ON THE RECONSTRUCTION OF CONTINUOUS-TIME MODELS FROM ESTIMATED DISCRETE-TIME MODELS OF STOCHASTIC PROCESSES [1]

Kaushik Mahata * and **Minyue Fu** *

* *Centre for Complex Dynamic Systems and Control, University of Newcastle, Callaghan, NSW 2308, Australia. Email:* Kaushik.Mahata@newcastle.edu.au, Minyue.Fu@newcastle.edu.au

Abstract: In this paper we study the the problem of reconstructing a continuous-time model from an identified discrete-time model of a continuous-time stochastic processes. We present new necessary and sufficient condition for the existence of the soluion. It is also shown that the solution is unique if it exists. The theory is applicable to multivariable processes. The ideas are then used to develop a continuous-time ARMA (CARMA) process modelling algorithm with guaranteed solution. The theoretical results are tested using numerical simulations.

Keywords: Continuous-time stochastic processes, stochastic modelling, spectrum estimation, linear matrix inequalities.

1. PRELIMINARIES

Suppose that we observe a continuous-time stationary stochastic process $y(t)$ given in the state space form in terms of the linear stochastic differential equation

$$dz(t) = Az(t)dt + Bde(t), \qquad y(t) = Cz(t),$$

where $z(t)$ is a d-dimensional state vector, $e(t)$ is a m-dimensional Weiner process with increamental covariance matrix $R_c dt$. The problem under consideration is to model the process $z(t)$ from sampled data, where the sampling interval is h. Suppose that we observe the process $y(t)$ and sample it (without any anti-aliasing filter) uniformly with a sampling interval h. Then the sampled signal can be expressed in a discrete-time state space form as (Söderström, 2002, Page 86), (Åström, 1970)

$$z_d\{(k+1)h\} = \mathrm{e}^{Ah} z_d(kh) + w(kh)$$
$$y(kh) = Cz_d(kh).$$

where $w(kh)$ is an *equivalent* discrete-time zero-mean white noise such that

$$\mathcal{E}\left\{w(k_1 h)w'(k_2 h)\right\} = R_d \delta_{k_1,k_2}.$$

Furthermore, R_d is related to R_c by the relation

$$R_d = \int_0^h \mathrm{e}^{As} B R_c B' \mathrm{e}^{A's} \mathrm{d}s. \qquad (1)$$

The discrete-time model above is the *discrete-time equivalent* to the underlying continuous-time model in the sense that the second order statistics of the discrete-time model is consistent with the continuous-time process at the sampling instants. In the rest of this presentation we assume the following.

Assumption 1. For any eigenvalue $p = p_r + \mathrm{i}p_i$ of the matrix A it holds that $-\pi < hp_i < \pi$.

[1] This work is supported by Australian Research Council

This is a band-limitedness type of assumption. Later we comment upon the relevance of the assumption in more details. Intutively, one would expect that the continuous-time state $z(t)$ and the discrete-time state $z_d(kh)$ will have the same covariance matrix. Indeed, if P is the covariance matrix of $z(t)$ then it must satisfy the continuous-time Lyapunov equation

$$AP + PA' + BR_cB' = 0, \tag{2}$$

and then it can be shown that P does also satisfy the discrete-time Lyapunov equation

$$R_d = P - \mathrm{e}^{Ah}P\mathrm{e}^{A'h}. \tag{3}$$

To give a quick proof, subtitute (2) in (1). We have

$$\begin{aligned} R_d &= -\int_0^h \mathrm{e}^{As}(AP + PA')\mathrm{e}^{A's}\mathrm{d}s \\ &= -\int_0^h f(s)\mathrm{d}s - \int_0^h f'(s)\mathrm{d}s, \end{aligned}$$

where we define

$$f(s) = \mathrm{e}^{As}AP\mathrm{e}^{A's} = A\mathrm{e}^{As}P\mathrm{e}^{A's} = \frac{\mathrm{d}}{\mathrm{d}s}\{\mathrm{e}^{As}\}P\mathrm{e}^{A's}.$$

Now integrating $f(s)$ by parts we get

$$-\int_0^h f(s)\mathrm{d}s = P - \mathrm{e}^{Ah}P\mathrm{e}^{A'h} + \int_0^h f'(s)\mathrm{d}s,$$

which gives (3) after a rearrangement.

