H_∞ CONTROL FOR STOCHASTIC SYSTEMS WITH DISTURBANCE PREVIEW *

HONGXIA WANG†, MINYUE FU‡, AND HUANSHUI ZHANG§

Abstract. The paper considers the H_∞ control problem for stochastic systems with disturbance preview, which is very challenging since it involves the preview problem and multiplicative noise simultaneously. The H_∞ control problem for deterministic systems with disturbance preview was once listed as one of the 53 open problems in mathematical and control and its methods can not be generalized to solve the corresponding stochastic problem because of the essential differences of the two classes of systems. Using the projection principle in indefinite space, we give a necessary condition of the solvable H_∞ preview control problem by using a pair of variables. The necessary condition is very useful for solving the minimax problem. An inertia condition of matrices, as the necessary and sufficient condition under which the H_∞ control for stochastic linear systems is solvable, is also proposed and testified. This condition generalizes the results for H_∞ control for deterministic systems with disturbance preview. Our results are demonstrated via a quarter vehicle active suspension system.

Key words. stochastic system, disturbance attenuation, minimax problem, H_∞ preview control

AMS subject classifications. 39A06, 93E15, 93D15

1. Introduction. Disturbance attenuation has been one of the core control design problems for applications [24, 8, 17, 15, 4]. With the rapid development of the sensor technology, more and more information becomes available in advance, leading to the great research interest on preview control. How to utilize the preview information on disturbances to effectively improve the disturbance attenuation performance is the problem of our concern. The H_∞ control problem for disturbance attenuation with preview information has been known to be a challenging one for a long time and was stated as Open Problem 51 in 1998 [3]. For deterministic systems, the problem was finally solved in 2005 for the continuous-time case [26] and the discrete-time case [27].

Other alternative solutions to the H_∞ control with disturbance preview for deterministic systems can be found in the literature as well. For example, the H_∞ control for deterministic systems with both input delay and disturbance preview was solved in [19, 20]. Under the assumption that the standard H_∞ problem (which corresponds to the system without input delay and preview) is solvable, an analytic solution to the problem was provided by deriving the explicit expressions of some abstract operators in [19, 20]. But as pointed out in [26], this assumption leads to a sufficient condition only because the achievable H_∞ performance level by using disturbance preview is typically lower (better) than that achievable by the standard H_∞ solution. In [29], using the so-called reorganization technique, the H_∞ preview problem was solved and

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a duality between the H_∞ smoothing and the H_∞ control with input-delay and disturbance preview was established. However, there has been no progress so far for stochastic systems.

The purpose of this paper is to generalize the work in [26, 27] to the stochastic setting. Stochastic systems involve parameter uncertainties in the system model which are random in nature. Examples of random physical parameters include impedance variations in electrical circuits [7], stiffness, damping and inertia changes in mechanical systems [16], and and gravitational field fluctuations in satellite dynamics [25].

Our motivation stems from the fact that technical tools used in [26, 27] are suitable for deterministic systems only. More precisely, [26, 27] gives a very elegant solvability condition for H_∞ control with disturbance preview and provides an analytic solution using two Riccati equations with the same dimension as the system without preview. This is made possible fundamentally due to the separation principle [1] for deterministic systems. Unfortunately, H_∞ control for stochastic systems with disturbance preview is inherently different from the deterministic case because the separation principle no longer holds [18, 6].

Several contributions are made in the paper. Firstly, the necessary condition of the H_∞ control for stochastic systems with preview disturbance is presented by a pair of variables admitting a forward-backward stochastic system and two stationary equations. The condition is a counterpart for bi-objective problem of the maximum principle for stochastic systems. Secondly, the affine link between the states of the forward-backward system is established. More precisely, the link is between the full-information (state and the disturbance preview) and the state of the backward system. Thirdly, an inertia condition which is necessary and sufficient for the solvability of H_∞ control problem for stochastic systems is provided. Fourthly, an analytic solution to the H_∞ preview control for stochastic systems is given.

Our results above are novel because the existing results [12, 13, 22, 21, 23] are for the H_∞ tracking for stochastic systems with reference signal preview. They are extensions of the work [5] rather than [27]. When the preview is on reference signal instead of disturbance in [12, 13, 22, 21, 23], as [26] pointed out, the preview information is treated in the H_2 setting rather than the H_∞ setting. In our case, the problem of H_∞ control with disturbance preview is much more involved than the H_∞ tracking problem with reference signal preview [12, 13, 22, 21, 23] . Technically speaking, our problem leads to a totally different solvability condition.

The rest of this paper is organized as follows. The problem to be solved is formulated in Section 2. A necessary condition for the solving H_∞ control with disturbance preview is presented in Section 3. The necessary condition is proved to be sufficient in Section 4. Some further discussion concerning how to use the disturbance preview to improve the closed-loop system performance is given in Section 5. Section 6 provides a quarter vehicle active suspension system to illustrate the application of our control law. Some concluding remarks are given in Section 7.

Notations: In the paper, w_k is a white noise with zero mean and variance σ, and it is defined on a complete probability measurable space $(\Omega, \mathcal{F}, P)$; $\mathcal{F}_k$ represents a σ-algebra generated by $\{w_i, i = 0, \cdots, k\}$; $E[X]$ is the expectation of the random variable X; $E[X|\mathcal{F}]$ is the conditional expectation of the random variable X given σ-algebra $\mathcal{F}$; l_2 is a space of expectation-square-summable and adapted sequences, i.e. for any $x \in l_2$, $\sum_{i=0}^{\infty} E[x_i' x_i] < \infty$ and x_i is $\mathcal{F}_{i-1}$-measurable. $l_{2[a,b]}$ means that every sequence here is defined over the interval $[a, b]$ [2]; For any $x, y \in l_{2[a,b]}$, $\langle x, y\rangle = \sum_{i=a}^{b} E[x_i' y_i]$ and $(l_{2[a,b]}, \langle\cdot,\cdot\rangle)$ is also a Hilbert space. If $i > j$, then $\sum_{i}^{j} a_k = 0$. For any integer n and $m = 1, \cdots, d$, $n_m = n + d - m$. For any matrix M, $M > 0 (M \geq 0)$ means that M is positive definite (semi-definite).

2. Problem statement.

The system to be considered in this paper is

$$x_{k+1} = A_k x_k + B_k u_k + C_k v_{k-d} \tag{2.1}$$

$$z_k = F_k x_k + D_k u_k \tag{2.2}$$

where x_k, u_k, z_k are state, control input, and the output to be regulated, and v_k is energy-bounded previewed exogenous disturbance with preview length $d > 0$, a integer; $A_k = A + w_k \bar{A}, B_k = B + w_k \bar{B}, C_k = C + w_k \bar{C}, F_k = F + w_k \bar{F}, D_k = D + w_k \bar{D}$; w_k is a scalar random white noise with zero mean and variance σ^2 and $A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D}, F$ and $\bar{F}$ are constant matrices with compatible dimensions.

In fact, it is shown that a large class of linear systems have their matrices A_k, B_k, C_k, D_k, F_k depending linearly on physical parameters [4]. When a physical parameter deviates from its nominal value due to various stochastic disturbances (e.g., thermal noises, vibration, impedance variations, etc.), it can be modeled as the nominal value plus some random noise. This will result in the multiplicative noise model considered in this paper.

Throughout the rest of this paper, we adopt the following assumption:

$$\bar{F} = 0, \bar{D} = 0, D'[D \ F] = [I \ 0] \tag{2.3}$$

which means that $E[z_k' z_k] = E[x_k' F' F x_k] + E[u_k' u_k]$. This will considerately reduce the complexity of required algebraic manipulations in the derivation of our some results and our idea is actually applicable to the general case without this assumption.

In the preview control setting, both the disturbance v_k and the control u_k are $\mathcal{F}_{k-1}$-adapted. Because v_k is available at time k but delayed, i.e., v_{k-d} is applied to the system at time k, u_k (being $\mathcal{F}_{k-1}$-adapted) would have the full information of a window of the "future" disturbance values $v_{k-d}, v_{k-d+1}, \ldots, v_k$. This future information makes the preview control particularly interesting in applications where adversaries (i.e. disturbances) have sluggish reactions which can be effectively modelled by time delays. However, how to utilize the future information to achieve the better control performance also makes the control problem technically challenging at the same time.

Given a control law u_k, the l_2 induced norm of the closed-loop mapping $L_{vz} : v \to z$ of (2.1)-(2.2) subject to the zero initial condition, i.e., $x_0 = 0, v_s = 0$ for $s = -d, \cdots, -1$, is given by

$$||L_{vz}|| = \sup_{v \in l_2} \frac{||z||_{l_{2[0,N]}}}{||v||_{l_{2[0,N-d]}}} \tag{2.4}$$

System (2.1)-(2.2) is said to satisfy a given H_∞ performance level $\gamma > 0$ if the following holds:

$$||L_{vz}|| < \gamma \tag{2.5}$$

The H_∞ preview control problem in this paper is to testify for a given $\gamma > 0$, whether there exists a full-information and adapted control law satisfying the H_∞ performance (2.5) and if exists, provides such a control law.

Remark 2.1. Adaptedness is one of the most significant differences between the deterministic and stochastic systems. Every variable appearing in the controlled stochastic system is required to be adapted. It also leads to the essential difference between backward stochastic systems and backward deterministic systems. Unlike the case of backward deterministic systems, it is very difficult to get an explicit and analytic solution for a delayed backward stochastic system.

3. Necessary condition of H_∞ control for stochastic systems with preview. In this section, we will see what happens when there is a full-information and adapted controller such that the H_∞ performance (2.5) holds for the given γ, which in turn will be helpful for us to find a criteria to testify if there exists such a controller such that (2.5) holds for a given γ in the next section.

Define

$$J(0,N) = ||z||^2_{l_{2[0,N]}} - \gamma^2 ||v||^2_{l_{2[0,N-d]}} \tag{3.1}$$

There is a relationship between the H_∞ control performance (2.5) and dynamic game

$$\max_v \min_u J(0,N) \tag{3.2}$$

because

$$\inf_u \sup_{v\in l_2} \frac{||z||_{l_{2[0,N]}}}{||v||_{l_{2[0,N-d]}}} \leq \sup_{v\in l_2} \inf_u \frac{||z||_{l_{2[0,N]}}}{||v||_{l_{2[0,N-d]}}} \tag{3.3}$$

Obviously, the upper value (the left of (3.3)) is not less than the lower value (the right of (3.3)) [2]. Hence, for a given $\gamma > 0$, if $\inf_u \sup_{v\in l_2} \frac{||z||_{l_{2[0,N]}}}{||v||_{l_{2[0,N-d]}}} < \gamma$, then $\sup_{v\in l_2} \inf_u \frac{||z||_{l_{2[0,N]}}}{||v||_{l_{2[0,N-d]}}} < \gamma$, and the latter can be converted into the solvable minimax problem (3.2). Moreover, the optimal u_k, v_k admit the identical equations with the H_∞ central controller (please refer to Chapter 9 of [14]) and the worst-case disturbance. Based on this, we propose a necessary condition for the solvable H_∞ preview control problem.

