frac{(\gamma+1)(2\gamma+1)}{2} \\ \frac{(\gamma+1)(2\gamma+1)}{2} & 2\gamma+1 \end{smallmatrix}\right]$.

Assume the topology of the four agents is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the adjacency and Laplacian matrices as

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L_{\mathcal{G}} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Clearly, the topology is connected and the non-zero eigenvalues of $L_{\mathcal{G}}$ are $\lambda_2 = 2, \lambda_3 = \lambda_4 = 4$.

Given the above assumptions and Theorem 1, the allowable delay bound under protocol (2) for consensus is $\tau^* = 0.194$. In contrast, using Theorem 2 and protocol (3), the delay bound is $\tau^* = 0.486$. In this example, take $\tau = 0.40$, and select gains $h_1 = \frac{1}{3}, h_2 = 1$ from Theorem 2. Furthermore, the tolerance set for parameter γ is computed as $\overline{\Omega} = [0.572, 0.594]$. Selecting $\gamma = 0.58$, Figures 1 and 2 show that the error states converge to zero asymptotically.

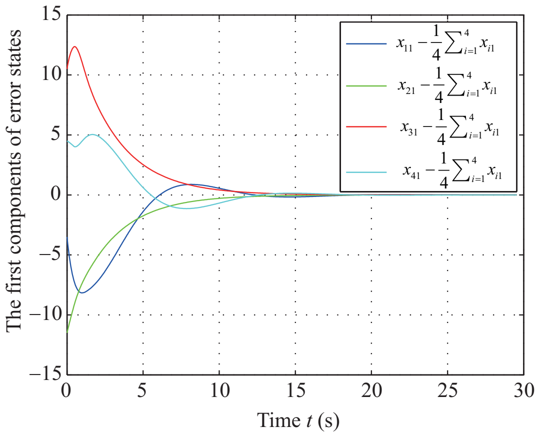

Figure 1 (Color online) Error states 1.

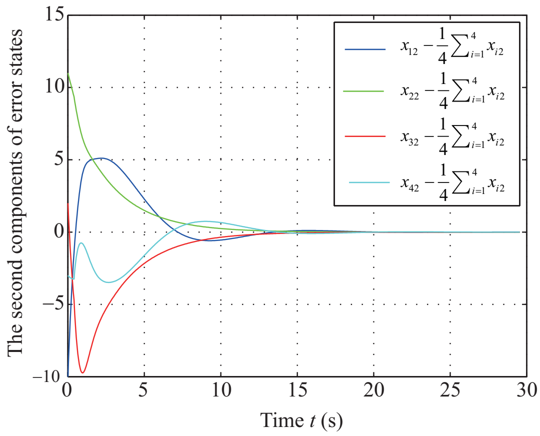

Figure 2 (Color online) Error states 2.

6 Conclusion

This study researches the consensus problem for general high-order multi-agent systems in the presence of communication delay. By employing an ARE, allowable delay bounds depending on the network topology and agent dynamics are derived for two designed protocols. It is observed in particular that when the network topology is complete and a part of agent's own historical input information is introduced in the protocol, any large yet bounded communication delay is tolerant for consensus. Future research on this subject includes extending the results in this study to the unknown and time-varying delay cases.

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Conflict of interest The authors declare that they have no conflict of interest.

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