One approach, see (Larsson and Mossberg, 2003) and references therein, for identifying a continuous-time stochastic process model is to identify the discrete-time equivalent model first. Then one seeks for a transformation which maps the identified discrete-time model to the equivalent continuous-time model. In (Söderström, 1991) the problem of reconstructing the continuous-time model from the identified discrete-time model is considered in details and a numerical algorithm was proposed. The procedure assumes that we have estimates of e^{Ah}, R_d and C from the data, and is theoretically [1] equivalent to the following steps:

(1) Compute P by solving the discrete-time Lyapunov equation in (3).
(2) Solve the spectral factorization problem

$$\begin{aligned} G(s)G'(-s) = \Phi_c(s) := -C(sI - A)^{-1} \times \\ (AP + PA')(-sI - A')^{-1}C'. \end{aligned} \tag{4}$$

to compute B and R_c.

In the following we remark on some interesting points associated with the above procedure. If we use only the second order statistics to identify the discrete-time process, R_d is not unique for a given choice of e^{Ah} and C. This is because there are many possible values of R_d for which the discrete-time spectrum

$$\Phi_d(z) := C(zI - \mathrm{e}^{Ah})^{-1}R_d(z^{-1}I - \mathrm{e}^{A'h})^{-1}C' \tag{5}$$

remains the same. For example, if we subtitute R_d in (5) by

$$\bar{R}_d = R_d + Q - \mathrm{e}^{Ah}Q\mathrm{e}^{A'h}$$

for any Hermitian matrix Q satisfying

$$QC = 0,$$

then the resuting spectrum remains unchanged. This is because

$$\begin{aligned} & C(zI - \mathrm{e}^{Ah})^{-1}\{Q - \mathrm{e}^{Ah}Q\mathrm{e}^{A'h}\} \times \\ & \qquad (z^{-1}I - \mathrm{e}^{A'h})^{-1}C' \\ &= C(zI - \mathrm{e}^{Ah})^{-1} \times \\ & \quad \{zz^{-1}Q - zQ\mathrm{e}^{A'h} + zQ\mathrm{e}^{A'h} - \mathrm{e}^{Ah}Q\mathrm{e}^{A'h}\} \times \\ & \qquad (z^{-1}I - \mathrm{e}^{A'h})^{-1}C' \\ &= C(zI - \mathrm{e}^{Ah})^{-1}zQC' + \\ & \qquad CQ\mathrm{e}^{A'h}(z^{-1}I - \mathrm{e}^{A'h})^{-1}C' \\ &= CQC' + C(zI - \mathrm{e}^{Ah})^{-1}\mathrm{e}^{Ah}QC' + \\ & \qquad CQ\mathrm{e}^{A'h}(z^{-1}I - \mathrm{e}^{A'h})^{-1}C' \\ &= 0. \end{aligned} \tag{6}$$

For example, in (3), R_d is a generically full rank matrix, and BR_cB' in (2) is of rank m, where m is the dimension of $z(t)$. On the other hand discrete-time model identification algorithms (prediction error method for example) often identifies the innovations model, where R_d will be of rank m and consequenntly, we see that $AP + PA'$ in (4) is generically indefinite and have a rank more than m. Therefore, a question about uniqueness aspect arises naturally. If we use two diffrent values of R_d giving the same discrete-time spectrum in the reconstruction routine, do we get the same continuous-time spectrum?

The second interesting point regarding the reconstruction routine is about soluability of the problem. We do not get an associated continuous-time spectrum for every discrete-time spectrum. This is problem is encountered when $\Phi_c(s)$ in (4) is not positive on the imaginary axis. So we are unable to get a minimum phase spectral factor. Several authors have investigated the conditions for solubility. An analysis for SISO second order process is given in (Söderström, 1990). See also (Wahlberg, 1988) for a discussion on *sampling zeros*.

In this note we consider these two questions. We show that for a given discrete-time spectrum the

[1] We causion that the procedure described here, as it stands, may not be sound numerically. However, there are ways to arrange the computation in a numerically sound way. Here we are not concerned about the numerical aspects. See (Söderström, 1991) for details

associated continuous-time (provided that exists) reconstruction is unique and is independent of the choice of R_d. Then we examine the solubility condition from a different angle. We give an LMI condition which determines whether the solution exists. This condition does not depend on system order or the dimension m. It is also possible to test the condition numerically using recent tools for dsemidefinite programming.

2. UNIQUESNESS AND EXISTENCE

The main result of this note is stated in form of the following proposition.