LEMMA 3.1. *Consider the system (2.1)-(2.2). If there exists a adapted controller such that (2.5) holds, then for $k \geq 0$, the H_∞ central controller and the worst-case disturbance obey the following relations*

$$0 = E[B_k' \lambda_k | \mathcal{F}_{k-1}] + u_k \tag{3.4}$$

$$0 = E[C_{k+d}' \lambda_{k+d} | \mathcal{F}_{k-1}] - \gamma^2 v_k \tag{3.5}$$

where

$$\lambda_{k-1} = E[A_k' \lambda_k | \mathcal{F}_{k-1}] + F' F x_k \tag{3.6}$$

$$\lambda_N = 0 \tag{3.7}$$

Lemma 3.1 will be proved with the aid of projection principle in Krein space [26]. It is stated as follows.

LEMMA 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces with bounded linear operators $J : \mathcal{X} \to \mathcal{Y}$ and $S : \mathcal{X} \to \mathcal{Y}$. Suppose $J = J'$ and $S'JS > \epsilon I$ for some $\epsilon > 0$. Then, given any $y \in \mathcal{Y}$, there exists a unique solution to the optimization problem*

$$\min_{x\in\mathcal{X}} ||Sx - y||^2_J = \min_{x\in\mathcal{X}} \langle (Sx - y), J(Sx - y)\rangle \tag{3.8}$$

This solution is defined by y and a bounded linear operator, $x^* = (S'JS)^{-1} S'Jy$. Equivalently, x^* is completely characterized by the equality $S'J(Sx^* - y) = 0$, i.e., $\forall x \in \mathcal{X}, \langle Sx, J(Sx^* - y)\rangle = 0$.

Now we are in the position to prove Lemma 3.1.

Proof. As mentioned earlier, if the H_∞ preview control for (2.1)-(2.2) is solvable, the game problem (3.2) is solvable.

From (3.1),

$$J(0,N) = E[\sum_{k=0}^{N} z_k' z_k - \gamma^2 \sum_{k=0}^{N-d} v_k' v_k]$$

$$= E[\sum_{k=0}^{N} x_k' F' F x_k + u_k' u_k - \gamma^2 \sum_{k=0}^{N-d} v_k' v_k] \tag{3.9}$$

Firstly, we consider the inner optimization $\min_u ||z||^2_{l_{2[0,N]}}$ of (3.2). Denote the input-output operators from the inputs u, v and initial data $(x_0, \hat{v}_0)$ to the output as $\mathcal{T}_u, \mathcal{T}_v$ and $\mathcal{T}_0$, respectively. According to Lemma 3.2, $\mathcal{T}_u$, the identity operator and $\mathcal{T}_v v + \mathcal{T}_0(x_0, \hat{v}_0)$ will play the roles of S, J and $-y$, respectively. The fact $||\mathcal{T}_u u||^2_{l_{2[0,N]}} > 0$ for $u \neq 0$ means $S'JS = S'S$ is uniformy positive. Hence, a unique optimal u, denoted by u^* minimizing $||z||^2_{l_{2[0,N]}}$ obeys

$$\langle \mathcal{T}_u u, \mathcal{T}_u u^* + \mathcal{T}_v v + \mathcal{T}_0(x_0, \hat{v}_0) \rangle = 0 \tag{3.10}$$

The above means that the optimal z is orthogonal to the output of any input u, which is also very useful for finding the optimal solution to the outer optimization. Denoting z^* as the optimal z corresponding any given v and initial data $(x_0, \hat{v}_0)$, (3.10) can be rewritten as

$$\langle u, \mathcal{T}_u' z^* \rangle = 0 \tag{3.11}$$

In order to obtain the relation (3.4), the adjoint operator $\mathcal{T}_u'$ of the operator $\mathcal{T}_u$ is characterized in the sequel.

Straightforward calculation shows that the k^{th} component of $\mathcal{T}_u u$ is as

$$(\mathcal{T}_u u)_k = F \sum_{i=0}^{k-2} F(k-2, i+1) B_i u_i + D u_{k-1} \tag{3.12}$$

$$\mathcal{T}_u u = \begin{bmatrix} D u_0 \\ F \sum_{i=0}^{0} F(0, i+1) B_i u_i + D u_1 \\ \vdots \\ F \sum_{i=0}^{k-1} F(k-1, i+1) B_i u_i + D u_k \\ \vdots \end{bmatrix} \tag{3.13}$$

where

$$F(k,i) = \begin{cases} A_k A_{k-1} \cdots A_i, k \geq i \\ I, k = i-1 \\ 0, k < i-1 \end{cases} \tag{3.14}$$

Similarly, we can give the k^{th} components of $\mathcal{T}_v v$ and $\mathcal{T}_0(x_0, \hat{v}_0)$ as follows

$$(\mathcal{T}_v v)_k = F \sum_{i=d}^{k-2} F(k-2, i+1) C_i v_{i-d} \tag{3.15}$$

$$(\mathcal{T}_0(x_0, \hat{v}_0))_k = F F(k-2, 0) x_0 + F \sum_{i=0}^{\min\{k-2, d-1\}} F(k-2, i+1) C_i v_{i-d} \tag{3.16}$$

Hence,

$$\mathcal{T}_u = \begin{bmatrix} D & 0 & \cdots & 0 \\ F(0,1)B_0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F(N-1,1)B_0 & F(N-1,2)B_1 & \cdots & D \end{bmatrix} \tag{3.17}$$

Denote the optimal state and output generated by the optimal control law u_k^* as x_k^* and z_k^*, respectively. The adaptedness of $\mathcal{T}_u' z^*$ and (3.17) together with the equality

$$\langle \mathcal{T}_u u, z^* \rangle = \langle u, \mathcal{T}_u' z^* \rangle \tag{3.18}$$

show the k^{th} component of $\mathcal{T}_u' z^*$

$$(\mathcal{T}_u' z^*)_k = D' z_{k-1}^* + E[B_{k-1}' \sum_{i=k}^{N} F(i-1,k)' F' z_i^* | \mathcal{F}_{k-2}] \tag{3.19}$$

In virtue of the assumption (2.3), the above relation can be reduced to

$$(\mathcal{T}_u' z^*)_k = u_{k-1}^* + E[B_{k-1}' \sum_{i=k}^{N} F(i-1,k)' F' F x_i^* | \mathcal{F}_{k-2}] \tag{3.20}$$

Let

$$\lambda_{k-1}^* = E[A_k' \lambda_k | \mathcal{F}_{k-1}] + F' F x_k \tag{3.21}$$

$$\lambda_N^* = 0. \tag{3.22}$$

Then (3.20) can be rewritten as

$$(\mathcal{T}_u' z^*)_k = u_{k-1}^* + E[B_{k-1}' \lambda_{k-1}^* | \mathcal{F}_{k-2}] \tag{3.23}$$

which together with (3.11) shows that the optimal u_{k-1}^* admits

$$0 = u_{k-1}^* + E[B_{k-1}' \lambda_{k-1}^* | \mathcal{F}_{k-2}] \tag{3.24}$$

Hence, (3.4) holds. Note, in particular, that u_k is $\mathcal{F}_{k-1}$ adapted.

Next we consider the outer optimization problem in (3.2) over v_k. Since the H_∞ control problem is solvable, the inequality (2.5) subject to a admissible and adapted control law u_k^* holds for any disturbance v_k and zero initial state, namely,

$$\sup_{v \in l_2} \frac{||z^*||^2_{l_{2[0,N]}}}{||v||^2_{l_{2[0,N-d]}}} < \gamma^2 \tag{3.25}$$

Therefore,

$$\gamma^2 ||v||^2_{l_{2[0,N-d]}} - ||z^*||^2_{l_{2[0,N-d]}} > 0 \tag{3.26}$$

Denoting $J = diag\{\gamma^2 I, -I\}$ and $Sv = (v, \mathcal{T}_v v + \mathcal{T}_u u^*)$, the inequality (3.26) implies $S'JS$ is a positive operator.

We now solve the outer optimization in (3.2) according to Lemma 3.2. Let $(0, \mathcal{T}_0(x_0, \hat{v}_0))$ and v play the roles of $-y$ and x in Lemma 3.2, then $J^*(0,N)$ in (3.2) can be rewritten as

$$J^*(0,N) = \langle Sv + (0, \mathcal{T}_0 x_0), J[Sv + (0, \mathcal{T}_0 x_0)] \rangle \tag{3.27}$$

where $J^*(0,N)$ means the J driven by u^*. The positive definiteness of $S'JS$ implies that $\max_v J^*(0,N)$ is solvable and the optimal v solving $\max_v J^*(0,N)$, denoted as $v^\#$, satisfying the relation below

$$S'J[Sv^\# + (0, \mathcal{T}_0 x_0)] = 0 \tag{3.28}$$

i.e.

$$S'J(v^\#, z^\#) = 0 \tag{3.29}$$

where $z^\#$ is the output driven by the optimal u^*, the optimal $v^\#$ and the any given initial data $(x_0, \hat{v}_0)$. Different from the inner optimization in (3.2), it is not easy to derive the adjoint operator S' from the equation (3.29) to characterize the optimal $v^\#$. We thus introduce a new operator $\tilde{S}$ as

$$\tilde{S}v = (v, \mathcal{T}_v v) \tag{3.30}$$

Here note that, as a candidate of z^*, $z^\#$ is generated by the optimal control u^*, the optimal v^* and any given initial data $(x_0, \hat{v}_0)$, which together with (3.10) shows $z^\#$ is orthogonal to the output $\mathcal{T}_u u$ for any u, one of which is $\mathcal{T}_u u^*$. Hence, $\langle z^\#, \mathcal{T}_u u^* \rangle = 0$. Based on it, (3.29) can read as

$$\begin{aligned} 0 &= \langle Sv, J(v^\#, z^\#) \rangle \\ &= \langle (v, \mathcal{T}_v v + \mathcal{T}_u u^*), J(v^\#, z^\#) \rangle \\ &= \gamma^2 \langle v, v^\# \rangle - \langle \mathcal{T}_v v + \mathcal{T}_u u^*, z^\# \rangle \\ &= \gamma^2 \langle v, v^\# \rangle - \langle \mathcal{T}_v v, z^\# \rangle \\ &= \langle \tilde{S}v, J(v^\#, z^\#) \rangle \end{aligned} \tag{3.31}$$

i.e.,

$$\tilde{S}'J(v^\#, z^\#) = 0 \tag{3.32}$$

Since $\tilde{S}'(v,z) = v + \mathcal{T}_v' z$,

$$0 = \tilde{S}'J(v^\#, z^\#) = \gamma^2 v^\# - \mathcal{T}_v' z^\# \tag{3.33}$$

which implies that the k^{th} component of $\mathcal{T}_v' z^\#$ equals to

$$\begin{aligned} (\mathcal{T}_v' z^\#)_k &= E[C_{k-1}' \sum_{i=k}^{N} F(i-1,k)' F' z_i^\# | \mathcal{F}_{k-2}]' \\ &= \gamma^2 v_{k-1-d}^\# \end{aligned} \tag{3.34}$$

Let

$$\lambda_{k-1}^\# = E[A_k' \lambda_k^\# | \mathcal{F}_{k-1}] + F' F x_k^\# \tag{3.35}$$

$$\lambda_N^\# = 0. \tag{3.36}$$

the equation (3.34) can be reduced to

$$E[C_{k-1}' \lambda_{k-1}^\# | \mathcal{F}_{k-2-d}]' = \gamma^2 v_{k-1-d}^\# \tag{3.37}$$

In the above, all the variables labeled by $^{\#}$ have a similar meaning as $z^{\#}$ and are optimal trajectories corresponding to the optimal $v^{\#}$ and the optimal u^{*}. Here, u^{*} can be denoted as $u^{\#}$ since u_k can obtain the information of v_k and u_k^{*} actually depends on $v^{\#}$ when v_k equals to $v^{\#}$.