Proposition 1. Assume that the solution to the reverse reconstruction problem exist, and

$$\Phi_d(z) = C(zI - e^{Ah})^{-1} R_i (z^{-1}I - e^{A'h})^{-1} C'$$

for $i = 1, 2$ such that $R_1 \neq R_2$. Let P_1 and P_2 be the matrices such that

$$R_i = P_i - e^{Ah} P_i e^{A'h}, \qquad i = 1, 2. \tag{7}$$

Let the corresponding reconstructed continuous-time spectrums be denoted by $\Phi_{ci}(s)$ for $i = 1, 2$:

$$\Phi_{ci}(s) = C(sI - A)^{-1}\{-AP_i - P_i A'\}(-sI - A')^{-1}.$$

Then

$$\Phi_{c1}(s) = \Phi_{c2}(s), \qquad \forall s.$$

Proof: Since $\Phi_d(z)$ is a positive definite function on the unit circle, there exists a unique positive semidefinite matrix R_* of rank m and an associated symmetric matrices $\{Q_i\}_{i=1}^{2}$

$$\begin{aligned} R_i - Q_i + e^{Ah} Q_i e^{A'h} &= R_*, \qquad (8)\\ e^{Ah} Q_i C' = 0, \qquad & CQ_iC' = 0, \end{aligned}$$

for $i = 1, 2$. Since e^{Ah} is generically nonsingular, it follows for $i = 1, 2$ that

$$Q_i C' = 0, \tag{9}$$

$$P_i - Q_i - e^{Ah}(P_i - Q_i)e^{A'h} = R_*, \tag{10}$$

where the last equality follows by combining (7) and (8). Now from (10) it readily follows that

$$\begin{aligned} &\{(P_1 - P_2) - (Q_1 - Q_2)\}\\ &\quad - e^{Ah}\{(P_1 - P_2) - (Q_1 - Q_2)\}e^{A'h} = 0, \end{aligned}$$

Since e^{Ah} has all roots inside the unit circle, it is evident that

$$\Delta := P_1 - P_2 = Q_1 - Q_2.$$

Note that $\Delta C' = 0$, and consequently,

$$\begin{aligned} &\Phi_{c1}(s) - \Phi c2(s)\\ &= C(sI - A)^{-1}\{-A\Delta - \Delta A'\}(-sI - A')^{-1}C'\\ &= C(sI - A)^{-1}\{s\Delta - A\Delta - s\Delta + -\Delta A'\} \times\\ &\qquad\qquad (-sI - A')^{-1}C'\\ &= C\Delta(-sI - A')^{-1}C' + C(sI - A)^{-1}\Delta C' = 0 \end{aligned}$$

for all s, and the proposition is proved. ∎

Next we examine the necessary and sufficient conditions for existence of the solution. The solution exists if and only if we can factorize the spectrum (4). The reconstructed continuous-time spectrum is given by

$$\begin{aligned} \Phi_c(s) &= -C(sI - A)^{-1}\{AP + PA'\}(-sI - A')^{-1}C'\\ &= CP(-sI - A')^{-1}C' + C(sI - A)^{-1}PC'. \end{aligned}$$

The function $\Phi c(s)$ admits a stable minimum phase spectral factor if and only if it is positive definite on the imaginary axis. The positivity condition can be given using Kalman Yakubovitz Popov lemma (Rantzer, 1996; Boyd *et al.*, 1994), which requires the existence of a symmetric matrix S such that

$$\begin{bmatrix} AS + SA' & (S + P)C' \\ C(S + P) & 0 \end{bmatrix} > 0 \tag{11}$$

In this section we explore the possibility of modifying the discrete-time estimation procedure such that we can *always* find a solution to the continuous-time problem. The idea is to use the constraint (11) in the discrete-time identification algorithm. Here we adopt the framework proposed in (Mari *et al.*, 2000), see also (Stoica *et al.*, 2000).

3. AN ESTIMATION ALGORITHM

In the previous section we examined the uniqueness and existence aspects of the reconstructed continuous-time spectrum from an estimated discrete-time spectrum. However, in a practical scenario, there is no guarantee that we shall be able to find a solution. In other words, often we arrive at a discrete-time solution when it is not possible to find S satisfying (11). Here we assume that the matrices e^{Ah} and C can be estimated from the data accurately from the data using a suitable method, see (Mari *et al.*, 2000) and the references therein. The idea is to estimate the positive real spectrum of the equivalent discrete-time process such that the associated continuous-time spectrum is positive definite on the imaginary axis. The positive real spectrum $f(s)$ admits a state-space description

$$f(z) = D + C(zI - e^{Ah})^{-1}B \tag{12}$$

Therefore the remaining unknowns are B and D. When we solve them we can reconstruct the equivalent discrete-time spectrum

$$\Phi_d(z) = f(z) + f'(z^{-1})$$
$$= \begin{bmatrix} (zI - \mathrm{e}^{A'h})^{-1}C' \\ I \end{bmatrix}' \begin{bmatrix} 0 & B \\ B' & D+D' \end{bmatrix} \times \begin{bmatrix} (z^{-1}I - \mathrm{e}^{Ah})^{-1}C' \\ I \end{bmatrix}$$
$$= \begin{bmatrix} (zI - \mathrm{e}^{A'h})^{-1}C' \\ I \end{bmatrix}' \times \begin{bmatrix} T - \mathrm{e}^{Ah}T\mathrm{e}^{A'h} & B - \mathrm{e}^{Ah}TC' \\ B' - CT\mathrm{e}^{A'h} & D+D'-CTC' \end{bmatrix} \times \begin{bmatrix} (z^{-1}I - \mathrm{e}^{Ah})^{-1}C' \\ I \end{bmatrix},$$

for any symmetric matrix T. The details of the calculations leading to the last equality are not shown here. However, this is easy to verify using a calculation similar to (6), see also (Genin *et al.*, 1999*b*; Genin *et al.*, 1999*a*) for a related discussion.

Recall that the discrete-time equivalent spectrum is strictly proper. Therefore, comparing with (5) we have

$$R_d = T - \mathrm{e}^{Ah}T\mathrm{e}^{A'h} \quad (13)$$
$$B = \mathrm{e}^{Ah}TC', \qquad D + D' = CTC'. \quad (14)$$

As we mentioned before, T is non-unique. If $T_1C' = 0$, then we can replace T in above by $T + T_1$. Hwoever, $\Phi_d(z)$ must be positive definite on the unit circle. Therefore, according to KYP Lemma, B and D must be such that we can find a symmetric matrix T which forces

$$\begin{bmatrix} T - \mathrm{e}^{Ah}T\mathrm{e}^{A'h} & B - \mathrm{e}^{Ah}TC' \\ B' - CT\mathrm{e}^{A'h} & D+D'-CTC' \end{bmatrix} \geq 0 \quad (15)$$

and (14) together. If we are given a combination of such T, B and D, then we can compute R_d as in (13). Note that our main goal is to identify the underlying continuous-time spectral density. Therefore, Proposition 1 enables us to use R_d in (13) to reconstruct the continuous-time spectrum using the the reconstrction algorithm described in Section 1. In that case the reconstructed continuous-time spectrum becomes

$$\Phi_c(s) := C(sI - A)^{-1}(-AT - TA')(-sI - A')^{-1}C'.$$

Immediately, we have an additional requirement on T imposed by (11) with P replaced by T. The observations are summarized in the following theorem.

Theorem 1. There exists a continuous-time counterpart of the equivalent discrete-time positive real spectrum in (12) if and only if one can find symmetric matrices T and S such that (14), (15) and

$$AS + SA' > 0, \qquad C(S+T) = 0$$

hold.

Proof: The proof is given above. ■

The condition $C(S+T) = 0$ is equivalent to $S + T = C_\perp X$ for some "fat" matrix X. Moreover, $S + T$ is symmetric. Hence we have $CX'C_\perp' = 0$. Since $C_\perp$ has a full column rank, this implies that

$$S + T = C_\perp \Omega C_\perp', \qquad \Omega = \Omega'.$$

Then the above conditions can be condensed as

$$AC_\perp \Omega C_\perp' + C_\perp \Omega C_\perp' A' > AT + TA'.$$

for some symmetric matrix Ω (which is of a size lower than that of S).