At present, all the variables $x_k, u_k, v_k, z_k, \lambda_k$ are unified and labeled by $^{\#}$. For notational simplicity, we omit the superscript $^{\#}$ in (3.35)-(3.36), we get (3.6)-(3.7), which means that the optimal u and v can be characterized by the unified (3.6)-(3.7). The conclusion in this lemma is thus proved. ∎

Remark 3.3. Lemma 3.1 proposes a necessary condition of the solvable minimax problem (3.2) by the projection principle in indefinite space, which is very helpful for characterizing the optimal trajectories of (3.2) by a unified pair of variables and thus pursuing the optimal solution to the minimax problem (3.2).

Remark 3.4. Lemma 3.1 is an analogue for the minimax problem of the maximum principle for the optimal problem [28].

Lemma 3.1 implicitly describes a necessary condition, in the form of equations satisfied by the H_∞ preview controller and the worst-case disturbance, of the solvable H_∞ preview control problem, and what follows is an explicit expression.

LEMMA 3.5. *Consider the system* (2.1)-(2.2)*. If there exists a adapted controller such that* (2.5) *holds, then*

- *R_k and Λ have the same inertias, i.e. the numbers of negative, positive and zero eigenvalues of R_k and Λ are equal, respectively;*
- *The H_∞ central controller u_k and the worst-disturbance v_k admit*

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = -R_k^{-1}[T_k x_k + \sum_{j=0}^{d-1} T_k^j v_{k+j-d}] \tag{3.38}$$

- *There holds*

$$\lambda_{k-1} = P_k x_k + \sum_{j=0}^{d-1} P_k^j v_{k+j-d} \tag{3.39}$$

$$\lambda_{k+d-1} = S_k x_k + \sum_{j=0}^{d-1} S_k^j v_{k+j-d} \tag{3.40}$$

In the above,

$$\Lambda = diag\{I, -\gamma^2 I\} \tag{3.41}$$

$$R_k = E\begin{bmatrix} B_k' P_{k+1} B_k & B_k' P_{k+1}^{d-1} \\ C_{k+d}' S_{k+1} B_k & C_{k+d}' S_{k+1}^{d-1} \end{bmatrix} + \Lambda \tag{3.42}$$

and P_k, P_k^j admit the following recursive relations

$$P_k = E[A_k' P_{k+1} A_k] + F'F - \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ (P_{k+1}^{d-1})' A \end{bmatrix}' R_k^{-1} T_k \tag{3.43}$$

$$P_k^0 = E[A_k' P_{k+1} C_k] - \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ (P_{k+1}^{d-1})' A \end{bmatrix}' R_k^{-1} T_k^0 \tag{3.44}$$

$$P_k^j = A' P_{k+1}^{j-1} - \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ (P_{k+1}^{d-1})' A \end{bmatrix}' R_k^{-1} T_k^j \tag{3.45}$$

with

$$T_k = \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ E[C_{k+d}' S_{k+1}]A \end{bmatrix} \tag{3.46}$$

$$T_k^0 = \begin{bmatrix} E[B_k' P_{k+1} C_k] \\ E[C_{k+d}' S_{k+1}]C \end{bmatrix} \tag{3.47}$$

$$T_k^j = \begin{bmatrix} B' P_{k+1}^{j-1} \\ E[C_{k+d}' S_{k+1}^{j-1}] \end{bmatrix}, j = 1, \cdots, d-1 \tag{3.48}$$

Therein, $P_{N+1}^j = 0$*,* S_k *and* S_k^j*, which are initialized by* $S_{N+1} = 0$ *and* $S_{N+1}^j = 0$*, contain the noises* $w_k, \cdots, w_{k+d-1}$ *will be explicitly given in the next lemma.*

Proof. The proof is stated in Appendix. ∎

Until now S_{k+1} and $S_{k+1}^j, j = 0, \cdots, d-1$ involved in Lemma 3.5 still remain to be given. To the end, it is necessary to define some notations.

$$\Phi_n^{k+1} = \Phi_{n+k}^1 \Phi_n^k + \sum_{f=0}^{k-1} \Phi_{n+k}^{1,f+d-k} \Pi_n^f \tag{3.49}$$

$$\Phi_n^{k+1,j} = \Phi_{n+k}^1 \Phi_n^{k,j} + \sum_{f=0}^{k-1} \Phi_{n+k}^{1,f+d-k} \Pi_n^{f,j} + \Phi_{n+k}^{1,j-k}$$

(3.50)

$$\Pi_n^k = \Pi_{n+k}^0 \Phi_n^k + \sum_{f=0}^{k-1} \Pi_{n+k}^{0,f+d-k} \Pi_n^f \tag{3.51}$$

$$\Pi_n^{k,j} = \Pi_{n+k}^0 \Phi_n^{k,j} + \sum_{f=0}^{k-1} \Pi_{n+k}^{1,f+d-k} \Pi_n^{f,j} + \Pi_{n+k}^{0,j-k}$$

(3.52)

with the initial values

$$\Phi_n^0 = I, \Phi_n^{0,j} = 0 \tag{3.53}$$

$$\Phi_n^1 = A_n - [B_n \ 0] R_n^{-1} T_n \tag{3.54}$$

$$\Phi_n^{1,j} = \delta_j C_n - [B_n \ 0] R_n^{-1} T_n^j \tag{3.55}$$

$$\Pi_n^0 = -[0 \ I] R_n^{-1} T_n, \tag{3.56}$$

$$\Pi_n^{0,j} = -[0 \ I] R_n^{-1} T_n^j \tag{3.57}$$

where R_n, T_n and $T_n^j, j = 0, \cdots, d-1$ are as in (3.42), (3.46)-(3.48), respectively. It should be pointed that we also need the notations $\Phi_n^j = 0$, $\Phi_n^{1,j} = 0$ and $\Pi_n^j = 0$ for $j < 0$.

With those notations above, the expressions of S_n and $S_n^j, j = 0, \cdots, d-1$ are provided below.

Lemma 3.6. *The coefficient matrices* S_n *and* S_n^j *appearing in the relation* (3.40) *with $k = n$ are given as*

$$S_n = P_{n+d}\Phi_n^d + \sum_{f=0}^{d-1} P_{n+d}^f \Pi_n^f \tag{3.58}$$

$$S_n^j = P_{n+d}\Phi_n^{d,j} + \sum_{f=0}^{d-1} P_{n+d}^f \Pi_n^{f,j} \tag{3.59}$$

Moreover, S_n and $S_n^j, j = 0, \cdots, d-1$ only involve noises $\{w_{n+d-1}, \cdots, w_n\}$.

Proof. Let the inputs u and v be the optimal for $\max_v \min_u J(0, N)$. Then the following representations can be obtained

$$x_{n+k+1} = \Phi_n^{k+1} x_n + \sum_{j=0}^{d-1} \Phi_n^{k+1,j} v_{j+n-d} \tag{3.60}$$

$$v_{n+k} = \Pi_n^k x_n + \sum_{j=0}^{d-1} \Pi_n^{k,j} v_{j+n-d} \tag{3.61}$$

by inductive derivation over $k = 0, \cdots, d-1$. From these two expressions and (3.40), we can get the expressions (3.58) and (3.59).

What follows is a brief proof for (3.60) and (3.61). According to Lemma 3.5, the optimal u_n, v_n for $\max_v \min_u J(0, N)$ is

$$u_n = -[I\ 0]R_n^{-1}(T_n x_n + \sum_{j=0}^{d-1} T_n^j v_{n+j-d}) \tag{3.62}$$

$$v_n = -[0\ I]R_n^{-1}(T_n x_n + \sum_{j=0}^{d-1} T_n^j v_{n+j-d}) \tag{3.63}$$

Observing (3.57), it is direct to find that the optimal v_n as in (3.63) is exactly (3.61) with $k = 0$. Substituting (3.62) into (2.1), there holds

$$x_{n+1} = \Phi_n^1 x_n + \sum_{j=0}^{d-1} \Phi_n^{1,j} v_{n+j-d} \tag{3.64}$$

which is (3.60) with $k = 0$.

Assuming (3.60) and (3.61) hold for $k = 0, \cdots, s-1$ and $s < d-1$, we will verify that (3.60) and (3.61) also hold for $k = s$.

Similar to (3.62) and (3.64), we have

$$v_{n+s} = -[0\ I]R_{k+s}^{-1}(T_{n+s} x_{n+s} + \sum_{j=0}^{d-1} T_{n+s}^j v_{n+s+j-d}) \tag{3.65}$$

$$x_{n+s+1} = \Phi_{n+s}^1 x_{n+s} + \sum_{j=0}^{d-1} \Phi_{n+s}^{1,j} v_{n+s+j-d} \tag{3.66}$$

It is easy to know that the subscript of $v_{n+s+j-d}$, namely, $n+s+j-d$ is less than $n+s$ in the second term in the right side of (3.65)-(3.66) because of $j = 0, \cdots, d-1$, which means that $v_{n+s+j-d}$ with $s+j-d>0$ can be re-expressed by the inductive assumption.

Applying the inductive assumption (3.60) with $k=s-1$ and (3.61) with $k=0,\cdots,s-1$ into (3.65)-(3.66) and using the notations (3.49)-(3.52), (3.60)-(3.61) with $k=s$ are obtained.