Theorem 1 can be seen as a necessary and sufficient condition on A, B, C and D so that the positive-real spectrum defined by (12) has an associated continuous-time conterpart. This condition must be incorporated in the estimation algorithm. In what follows next we use the results to describe an estimation algorithm for identification of SISO CARMA processes. The procedure can be generalized easily for multivariable situation. However, we restrict our discussion to SISO situation for simplicity. Let us denote the covariances of the observed process as

$$r_\tau = \mathcal{E}\{y(t+\tau h)y'(t)\},$$

which can be estimated from the data in the standard way. Then it is wellknown that

$$r_0 = D + D', \qquad r_\tau = Ce^{Ah(\tau-1)}B.$$

In the follwing we use $\hat{r}_\tau$ to denote the estimate of r_τ. Now imposing the equality constraints (14) we can eliminate B and D:

$$r_\tau = C\mathrm{e}^{Ah\tau}TC', \qquad \Rightarrow \boldsymbol{r}_q = O_qTC', \quad (16)$$

where we assume that we estimate covariances upto lag $q-1$ and

$$\boldsymbol{r}_q = [\, r_0 \; r_1 \; \cdots \; r_q \,]'$$
$$O_q = [\, C' \; e^{A'h}C' \; \cdots \; e^{A'h(q-1)}C' \,]'$$

Here we choose to work in observer canonical form (Kailath, 1980, Page 107), where we choose e^{Ah} and C in the following form:

$$C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}', \qquad \mathrm{e}^{Ah} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\gamma_d & -\gamma_{d-1} & \cdots & -\gamma_1 \end{bmatrix}.$$

Clearly, d is the order of the process and $\{\gamma_k\}_{k=1}^d$ are the coefficients of the characteristic polynomial of e^{Ah}. Now from (16) we can verify using Caley-Hamilton theorem that

$$\gamma_d r_\tau + \gamma_d r_{\tau+1} + \cdots + \gamma_1 r_{\tau+d-1} + r_{\tau+d} = 0, \quad (17)$$

for all $\tau \geq 0$. Provided we have $q \geq 2n$ we can set up a linear regrassion problem by considering

(17) for $\tau = 0, 1, \ldots, q-d$, and replacing the covariances $\{r_\tau\}_{\tau=0}^{q}$ by corresponding estimates $\{\hat{r}_\tau\}_{\tau=0}^{q}$. For the estimate of e^{Ah} we can estimate A using matrix logarithm. Note that for the simplified SISO case there is no need to estimate C. For the mutivariable case, however, we have to estimate C and the associated procedure is bit more complicated. On the other hand estimating e^{Ah} and C for mutivariable problem can be carried out by any of the standard procedure, see for example (Mari *et al.*, 2000; Van Overschee and De Moor, 1993), and the reference therein. For single variate data we present a straightforward approach shortly.

Now we are ready to set up a semidefinite programming problem to estimate T. Here we solve the problem

$$\begin{aligned} &\text{minimize} \quad \ell \\ &\text{subject to} \quad T - e^{Ah} T e^{A'h} > 0 \\ &\qquad AC_\perp \Omega C_\perp' + C_\perp \Omega C_\perp' A' > AT + TA' \\ &\qquad \begin{bmatrix} \ell & \boldsymbol{r}_q' - CTO_q' \\ \boldsymbol{r}_q - O_q TC' & W \end{bmatrix} > 0 \end{aligned}$$

for a suitable known positive definite weighting matrix W.

4. NUMERICAL SIMULATION RESULTS

The algorithm porposed in the previous section is tested in a numerical simulation study. We consider a scalar continuous-time stochastic process with a spectrum

$$\Phi_c(s) = \frac{c(s)c(-s)}{a(s)a(-s)},$$

where

$$a(s) = s^3 + 0.3s^2 + 9s + 0.9, \qquad c(s) = s^2 + 0.5s + 6.$$

The correlation function of the chosen process has an occilatory behaviour and a large time constant. To obtain a reliable estimate of such a process it is generally required to have a large observation time. Here we work with a data of 500 sec length, sampled at a frequency of 20 Hz. We estimate the correlation function upto 5 sec from the data, and use it in our estimation routine.

The estimation results obtained from 100 Monte-Carlo simulations are shown in Figure 1, where the true spectrum is compared with estimated mean value ± standard deviation. As can be seen in Figure 1, the estimated spectrum is unbiased and is accurate. We point out here that the weighting matrix W in this case is chosen as an identity matrix. In estimating the matrix lagarithm there are possibilities of numerical problem, and if e^{Ah} has eigenvalues close to each other, this problem