Reminding of the relation (A.22), we have

$$\lambda_{n+d-1} = P_{n+d}x_{n+d} + \sum_{j=0}^{d-1} P_{n+d}^{j} v_{n+j} \tag{3.67}$$

From (3.60)-(3.61),

$$x_{n+d} = \Phi_n^d x_n + \sum_{j=0}^{d-1} \Phi_n^{d,j} v_{n+j-d} \tag{3.68}$$

$$v_{n+j} = \Pi_n^j x_n + \sum_{i=0}^{d-1} \Pi_n^{j,i} v_{n+i-d} \tag{3.69}$$

Inserting both of them into (3.67), one will get (3.58)-(3.59). In terms of the recursive relations (3.49)-(3.52), we can see that $\Phi_n^d, \Phi_n^{d,j}$ and $\Pi_n^f, \Pi_n^{f,j} f=0,\cdots,d-1$ only include the noises $\{w_{k+d-1},\cdots,w_n\}$ and $\{w_{k+f},\cdots,w_n\}$, respectively. As a consequence, S_n and $S_n^j, j=0,\cdots,d-1$ only involve the noises $\{w_{n+d-1},\cdots,w_n\}$.□

Lemma 3.6 shows that there are links between $P_{k+1}, P_{k+1}^j, j=0,\cdots,d-1$ and $S_{k+1}, S_{k+1}^j, j=0,\cdots,d-1$. The links will help us to get explicit expressions of $E[C'_{k+d}S_{k+1}]$ and $E[C'_{k+d}S_{k+1}^j], j=0,\cdots,d-1$ appearing in (3.42), (3.46)-(3.48) in Lemma 3.5.

Lemma 3.7. *The following relations hold for $k=0,\cdots,N$ and $j=1,\cdots,d$:*

$$E[C'_{k+d}S_{k+1}] = (P_{k+1}^{d-1})' \tag{3.70}$$

$$E[C'_{k+d}S_{k+1}^{j-1}] = m_k^j + \delta_{d-j}(C'P_{k_1}C + \sigma\bar{C}'P_{k_1}\bar{C}) \tag{3.71}$$

with

$$m_k^j = -\sum_{i=1}^{j}(T_{k+i}^{d-i})'R_{k+i}^{-1}T_{k+i}^{j-i} + \sum_{i=1}^{d-1}\delta_{i-j}(P_{k+1+i}^{d-i-1})'C \tag{3.72}$$

$$k_1 = k+d-1 \tag{3.73}$$

where δ_i is a Kronecker operator with the center in 0.

Proof. The proof of Lemma 3.7 is based on Lemma 3.6 and inductive derivation over $k=N,\cdots,0$.

As $k=N$, (3.70) and (3.71) are trivial since the initial matrices value $S_{N+1}=0$ and $P_{N+1}^j=0, S_{N+1}^j=0$ with $j=0,\cdots,d-1$.

Assume (3.70) and (3.71) hold for all $k\ge n$. Then (3.42), (3.46)-(3.48) can be rewritten as

$$R_k = \begin{bmatrix} E[B_k' P_{k+1} B_k] & B' P_{k+1}^{d-1} \\ (P_{k+1}^{d-1})' B & m_k^{d-1} + E[C_k' P_{k_1} C_k] \end{bmatrix} + \Lambda \tag{3.74}$$

$$T_k = \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ E[C_{k+d}' S_{k+1}] A \end{bmatrix} \tag{3.75}$$

$$T_k^0 = \begin{bmatrix} E[B_k' P_{k+1} C_k] \\ E[C_{k+d}' S_{k+1}] C \end{bmatrix} \tag{3.76}$$

$$T_k^j = \begin{bmatrix} B' P_{k+1}^{j-1} \\ E[C_{k+d}' S_{k+1}^{j-1}] \end{bmatrix}, j = 1, \cdots, d-1 \tag{3.77}$$

Consequently, (3.43)-(3.45) can be reformulated as

$$P_k = A' P_{k+1} A + \sigma \bar{A}' P_{k+1} \bar{A} - T_k' R_k^{-1} T_k + F' F \tag{3.78}$$

$$P_k^0 = A' P_{k+1} B + \sigma \bar{A}' P_{k+1} \bar{B} - T_k' R_k^{-1} T_k^0 \tag{3.79}$$

$$P_k^j = A' P_{k+1}^{j-1} - T_k' R_k^{-1} T_k^j \tag{3.80}$$

What follows is to prove (3.70)-(3.71) also hold in the case of $k = n-1$.

These two equalities

$$\begin{aligned} E[C_{n_1}' S_n] = (P_{n_m}^{m-1}) E[\Phi_{n_m}^{d-m}] + \sum_{f=0}^{d-1-m} [C' P_{n+d}^f \\ - \sum_{i=1}^{m} (T_{n_i}^{i-1})' R_{n_i}^{-1} T_{n_i}^{f+i}] E[\Pi_n^f] \end{aligned} \tag{3.81}$$

$$\begin{aligned} E[C_{n_1}' S_n^j] = (P_{n_m}^{m-1}) E[\Phi_{n_m}^{d-m,j}] + \sum_{f=0}^{d-1-m} [C' P_{n+d}^f \\ - \sum_{i=1}^{m} (T_{n_i}^{i-1})' R_{n_i}^{-1} T_{n_i}^{f+i}] E[\Pi_n^{f,j}] \\ - \sum_{i=d-m}^{j} (T_{n+i}^{d-i-1})' R_{n+i}^{-1} T_{n+i}^{j-i} + \sum_{i=d-m}^{d-2} \delta_{j-i} (P_{n+i+1}^{d-i-2})' C \\ + \delta_{d-1-j} E[C_n' P_{n+d} C_n] \end{aligned} \tag{3.82}$$

are very useful for our proof. They can be proved by inductive derivation over $m = 1 \cdots, d$ and straightforward expectation calculation based on Lemma 3.6 and matrices (3.49)-(3.57), so we omit it here.

Let $m = d$ in (3.81) and (3.82), we will see (3.70) and (3.71) hold for $k = n-1$. Now the proof is completed. ∎

According to Lemma 3.7, some matrices appearing in Lemma 3.5 are simplified further in the following remark.

Remark 3.8. Those notations related to $E[C_{k+d}' S_{k+1}]$ as well as $E[C_{k+d}' S_{k+1}^j]$, appearing in Lemma 3.5 can be rewritten as

$$T_k = \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ (P_{k+1}^{d-1})' A \end{bmatrix} \tag{3.83}$$

$$T_k^0 = \begin{bmatrix} E[B_k' P_{k+1} C_k] \\ (P_{k+1}^{d-1})' C \end{bmatrix} \tag{3.84}$$

$$T_k^j = \begin{bmatrix} B' P_{k+1}^{j-1} \\ (P_{k+j+1}^{d-j-1})' C - \sum_{f=1}^{j} (T_{k+f}^{d-f})' R_{k+f}^{-1} T_{k+f}^{j-f} \end{bmatrix} \tag{3.85}$$

$$R_k = \begin{bmatrix} E[B_k' P_{k+1} B_k] & (P_{k+1}^{d-1})' B \\ B' P_{k+1}^{d-1} & E[C_{k+d}' P_{k+d+1}' C_{k+d}] \end{bmatrix} + diag\{I, -\gamma^2 I - \sum_{f=1}^{d} (T_{k+f}^{d-f})' R_{k+f}^{-1} T_{k+f}^{d-f}\} \tag{3.86}$$

Further, (3.43)-(3.45) are expressed as

$$P_k = A' P_{k+1} A + \sigma \bar{A}' P_{k+1} \bar{A} - T_k' R_k^{-1} T_k + F' F \tag{3.87}$$

$$P_k^0 = A' P_{k+1} C + \sigma \bar{A}' P_{k+1} \bar{C} - T_k' R_k^{-1} T_k^0 \tag{3.88}$$

$$P_k^j = A' P_{k+1}^{j-1} - T_k' R_k^{-1} T_k^j \tag{3.89}$$

Remark 3.8 provides a more direct but equivalent result than that in Lemma 3.5, which is very useful in the next section.

4. Sufficient condition of H_∞ control for stochastic systems with preview.

In the section, we will verify that the necessary condition in Lemma 3.5 is also sufficient for the solvability of the H_∞ control problem with disturbance preview.

Although the same notations as the last section are introduced at the beginning of this section, please note that their meanings are actually different because R_k and $T_k^j, j = 1, \cdots, d-1$ appearing in (4.1)-(4.3) and (3.87)-(3.89) are different.

Before our proof begins, we need to define some notations.

$$P_k = A' P_{k+1} A + \sigma \bar{A}' P_{k+1} \bar{A} - T_k' R_k^{-1} T_k + F' F \tag{4.1}$$

$$P_k^0 = A' P_{k+1} C + \sigma \bar{A}' P_{k+1} \bar{C} - T_k' R_k^{-1} T_k^0 \tag{4.2}$$

$$P_k^j = A' P_{k+1}^{j-1} - T_k' R_k^{-1} T_k^j \tag{4.3}$$

(4.4)

$$R_k = \begin{bmatrix} E[B_k' P_{k+1} B_k] & (P_{k+1}^{d-1})' B \\ B' P_{k+1}^{d-1} & \beta_{k+1}(d-1, d-1) \end{bmatrix} + \Lambda \tag{4.5}$$

$$T_k = \begin{bmatrix} E[B_k' P_{k+1} A_k] \\ (P_{k+1}^{d-1})' A \end{bmatrix} \tag{4.6}$$

$$T_k^0 = \begin{bmatrix} E[B_k' P_{k+1} C_k] \\ (P_{k+1}^{d-1})' C \end{bmatrix} \tag{4.7}$$

$$T_k^j = \begin{bmatrix} B' P_{k+1}^{j-1} \\ \beta_{k+1}(d-1, j-1) \end{bmatrix} \tag{4.8}$$

with

$$\beta_k(i,j) = \beta_{k+1}(i-1,j-1) - (T_k^i)' R_k^{-1} T_k^j \tag{4.9}$$
$$\beta_k(j,i) = \beta_k(i,j)' \tag{4.10}$$
$$\beta_k(0,j) = C' P_{k+1}^{j-1} - (T_k^0)' R_k^{-1} T_k^j \tag{4.11}$$
$$\beta_k(0,0) = E[C_k' P_{k+1} C_k] - (T_k^0)' R_k^{-1} T_k^0 \tag{4.12}$$

For $i = 0, \cdots, d-1$ and $j = 0, \cdots, d-1$, the initial matrices value of P_k^j and $\beta_k(i,j)$ are given as $P_{N+1}^j = 0$ and $\beta_{N+1}(i,j) = 0$.

Remark 4.1. In fact, the relationships (4.2)-(4.3) together with their initial values means that $P_k^j = 0$ if $k+j-d > N-d$. Similarly, $\beta_k(i,j) = 0$ if $k+\max\{i,j\}-d > N-d$ follows from the relation (4.9) and the initial value of $\beta_k(i,j)$.

Now a condition is provided to guarantee the solvability of the H_∞ preview control problem for a given γ.

LEMMA 4.2. *For a given $\gamma > 0$. If (4.1)-(4.3) admit solutions such that R_k and Λ have the same inertias, then the H_∞ control problem (2.5) subject to (2.1) is solvable. Moreover, the H_∞ central controller u_k and the worst-disturbance v_k admit*

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = -R_k^{-1}[T_k x_k + \sum_{j=0}^{d-1} T_k^j v_{k+j-d}] \tag{4.13}$$

Proof. Define a value function by

$$V(k,\bar{x}_k) = E[x_k' P_k x_k + 2\sum_{j=0}^{d-1} x_k' P_k^j v_{k+j-d} + \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} v_{k+j-d}' \beta_k(i,j) v_{k+i-d}] \tag{4.14}$$

where $\bar{x}_k = col\{x_k, v_{k-1}, \cdots, v_{k-d}\}$.

Then we have

$$\begin{aligned} V(k+1,\bar{x}_{k+1}) &= E[x_{k+1}' P_{k+1} x_{k+1} \\ &+2\sum_{j=0}^{d-1} x_{k+1}' P_{k+1}^j v_{k+1+j-d} + \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} v_{k+1+i-d}' \beta_{k+1}(i,j) v_{k+1+j-d}] \end{aligned} \tag{4.15}$$

Plugging (2.1) into (4.15) and Completing square over $col\{u_k, v_k\}$ will yield

$$\begin{aligned} &V(k+1,\bar{x}_{k+1}) \\ &= E[x_k'(A_k' P_{k+1} A_k - T_k' R_k^{-1} T_k) x_k + \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix} \\ &\quad -u_k' u_k + \gamma^2 v_k' v_k \\ &\quad +2x_k'(A_k' P_{k+1} C_k - T_k' R_k^{-1} T_k^0) v_{k-d} + 2x_k' \sum_{j=1}^{d-1} (A_k' P_{k+1}^{j-1} - T_k' R_k^{-1} T_k^j) v_{k+j-d} \\ &\quad +v_{k-d}' C_k' P_{k+1} C_k v_{k-d} - \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} v_{k+i-d}' (T_k^i)' R_k^{-1} T_k^j v_{k+j-d} \\ &\quad +2\sum_{j=1}^{d-1} v_{k-d}' C_k' P_{k+1}^{j-1} v_{k+j-d} + \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} v_{k+i-d}' \beta_{k+1}(i-1,j-1) v_{k+j-d}] \end{aligned} \tag{4.16}$$

where

$$\begin{bmatrix} \bar{u}_k^* \\ \bar{v}_k^* \end{bmatrix} = R_k^{-1}(T_k x_k + \sum_{j=0}^{d-1} T_k^j v_{k+j-d}) \tag{4.17}$$

Applying (4.1)-(4.3), (4.9) and (4.11)-(4.12) in (4.16) yields

$$\begin{aligned} & V(k+1, \bar{x}_{k+1}) \\ = & E[x_k'(P_k - F'F)x_k + \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}) \\ & -u_k' u_k + \gamma^2 v_k' v_k \\ & +2x_k' \sum_{j=0}^{d-1} P_k^j v_{k+j-d} + \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} v_{k+i-d}' \beta_k(i,j) v_{k+j-d}] \end{aligned}$$

Now it is straightforward to obtain

$$\begin{aligned} & V(k, \bar{x}_k) - V(k+1, \bar{x}_{k+1}) \\ = & E(x_k' F' F x_k + u_k' u_k - \gamma^2 v_k' v_k - \sum_{k=0}^{N} \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}) \\ = & E(z_k' z_k - \gamma^2 v_k' v_k - \sum_{k=0}^{N} \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}) \end{aligned} \tag{4.18}$$

Adding (4.18) from $k = 0$ to $k = N$, we have

$$\begin{aligned} & V(0, \bar{x}_0) - V(N+1, \bar{x}_{N+1}) \\ = & \sum_{k=0}^{N} E[z_k' z_k - \gamma^2 v_k' v_k] + \sum_{k=0}^{N} \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix} \end{aligned} \tag{4.19}$$

As $k = N+1$, $V(N+1, \bar{x}_{N+1}) = x_{N+1} P_{N+1} x_{N+1}$ from (4.14) and Remark 4.1; On the other hand, as $k > N-d$, $R_k = diag\{E[B_k' P_{k+1} B_k + I], -\gamma^2 I\}$ from (4.5) and Remark 4.1; $v_k^* = 0$ because the blocks in T_k and $T_k^j, j = 0, \cdots, d-1$ corresponding to v_k are null, which originates from Remark 4.1, as $k > N-d$, $P_k^{d-1} = 0$ and $\beta_k(d-1, j) = 0$.

Now it is easy to get from (4.19)

$$J = V(0, \bar{x}_0) + \sum_{k=0}^{N} \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix}' R_k \begin{bmatrix} u_k + \bar{u}_k^* \\ v_k + \bar{v}_k^* \end{bmatrix} + \gamma^2 \sum_{k=N-d+1}^{N} v_k' v_k \tag{4.20}$$

Given that R_k and Λ have the same inertia, (4.20) shows that $J < 0$ holds when the initial data $\bar{x}_0 = 0$ and $u_k = \bar{u}_k^*$. ∎

At the moment, we associate the sufficient condition in Lemma 4.2 with the necessary condition in Lemma 3 and give the following necessary and sufficient condition for the solvability of the H_∞ preview control.

THEOREM 4.3. *For a given $\gamma > 0$, the H_∞ preview control problem (2.5) subject to (2.1) is solvable if and only if (3.87)-(3.89) with 3.83-3.86 admit solutions such* *that $diag\{\Omega_k, \Delta_k\}$ and Λ have the same inertias. Moreover, the H_∞ preview control law is given as*

$$u_k = -\Omega_k^{-1}(E[B_k' P_{k+1} A_k] x_k + E[B_k' P_{k+1} C_k] v_{k-d} + \sum_{j=1}^{d} B' P_{k+1}^{j-1} v_{k+j-d}) \tag{4.21}$$

In the above,

$$\Omega_k = I + B' P_{k+1} B + \sigma \bar{B}' P_{k+1} \bar{B} \tag{4.22}$$

$$\Delta_k = -\gamma^2 I + C' P_{k+d+1} C + \sigma \bar{C}' P_{k+d+1} \bar{C} - \sum_{f=1}^{d} (T_{k+f}^{d-f})' R_{k+f}^{-1} T_{k+f}^{d-f} - (P_{k+1}^{d-1})' B \Omega_k^{-1} B' P_{k+1}^{d-1} \tag{4.23}$$

Proof. The straightforward calculation shows the explicit expressions of $\beta_k(i,j)$ in the aforementioned as follows. In the case of $i < j$, from (4.9) and (4.11),

$$\beta_k(i,j) = C' P_{k+i+1}^{j-i-1} - \sum_{f=0}^{i} (T_{k+f}^{i-f})' R_{k+f}^{-1} T_{k+f}^{j-f} \tag{4.24}$$

In the case of $i = j$, from (4.9) and (4.12),

$$\beta_k(i,j) = E[C_{k+i}' P_{k+i+1} C_{k+i}] - \sum_{f=0}^{i} (T_{k+f}^{i-f})' R_{k+f}^{-1} T_{k+f}^{i-f} \tag{4.25}$$

As for the case of $i > j$, the explicit expression will be given by (4.10).

With the explicit expression of $\beta_k(i,j)$, R_k and $T_k^j, j = 1, \cdots, d-1$ can be read as

$$T_k^j = \begin{bmatrix} B' P_{k+1}^{j-1} \\ (P_{k+j+1}^{d-j-1})' C - \sum_{f=1}^{j} (T_{k+f}^{d-f})' R_{k+f}^{-1} T_{k+f}^{j-f} \end{bmatrix} \tag{4.26}$$

$$R_k = \begin{bmatrix} E[B_k' P_{k+1} B_k] & (P_{k+1}^{d-1})' B \\ B' P_{k+1}^{d-1} & E[C_{k+d}' P_{k+d+1}' C_{k+d}] \end{bmatrix} + diag\{I, -\gamma^2 I - \sum_{f=1}^{d} (T_{k+f}^{d-f})' R_{k+f}^{-1} T_{k+f}^{d-f}\} \tag{4.27}$$

Now it is clear that (4.1)-(4.3) can be reformulated as (3.87)-(3.89), which together with Lemma 3.5 and Lemma (4.2) shows H_∞ control problem is solvable if and only if (3.87)-(3.89) have solutions such that R_k and Λ have the same inertia. In order to obtain a preview control law, after making a LDU decomposition for R_k, (4.20) can be rewritten as

$$J(0,N) = V(0, \bar{x}_0) + \sum_{k=0}^{N} (u_k + \check{u}_k^*)' \Omega_k (u_k + \check{u}_k^*) + \sum_{k=0}^{N-h} (v_k + \hat{v}_k^*)' \Delta_k (v_k + \hat{v}_k^*)' \tag{4.28}$$

with

$$\breve{u}_k^* = \Omega_k^{-1}(E[B_k' P_{k+1} A_k]x_k + E[B_k' P_{k+1} C_k]v_{k-d} + \sum_{j=1}^{d} B' P_{k+1}^{j-1} v_{k+j-d}) \tag{4.29}$$

and $\hat{v}_k^* = \bar{v}_k^*$ as in (4.17). Consequently, the H_∞ preview control law can be chosen as $-\breve{u}_k^*$, i.e. , (4.21). ∎

To compare the performances of the H_∞ preview control and the standard H_∞ full-information control, we present the following theorem.

THEOREM 4.4. *For a given $\gamma > 0$, the H_∞ full-information control problem* (2.5) *subject to* (2.1) *with $d = 0$ is solvable if and only if*

$$P_k = A' P_{k+1} A + \sigma \bar{A}' P_{k+1} \bar{A} - T_k' R_k^{-1} T_k + F' F \tag{4.30}$$

admit solutions such that diag$\{\Omega_k, \Delta_k\}$ and diag$\{I, -\gamma^2 I\}$ have the same inertia. Moreover, the H_∞ full-information control law is given as

$$u_k = -\Omega_k^{-1}(E[B_k' P_{k+1} A_k]x_k + E[B_k' P_{k+1} C_k]v_k) \tag{4.31}$$

In the above,

$$R_k = \begin{bmatrix} E[B_k' P_{k+1} B_k] + I & E[B_k' P_{k+1} C_k] \\ E[C_k' P_{k+1} B_k] & -\gamma^2 I + E[C_k' P_{k+1} C_k] \end{bmatrix} \tag{4.32}$$

$$T_k = \begin{bmatrix} B' \\ C' \end{bmatrix} P_{k+1} A + \begin{bmatrix} \bar{B}' \\ \bar{C}' \end{bmatrix} P_{k+1} \bar{A} \tag{4.33}$$

$$\Omega_k = I + B' P_{k+1} B + \sigma \bar{B}' P_{k+1} \bar{B} \tag{4.34}$$

$$\Delta_k = -\gamma^2 I + E[C_k' P_{k+1} C_k] - E[B_k P_{k+1} C_k]' \Omega_k^{-1} E[B_k P_{k+1} C_k] \tag{4.35}$$

Proof. The necessity and sufficiency can be proved by applying the similar lines to Lemma 3.1 and Lemma 4.2, respectively, we thus omit them. ∎

Remark 4.5. The result generalizes the deterministic H_∞ control theory in state space [14] and the idea is different from that of the existing literature [4] and [10]. Specifically, [4] and [10] solved the H_∞ control problem for stochastic systems by obtaining the stochastic version of bounded real lemma. Moreover, [4] and [10] assume that the controller is linear state-feedback, and the results are given by linear matrices inequality.