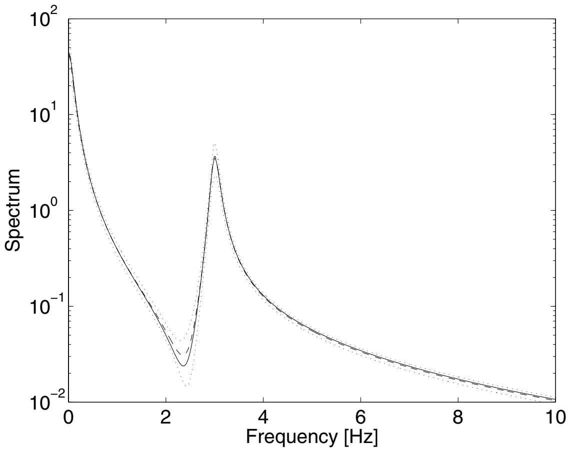

Fig. 1. Comparison of the mean of estimated spectrum (dashed line) and the true spectrum (solid line). The mean ± standard deviation of the estimated spectrum is shown in dotted lines.

is severe. In order to reduce the numerical the estimated correlation function (sampled at 20 Hz) is decimated and the down sampling is done at $2Hz$. The estimation of $\{\gamma_k\}_{k=1}^{d}$ in (17) is done with the decimated correlation function data at 2 Hz in 10 parallel channels. The estimation of T use the correlation data at a sampling interval of 0.5 sec upto 10 sec [2] . In fact it is natural to expect significant improvement in the accuracy if we use the whole correlation data and proper weighting matrix W, which is a part of the ongoing research. Finally we give the parametric estimation results in Table 1. Here the a_k and c_k denote coefficients of s^{d-k} (recall that $d = 3$ in our example) of the polynomials $a(s)$ and $c(s)$, respectively.

Parameter	True vale	Mean	Std Deviation
a_1	0.3	0.3103	0.0492
a_2	9.0	9.0179	0.1283
a_3	0.9	0.9823	0.2255
c_1	1.0	0.9855	0.0443
c_2	0.5	0.5310	0.1972
c_3	6.0	5.9908	0.4065

Table 1. Parameter estimation performance.

5. CONCLUSIONS

In this paper we have addressed the problem of reconstructing continuous-time model from an equivalent discrete-time model. A necessary and sufficient condition for existence of the solution is given. It is also shown that the solution is unique if that exists. Using the theory developped we have shown a way to identify the continuous-time model directly from the covariance data. In our

[2] This is done to reduce the computational burden.

approach it is not required to compute the underlying discrete-time model. We estimate the state covariance matrix, which is the common property of both the continuous-time and the discrete-time model. Using the proposed algorithm it is guaranteed to get a solution and the amount of computation involved is mild. The numerical simulation results confirm that the proposed method is capable to delivering accurate and reliable result. The statistical properties of the proposed method is not considered in this paper, and is a part of the ongoing research.

6. REFERENCES

Åström, K. J. (1970). *Introduction to Stochastic Control*. Academic Press. New York, NY.

Boyd, S., L. E. Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM.

Genin, Y., Y. Nesterov and P. Van Dooren (1999*a*). The analytic center of LMI's and Riccati equations. Paper F1018.

Genin, Y., Y. Nesterov and P. Van Dooren (1999*b*). Positive transfer functions and convex optimization. In: *European Control Conference*. Paper F143.

Kailath, T. (1980). *Linear Systems*. Prentice Hall.

Larsson, E. and M. Mossberg (2003). On possibilities for estimating continuous-time arma parameters. In: *IFAC Symposium on System Identification*. Rotterdam, The Netherlands.

Mari, J., P. Stoica and T. McKelvey (2000). Vector arma estimation: a reliable subspace approach. *IEEE Transactions on Signal Processing* **48:7**, 2092–2104.

Rantzer, A. (1996). On the Kalman-Yakubovitz-Popov lemma. *Systems & Control Letters* **28**, 7–10.

Söderström, T (1990). On zero locations for sampled stochastic systems. *IEEE Transactions on Automatic Control* **AC-35**, 1249–1253.

Söderström, T. (1991). Computing stochastic continuous-time models from ARMA models. *International Journal of Control* **53**, 1311–1326.

Söderström, T. (2002). *Discrete-time Stochastic Systems, 2 ed*. Springer-Verlag. London.

Stoica, P., T. McKelvey and J. Mari (2000). MA estimation in polynomial time. *IEEE Transactions on Signal Processing* **48:7**, 1999–2012.

Van Overschee, P. and B. De Moor (1993). Subspace algorithms for the stochastic identification problem. *Automatica* **29:3**, 649–660.

Wahlberg, B. (1988). Limit results for sampled systems. *International Journal of Control* **48**, 1267–1283.