5. Further discussions. In the section, we provide some explanations concerning the relationship between the achievable performance γ and the preview length d. The derivation of the necessary and sufficient condition in the last two sections offers some evidences supporting our explanations.

From Theorem 4.3, we know γ is determined by the constraint $\Delta_k < 0$. It together with (4.23) means that γ nonlinearly depends on all of coefficient matrices in the system and the weighted matrices in performance index.

According to (4.23), there holds

$$\begin{aligned}\Delta_k = &-\gamma^2 I + E[C'_{k+d}P_{k+d+1}C_{k+d}] \\ &-E[B_{k+d}P_{k+d+1}C_{k+d}]'\Omega^{-1}_{k+d}E[B_{k+d}P_{k+d+1}C_{k+d}] \\ &-C'P^{d-1}_{k+d+1}\Delta^{-1}_{k+d}(P^{d-1}_{k+d+1})'C \\ &-\sum_{f=1}^{d-1}(T^{d-f}_{k+f})'R^{-1}_{k+f}T^{d-f}_{k+f} - (P^{d-1}_{k+1})'B\Omega^{-1}_k B'P^{d-1}_{k+1}\end{aligned} \tag{5.1}$$

Since $\max_v \min_u J(k,N) \geq \min_u J(k,N)$ for any $v_i, i = k, \cdots, N$ and a candidate of $\min J_u(k,N) \geq 0$ with $v_i = 0, i = k, \cdots, N$, $\max_v \min_u J(k,N) \geq 0$. It shows $P_k \geq 0$ and $\beta_k(i,i) \geq 0$. Associated with (4.25), there hold

$$E[C'_{k+i+1}P_{k+i+2}C_{k+i+1}] \geq 0$$

$$E[C'_{k+i+1}P_{k+i+2}C_{k+i+1}] \geq \sum_{f=0}^{i}(T^{i-f}_{k+1+f})'R^{-1}_{k+1+f}T^{i-f}_{k+1+f}$$

At the moment, it is direct that in order to guarantee that there exists $\gamma > 0$ such that $\Delta_k < 0$ and

$$\beta_{k+1}(d-1,d-1) > (P^{d-1}_{k+1})'B\Omega^{-1}_k B'P^{d-1}_{k+1}. \tag{5.2}$$

Observing Δ_k in Theorem 4.4 and Δ_k in Theorem 4.3, we find that there is possibility to find a smaller γ for the H_∞ preview control problem than γ for the H_∞ control for delay-free stochastic systems since the last three terms appear in Δ_k in (5.1).

An intuitive analysis is given from the game theory in the sequel. As the two players, the control u and the disturbance v try to minimize and maximize the performance $J(0,N)$, respectively. The term $v'_k(T^{d-f}_{k+f})'R^{-1}_{k+f}T^{d-f}_{k+f}v_k$ can be regarded as the contribution of these two players' decision using the information v_k at instant $k+f$ to the game value. This contribution will be very small in that they play the game. Yet the player u contributes an additional value $v'_k(P^{d-1}_{k+1})'B\Omega^{-1}_k B'P^{d-1}_{k+1}v_k$ to the game value at k instant, which may surpass the player v's contribution $v'_kC'P^{d-1}_{k+d+1}\Delta^{-1}_{k+d}(P^{d-1}_{k+d+1})'Cv_k$ at $k+d$ instant because v_k is the historical information at $k+d$ and plays a increasingly weaker role as d increases. Based on this and (5.1), there are two conclusions. One is that H_∞ preview control can suppress the external disturbance better than the standard H_∞ full-information control, i.e. the former has a smaller disturbance suppression level γ. The other one is the dependence of achievable performance on the preview length. Specifically, the larger the preview length d is, the smaller γ is. Yet we should also notice that the performance γ may saturate for a certain finite preview length, which may result from that the early historical information may not be useful. Our two conclusions and the saturation phenomenon are supported by Figure 1.

6. Example. In this section, we provide an example to illustrate the H_∞ control for stochastic systems with disturbance preview.

Figure 1 [11] is a schematic of the quarter vehicle active suspension configuration. It is broadly representative of the fundamental suspension problem of isolating the vibration from the road. In this figure, m_s is the sprung mass, which represents the vehicle chassis; m_u is the unsprung mass, which represents mass of the wheel

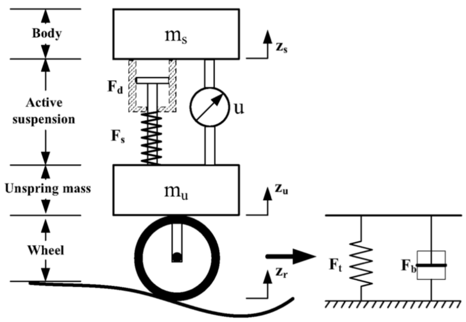


FIG. 1. *the quarter vehicle active suspension*

assembly; F_d and F_s are damping force and elastic force from the suspension system, respectively, and c_s and k_s are corresponding damping and stiffness, respectively; F_b and F_t are damping force and elastic force from the tire, respectively, and k_u and c_u stand for compressibility and damping of the pneumatic tyre, respectively; z_s and z_u are the displacements of the sprung and unsprung masses, respectively; u is the active input of the suspension system; z_r is the roadway elevation at vehicle, and it can be measured by the sensor mounting the suspension in advance and is thereby the same as that at the sensor position but delayed by a time (equal to the distance of the sensor in front of the vehicle divided by the vehicle velocity).

The dynamic equations of the sprung and unsprung masses are given by

$$m_s\ddot{z}_s + c_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) = u \tag{6.1}$$

$$m_s\ddot{z}_s + c_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) + c_u(\dot{z}_u - \dot{z}_r) + k_u(z_u - z_r) = -u \tag{6.2}$$

Define the following state variables:

$$x_1 = z_s - z_u \tag{6.3}$$

$$x_2 = z_u - z_r \tag{6.4}$$

$$x_3 = \dot{z}_s \tag{6.5}$$

$$x_4 = \dot{z}_u \tag{6.6}$$

where x_1 denotes the suspension deflection, x_2 is the tire deflection, x_3 is the sprung mass speed, and x_4 denotes the unsprung mass speed. We define disturbance input $v = \dot{z}_r$, which describes the roughness of the road. Then, by defining $x = [z_s - z_u, (\dot{z}_s - \dot{z}_u), \dot{z}_s, \dot{z}_u]'$, the dynamic equations in (6.1)-(6.2) can be rewritten in the following state-space form

$$\dot{x} = A_c x + B_c u + C_c v \tag{6.7}$$

where

$$A_c = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix} \tag{6.8}$$

$$B_c = \begin{bmatrix} 0 & 0 & \frac{1}{m_s} & -\frac{1}{m_u} \end{bmatrix}' \tag{6.9}$$

$$C_c = \begin{bmatrix} 0 & -1 & 0 & -\frac{c_t}{m_u} \end{bmatrix}' \tag{6.10}$$

In designing the control law for a suspension system, we need to consider ride comfort. It is widely accepted that ride comfort is closely related to the body acceleration. Therefore, when we design the controller, one of our main objectives is to reduce the body acceleration, that is, $\dot{x}_3$. In addition, in order to make sure the vehicle safety, we should ensure the firm uninterrupted contact of wheels to road, and the dynamic tire load $k_t x_2$ should be small so that $|k_t x_2| < (m_s + mu)g$. Because of mechanical structure, the suspension stroke x_1 should not exceed certain allowable maximum and it should be small either. Therefore, when we design the control law, our main objective is to guarantee that the regulated signal $z = \begin{bmatrix} \rho_1 \dot{x}_3 & \rho_2 \frac{k_t x_2}{(m_s+m_u)g} & \rho_3 x_1 \end{bmatrix}'$, a weighted column vector reflecting suspension body acceleration, the safety index (proportional to the tire deflection) and the body displacement (suspension stroke), is less than the weighted roughness of the road in the sense $||z|| < \gamma ||v||$, where $\rho_i \geq 0, i = 1, 2, 3$, are weights and are used for adjusting design preference. Now according to (6.7), z admits

$$z = F_c x + D_c u \tag{6.11}$$

where

$$F_c = \begin{bmatrix} -\rho_1 \frac{k_s}{m_s} & 0 & -\rho_1 \frac{c_s}{m_s} & \rho_1 \frac{c_s}{m_s} \\ 0 & \rho_2 \frac{k_t}{(m_s+m_u)g} & 0 & 0 \\ \rho_3 & 0 & 0 & 0 \end{bmatrix} \tag{6.12}$$

$$D_c = \begin{bmatrix} \rho_1 \frac{1}{m_s} & 0 & 0 \end{bmatrix}' \tag{6.13}$$

It is clear that system (6.7) has its matrices (A_c, B_c, C_c) depending on the physical parameters k_s, k_u, c_s, c_t, m_s. When they randomly deviates from their nominal values as a result of oscillatory motion and the change with the operation conditions, k_s, k_u, c_s, c_t, m_s can be modeled as $k_s + w_{ks}(t), k_u + w_{ku}(t), c_s + w_{cs}(t), c_t + w_{ct}(t), m_s + w_{ms}(t)$, here, $w_{ks}(t), w_{ku}(t), w_{cs}(t), w_{ct}(t), w_{ms}(t)$ are independent white processes with varinces $\sigma_{ks}, \sigma_{ku}, \sigma_{cs}, \sigma_{ct}, \sigma_{ms}$, respectively. The simple derivation shows that $\frac{\sigma}{\sigma_{ks}} w_{ks}(t)$, $\frac{\sigma}{\sigma_{ku}} w_{ku}(t)$, $\frac{\sigma}{\sigma_{cs}} w_{cs}(t)$, $\frac{\sigma}{\sigma_{ct}} w_{ct}(t)$, $\frac{\sigma}{\sigma_{ms}} w_{ms}(t)$ are white processes with variance σ. In particular, the approximation $\frac{1}{m_s + w_{ms}(t)} \doteq \frac{1}{m_s}(1 - \frac{w_{ms}(t)}{m_s})$ is used. This is the reason that we study the model with the multiplicative noise in this paper.

We borrow the quarter-vehicle suspension model parameters from [9] and list it in Table 1. Via the discretization of the vehicle suspension (6.7) and consideration of the

TABLE 1
vehicle suspension parameters

m_s	m_u	k_s	c_s	k_t	c_t
973kg	114kg	42720N/m	101115N/m	1095Ns/m	14.6Ns/m

parameter random uncertainty mentioned above, we obtain a discrete time stochastic system in the form of (2.1)-(2.2) with

$$A = \begin{bmatrix} 0.9251 & 0.1582 & 0.0176 & -0.0164 \\ 0.0669 & 0.8403 & 0.0022 & 0.0167 \\ -0.7711 & -0.2260 & 0.9722 & 0.0263 \\ 6.1633 & -14.8141 & 0.2248 & 0.6133 \end{bmatrix} \tag{6.14}$$

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0002 & 0 & 0.0004 & 0.0002 \\ 0.0002 & 0.0002 & 0.0004 & 0.0004 \end{bmatrix} \tag{6.15}$$

$$B = 10^{-3} \times \begin{bmatrix} 0.0018 \\ -0.0016 \\ 0.0180 \\ -0.1443 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0.0001 \\ 0 \end{bmatrix} \tag{6.16}$$

$$C = \begin{bmatrix} -0.0011 \\ -0.0189 \\ 0.0015 \\ 0.1618 \end{bmatrix}, \bar{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.003 \end{bmatrix} \tag{6.17}$$

$$F = \begin{bmatrix} -4.3905 & 0 & -0.1125 & 0.1125 \\ 0 & 0.9492 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0.001 \\ 0 \\ 0 \end{bmatrix} \tag{6.18}$$

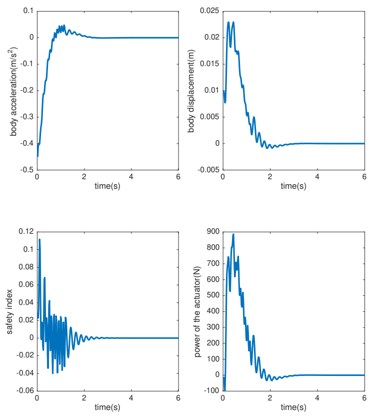
where the sample period $T = 0.02$, and $\rho_1 = 0.1, \rho_2 = 0.1, \rho_3 = 0.8$.

In the following, applying the more general version of Theorem 4.3 to the system (2.1)-(2.2) with (6.14)-(6.18), we will illustrate the performance of the closed-loop discrete-time suspension system with disturbance preview and random parameter uncertainty. Evaluation of the vehicle suspension performance is based on the examination of the sprung mass acceleration $\dot{x}_3$ (body acceleration), the safety index z_2 (tire deflection x_2), the sprung mass displacement x_1 (body displacement) and the H_∞ level γ. A controller is to be designed such that the regulated signal z is bounded by the weighted disturbance. In order to evaluate the suspension characteristics with respect to ride comfort and safety, the variability of the road profiles is taken into account. In the context of the vehicle suspension performance, road disturbances can be generally assumed as shock. Shocks are events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pothole on an otherwise smooth road. In the following, a kind of road profile is used to validate the performance of the presented control approach. Now consider the case of an isolated bump in an otherwise smooth road surface given by

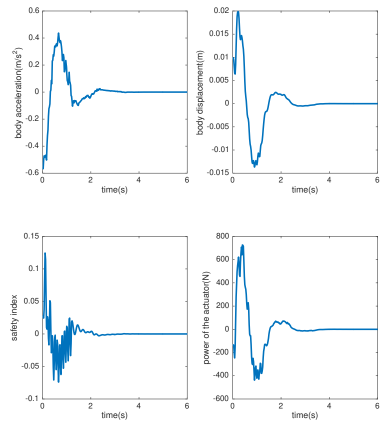
$$z_r = \frac{A}{2}(1 - cos(2\pi \frac{L}{V} t)) \tag{6.19}$$

where A and L are the height and the length of the bump. Assume $A = 80mm, L = 15m$ and the vehicle forward velocity $V = 45(km/h)$.

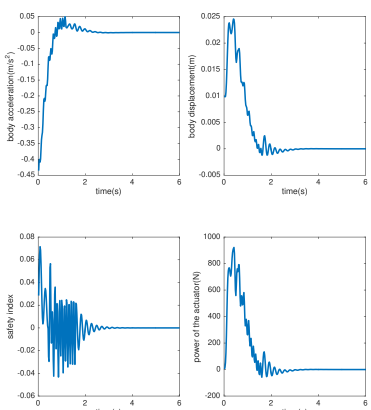
As Figure 2 shown, the random uncertainty deteriorates the suspension performance, in other word, the body acceleration z_1, body displacement x_1, safety index z_2 and the H_∞ performance γ increase as the random uncertainty of the suspension increases (i.e. σ becomes larger). On the other hand, the more the disturbance preview (larger d), the better the suspension performance, which means that the body



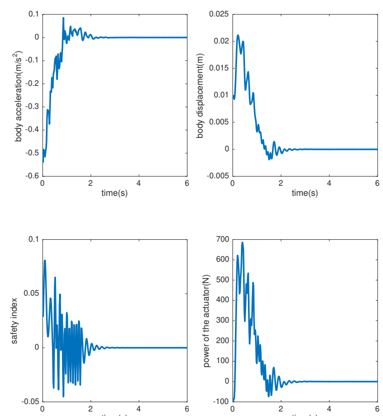
(a) $d = 0, \sigma = 0$



(b) $d = 0, \sigma = 0.05$



(c) $d = 20, \sigma = 0$



(d) $d = 20, \sigma = 0.05$

FIG. 2. *Bump response of the vehicle active suspension*

acceleration z_1, body displacement x_1, safety index z_2 and the H_∞ cost γ are smaller when more disturbance preview is utilized by the controller.

We also depict the curve of the optimal γ versus the preview length d for $N = 300$ and several different σ in Figure 3. From Figure 3, the curve for $\sigma = 0$ is in agreement with the one provided by the method in [27]. Besides, we also observe two phenomenona from Figure 3. One is the same conclusion as in Figure 2. The other is that using too much disturbance preview will not improve the suspension performance γ abidingly and the H_∞ performance will saturate after a certain length d.

7. Conclusions. In the paper, we obtain an analytic solution to the H_∞ preview control problem, which is an outstanding problem. It is shown that the problem is solvable if and only if a group of equations have solutions and an inertia condition holds. The proof depends heavily on how to characterize the necessary condition

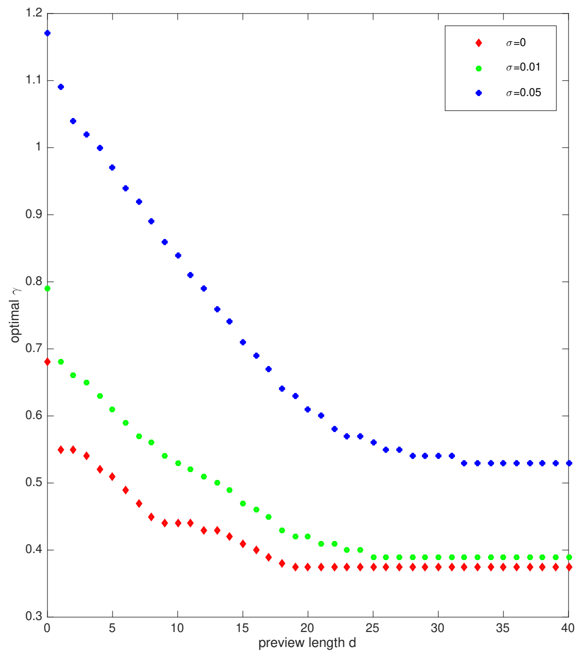


FIG. 3. *Optimal H_∞ performance versus the length of preview in our example*

better as the problem is solvable. We characterize it by a pair of stochastic difference equations with the aid of the projection principle in indefinite space which is helpful to get an explicit link between the two variables in the pair. The idea can also be used to solve the standard H_∞ control for stochastic systems completely and provide a solvability condition very similar to that for the deterministic counterpart. In fact, the idea can be applied to solve the game problems for stochastic systems with input delays too.

Appendix A. Proof of Lemma 3.5. We will present the proof of Lemma 3.5 here by using dynamic programming, which provides an effective means of obtaining the optimal solution to the minimax problem by solving a sequence of static games in reverse time.

For using dynamic programming, we define a similar notation as in the proof of Lemma 3.1. Let

$$J(i,N) = ||z||^2_{l_{l_{2[i,N]}}} - \gamma^2 ||v||^2_{l_{l_{2[i,N-d]}}} \tag{A.1}$$

Then

$$\begin{aligned} J(i,N) &= E[\sum_{k=i}^{N} z_k' z_k - \gamma^2 \sum_{k=i}^{N-d} v_k' v_k] \\ &= E[\sum_{k=i}^{N} (x_k' F' F x_k + u_k' u_k) - \gamma^2 \sum_{k=i}^{N-d} v_k' v_k] \end{aligned} \tag{A.2}$$

In fact, in the case of $i > N - d$,

$$\text{(A.3)} \quad J(i,N) = ||z||^2_{l_{2[i,N]}} = E\sum_{k=i}^{N} z_k' z_k = E\sum_{k=i}^{N}[x_k' F' F x_k + u_k' u_k]$$

since $\sum_{k=i}^{N-d} v_k' v_k = 0$.

With the same reason, if there is an adapted controller such that (2.5) holds for some $\gamma > 0$, (3.2) is solvable. According to Lemma 3.1, the optimal u_k and v_k can be characterized by (3.4) and (3.5). It should be stressed that the delay in disturbance input v_k leads to a special characterization (3.5) of the optimal v_k, where there is a time-lag between adapted processes v_k and λ_{k+d}. It will be very difficult to obtain the solvability and the optimal inputs.

In order to re-express the optimal game value, we derive a relation as follows

$$\text{(A.4)} \quad \begin{aligned} & E[x_k'\lambda_{k-1} - x_{k+1}'\lambda_k] \\ = & E[x_k'(E[A_k'\lambda_k|\mathcal{F}_{k-1}] + F'Fx_k) \\ & -(A_k x_k + B_k u_k + C_k v_{k-d})'\lambda_k] \\ = & E[x_k' F' F x_k - (B_k u_k + C_k v_{k-d})'\lambda_k] \end{aligned}$$

Applying (3.4) in the relation (A.4) leads to

$$\text{(A.5)} \quad \begin{aligned} & E[x_k'\lambda_{k-1} - x_{k+1}'\lambda_k] \\ = & E[x_k' F' F x_k - C_k' v_{k-d}'\lambda_k + u_k' u_k] \end{aligned}$$

Adding from $k = n+1$ to $k = N$ on the two sides of the equation (A.4), we have

$$\text{(A.6)} \quad \begin{aligned} & E[x_{n+1}'\lambda_n - x_{N+1}'\lambda_N] \\ = & \sum_{k=n+1}^{N} E[x_k' F' F x_k - C_k' v_{k-d}'\lambda_k + u_k' u_k] \end{aligned}$$

Denote the optimal game value $\max_v \min_u J(n+1,N)$ as $J^*(n+1,N)$ and apply (3.5) for $k \geq n+1$, then

$$\text{(A.7)} \quad J^*(n+1,N) = E[x_{n+1}'\lambda_n] + \sum_{k=n+1}^{\min\{n+d,N\}} E[v_{k-d}' C_k' \lambda_k]$$

Let

$$\text{(A.8)} \quad \bar{J}(n,N) = J^*(n+1,N) + z_n' z_n - \gamma^2 v_n' v_n$$

According to the dynamic programming principle, global optimization is the same as local one, i.e. if $\max\min ||z||^2_{l_{2[0,N]}} - ||v||^2_{l_{2[0,N-d]}}$ is solvable, $\max\min ||z||^2_{l_{2[i,N]}} - ||v||^2_{l_{2[i,N-d]}}$ is inevitably solvable, here $0 \leq i \leq N$. Moreover, the optimal solution of the later is in accordance with the former's in the overlapped interval $[i, N]$. Hence, $\bar{J}(n,N)$ is solvable over u_n, v_n.

With the above preparations, we now prove the three conclusions in the lemma using the inductive method on k.

First consider the case of $k = N$. Applying (2.1), we have

$$J(N,N) = E[z_N' z_N + x_{N+1}' P_{N+1} x_{N+1}] = E[x_N'(F'F + A_N' P_{N+1} A_N) x_N + u_N'(I + B_N' P_{N+1} B_N) u_N + v_{N-d}' C_N' P_{N+1} C_N v_{N-d} + 2x_N' A_N' P_{N+1}(B_N u_N + C_N v_{N-d}) + 2u_N' B_N' P_{N+1} C_N v_{N-d}] \tag{A.9}$$

Because (3.2) is solvable, so is $\max_v \min_u J(N,N)$. Given that $J(N,N)$ only contains a variable u_N to be determined, $\max_v \min_u J(N,N)$ actually becomes $\min_u J(N,N)$. Hence, $E[I + B_N' P_{N+1} B_N] > 0$ and $R_N = diag\{E[I + B_N' P_{N+1} B_N], -\gamma^2 I\}$ has the same inertias with Λ.

According to (3.4), the optimal u_N can be given as

$$u_N = -E[I + B_N' P_{N+1} B_N]^{-1}(E[B_N' P_{N+1} A_N] x_N + E[B_N' P_{N+1} C_N] v_{N-d}) \tag{A.10}$$

which associates with $v_N = 0$ shows (3.38) holds because of the facts $S_{N+1} = 0, P_{N+1}^j = 0, S_{N+1}^j = 0, j = 0, \cdots, d-1$.

Inserting (A.10) and (2.1) into (3.6),

$$\begin{aligned} \lambda_{N-1} &= E[A_N' \lambda_N | \mathcal{F}_{N-1}] + F'F x_N \\ &= E[A_N' P_{N+1}(A_N x_N + B_N u_N + C_N v_{N-d}) | \mathcal{F}_{N-1}] + F'F x_N \\ &= E[A_N' P_{N+1}(A_N x_N + C_N v_{N-d} - B_N E[I + B_N' P_{N+1} B_N]^{-1}(E[B_N' P_{N+1} A_N] x_N + E[B_N' P_{N+1} C_N] v_{N-d}))] + F'F x_N \\ &= (E[A_N' P_{N+1} A_N] + F'F - E[A_N' P_{N+1} B_N] \times E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} A_N]) x_N \\ &\quad + (E[A_N' P_{N+1} C_N] - E[A_N' P_{N+1} B_N] \times E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} C_N] v_{N-d}) \end{aligned} \tag{A.11}$$

The direct algebra calculation from (3.43)-(3.45), (3.46)-(3.48) and the initial matrices values $S_{N+1} = 0, S_{N+1}^j = 0, P_{N+1}^j = 0, j = 0, \cdots, d-1$ gives $P_N = E[A_N' P_{N+1} A_N] + F'F - E[A_N' P_{N+1} B_N] E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} A_N]$, $P_N^0 = (E[A_N' P_{N+1} C_N] - E[A_N' P_{N+1} B_N] E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} C_N]$, and $P_N^j = 0, j = 1, \cdots, d-1$. Comparing them with (A.11), we can see that (3.39) holds as $k = N$.

What follows is to prove (3.40) holds for $k = N$. If the delay $d = 1$, then

$$\lambda_{N+h-1} = \lambda_N = P_{N+1} x_{N+1}. \tag{A.12}$$

Plugging (2.1) and (A.10) into (A.12) yields

$$\begin{aligned} \lambda_{N+d-1} &= P_{N+1}(A_N - B_N E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} A_N]) x_N \\ &\quad + P_{N+1}(C_N B_N E[I + B_N' P_{N+1} B_N]^{-1} E[B_N' P_{N+1} A_N]) v_{N-d} \end{aligned} \tag{A.13}$$

which indicates that λ_{N+d-1} is in the form as (3.40) and the related coefficients only involves w_N. If the delay $d>1$, then

$$\lambda_{N+d-1}=0, \tag{A.14}$$

so it is trivial and (3.40) holds for $k=N$.

Inductively, assume those three conclusions in the lemma holds for all $k\geq n+1$, we will verify that those three conditions hold for $k=n$.

Since the case for $n\geq N-d$ is simpler and it can be handled with the similar lines with the case for $n\leq N-d$, we assume $n\leq N-d$. Plugging (A.7), (3.6) with $k=n$ and (2.1) in $\bar{J}(n,N)$ yields

$$\begin{aligned}\bar{J}(n,N) &= J^*(n+1,N)+z_n'z_n-\gamma^2v_n'v_n \\ &= E[x_{n+1}'\lambda_n]+\sum_{k=n+1}^{\min\{N,n+d\}}E[v_{k-d}'C_k'\lambda_k]] \\ &\quad +z_n'z_n-\gamma^2v_n'v_n \\ &= E[(A_nx_n+B_nu_n+C_nv_{n-d})' \\ &\quad \times(P_{n+1}x_{n+1}+\sum_{i=0}^{d-1}P_{n+1}^iv_{n+1+i-d})] \\ &\quad +\sum_{k=n+1}^{\min\{N,n+d\}}E[v_{k-d}'C_k'\lambda_k] \\ &\quad +x_n'F'Fx_n+u_n'u_n-\gamma^2v_n'v_n\end{aligned} \tag{A.15}$$

As we focus on the quadratic term over the vector $col\{u_n,v_n\}$ in $\bar{J}(n,N)$, there holds

$$\begin{aligned}\bar{J}(n,N) &= E[(B_nu_n)'(P_{n+1}B_nu_n+P_{n+1}^{d-1}v_n)] \\ &\quad +u_n'u_n+E[v_n'C_{n+d}'\lambda_{n+d}]-\gamma^2v_n'v_n+\cdots \\ &= E[(B_nu_n)'(P_{n+1}B_nu_n+P_{n+1}^{d-1}v_n) \\ &\quad +u_n'u_n-\gamma^2v_n'v_n] \\ &\quad +E[v_n'C_{n+d}'(S_{n+1}B_nu_n+S_{n+1}^{d-1}v_n)] \\ &\quad +\cdots \\ &= E\begin{bmatrix}u_n\\v_n\end{bmatrix}'R_n\begin{bmatrix}u_n\\v_n\end{bmatrix}+\cdots\end{aligned} \tag{A.16}$$

Observing the above expression, if the inertias of R_n is not equal to that of the matrix $diag\{I,-\gamma^2I\}$, one can come to a conclusion that $\max_v\min_u\bar{J}(n,N)$ is not solvable, which conflicts with our previous result about it. Therefore, the inertia of R_n equals to that of $diag\{I,-\gamma^2I\}$ as $\max_v\min_u J(n,N)$ is solvable.

In light of (3.4)-(3.5) and the relation (3.39)-(3.40), there holds

$$-u_n=E[B_n(P_{n+1}x_{n+1}+\sum_{j=0}^{d-1}P_{n+1}^jv_{n+1+j-d})|\mathcal{F}_{n-1}] \tag{A.17}$$

$$\gamma^2v_n=E[C_{n+d}(S_{n+1}x_{n+1}+\sum_{j=0}^{d-1}S_{n+1}^jv_{n+1+j-d})|\mathcal{F}_{n-1}] \tag{A.18}$$

Plugging (2.1) into them generates

$$0 = R_n \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} E[B_n' P_{n+1} A_n] \\ E[C_{n+d}' S_{n+1} A_n] \end{bmatrix} x_n + \sum_{j=1}^{d-1} \begin{bmatrix} E[B_n P_{n+1}^{j-1}] \\ E[C_{n+d}' S_{n+1}^{j-1}] \end{bmatrix} v_{n+j-d} + \begin{bmatrix} E[B_n' P_{n+1} C_n] \\ E[C_{n+d}' S_{n+1} C_n] \end{bmatrix} v_{n-d} \tag{A.19}$$

where we use the fact that S_{n+1} and S_{n+1}^j, $j = 0, \cdots, d-1$ only involve the noises $\omega_{n+d}, \omega_{n+d-1}, \cdots, \omega_{n+1}$. Now applying the notations (3.46)-(3.48), the optimal u_k, v_k admits (3.38).

In the sequel, we will verify the relationships (3.39)-(3.40) hold for $k = n$. By virtue of (3.6),

$$\lambda_{n-1} = E[A_n' \lambda_n | \mathcal{F}_n] + F' F x_n \tag{A.20}$$

$$\lambda_{n+d-1} = E[A_{n+d}' \lambda_{n+d} | \mathcal{F}_{n+d}] + F' F x_{n+d} \tag{A.21}$$

From the inductive assumption, (3.39)-(3.40) hold for $k = n+1$, consequently,

$$\lambda_{n-1} = E[A_n' (P_{n+1} x_{n+1} + \sum_{j=0}^{d-1} P_{n+1}^j v_{n+1+j-d}) | \mathcal{F}_n] + F' F x_n \tag{A.22}$$

$$\lambda_{n+d-1} = E[A_{n+d}' (P_{n+d+1} x_{n+1} + \sum_{j=0}^{d-1} P_{n+d+1}^j v_{n+1+j-d}) | \mathcal{F}_{n+d}] + F' F x_{n+d} \tag{A.23}$$

Substituting the system (2.1) with $k = n$ and the expression (3.38) of the optimal u_k, v_k with $k = n$ into the equality (A.22) and applying the recursive relations (3.43)-(3.45), one can derive that (3.39) holds for $k = n$. Apply (2.1) with $k = n, \cdots, n+d-1$ and (3.38) with $k = n, \cdots, n+d-1$ in (A.23) until there only contain those terms over $x_n, v_{n-1}, \cdots, v_{n-d}$ and then rearrange them, a relation like (3.40) can be obtained, and therein all of coefficient matrices indeed involve the noises $\{w_n, \cdots, w_{n+d-1}\}$. At this moment, the case for $k = n$ has been clarified. The inductive proof is completed.

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