Robust $\mathcal{H}_\infty$ Filtering for Continuous Time Varying Uncertain Systems with Deterministic Input Signals

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Abstract—**Many dynamical systems involve not only process and measurement noise signals but also parameter uncertainty and known input signals. When $\mathcal{L}_2$ or $\mathcal{H}_\infty$ filters that were designed based on a "nominal" model of the system are applied, the presence of parameter uncertainty will not only affect the noise attenuation property of the filter but also introduce a bias proportional to the known input signal, and the latter may be very appreciable. In this paper, we introduce a finite-horizon robust $\mathcal{H}_\infty$ filtering method that provides a guaranteed $\mathcal{H}_\infty$ bound for the estimation error in the presence of both parameter uncertainty and a known input signal. This method is developed by using a game-theoretic approach, and the results generalize those obtained for cases without parameter uncertainty or without a known input signal. It is also demonstrated, via an example, that the proposed method provides significantly improved signal estimates.**

I. Introduction

In contrast with the conventional $\mathcal{L}_2$ estimation algorithms that minimize the variance of the estimation error (see, e.g., [1]), recent advancement in signal estimation has focused on the development of $\mathcal{H}_\infty$ estimation methods ([2]–[9], [11]–[18]) that aim at minimizing the peak of the spectral density of the estimation error. The motivation for the $\mathcal{H}_\infty$ approach is that the statistical assumptions and information on the noise sources are often inaccurate, or unavailable, and also that the $\mathcal{L}_2$ estimation methods are sensitive to parameter variations in the signal process; see [14] for a survey of the $\mathcal{H}_\infty$ estimation algorithms.

As in the $\mathcal{L}_2$ case, $\mathcal{H}_\infty$ filters are usually designed based on a "nominal" model of the signal process. For this reason, we shall refer to them as "nominal" $\mathcal{H}_\infty$ filters. Although the "nominal" $\mathcal{H}_\infty$ filter has been shown to be less sensitive to parameter variations in the signal process than the corresponding $\mathcal{L}_2$ filter (see, e.g., [13] and Section II), no guaranteed performance is provided when the true signal process deviates from the assumed model.

To solve the above problem, a robust $\mathcal{H}_\infty$ filtering method has been developed in [6], [15], and [16] to guarantee a prescribed $\mathcal{H}_\infty$ performance in the presence of parameter uncertainty. It will be demonstrated, via an example, in Section II that such a robust $\mathcal{H}_\infty$ filter far outperforms the corresponding "nominal" $\mathcal{H}_\infty$ and the $\mathcal{L}_2$ filters.

The focal point of this paper is to address the robust $\mathcal{H}_\infty$ filtering problem for signal processes with both parameter uncertainty and a known input signal. Note that if the signal process does not involve parameter uncertainties, it is well known that for both $\mathcal{L}_2$ and $\mathcal{H}_\infty$ filters the existence of a known input does not affect the estimation error, i.e., the contribution of the known input signal can be completely cancelled (see, e.g., [1], [11]). This significant feature is, however, no longer valid in the presence of uncertain parameters. As a result, the estimation error will have, in general, components due to both the process and the measurement noise signals and the known system input. In Section II, we shall show, via an example, that the second component may be far more appreciable than the first one, when the filter is designed based on the nominal values of the parameters.

In this paper, we generalize the robust $\mathcal{H}_\infty$ filtering approach of [6], [15], and [16] to cope with the case where the process has a known input signal. The goal of the filter is to provide a uniformly small estimation error for any process and measurement noise signals and for any initial state in the presence of parameter uncertainty and a known input signal. The problem will be solved in the finite-horizon setting. As in [6] and [16], one of the key ideas is to convert the parameter uncertainty into a fictitious $\mathcal{L}_2$ noise signal and to formulate an auxiliary problem that does not involve any parameter uncertainty. It will be shown that the solution to the auxiliary problem, if it exists, guarantees, when applied to the original problem, a prescribed performance in an $\mathcal{H}_\infty$ sense. A game-theoretic approach is used to solve the auxiliary problem, which gives a solution in terms of Riccati differential equations. Three types of input signals are considered: causal, causal with known average, and noncausal. Causal signals are those that can be measured but not predicted, whereas noncausal signals are known *a priori*. Different filters are derived for the three cases.

The results of this paper will be demonstrated in an example that illustrates the significant improvement that can be achieved in signal estimation with the new technique.

It should be noted that the filtering methodology of this paper resolves one of the major difficulties that was encountered with $\mathcal{H}_\infty$ filtering, namely, the filter design in presence of measurable disturbances. This difficulty is easily resolved by the theory of this paper by considering the measured part of the disturbances as a known input signal.

Manuscript received March 18, 1993; revised February 15, 1994. This work was supported by the Australian Research Council and the C. & M. Maus Chair of Computer Systems Engineering at Tel-Aviv University, Israel. The associate editor coordinating the review of this paper and approving it for publication was Prof. Daniel Fuhrmann.

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IEEE Log Number 9408228.

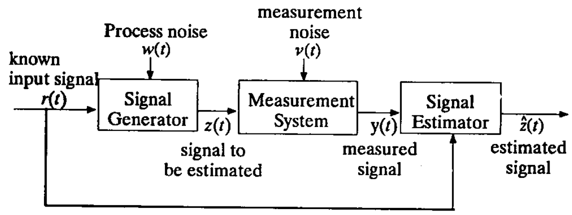

Fig. 1. Signal-generating mechanism.

II. MOTIVATION

To motivate the robust $\mathcal{H}_\infty$ filtering problem that is studied in this paper, we show via an example that filter designs that do not take into account parameter uncertainty may render a very poor signal estimate.

Consider the signal-generating system in Fig. 1 with the following mathematical model:

$$\dot{x} = \begin{bmatrix} 0 & -1+\delta \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} -0.4545 \\ 0.9090 \end{bmatrix} w + \begin{bmatrix} g \\ 0 \end{bmatrix} r \quad (2.1)$$

$$y = [0 \quad 100]x + v \quad (2.2)$$

$$z = [0 \quad 100]x \quad (2.3)$$

where x is the state, w is the process noise, r is a known deterministic input signal, y is the measurement, v is the measurement noise, z is the signal to be estimated, δ represents an uncertain parameter that satisfies $|\delta| \le \bar{\delta} = 0.3$, and g is a known input gain to be specified later.

Both infinite horizon Kalman and "nominal" $\mathcal{H}_\infty$ filters are designed for the nominal system that has been chosen to correspond to $\delta = 0$. These filters are of the form

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -0.5 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} g \\ 0 \end{bmatrix} r(t) + K[y(t) - M\hat{x}(t)]$$
$$\hat{z}(t) = [0 \quad 100]\hat{x}(t)$$

with $M = [0 \quad 100]$, where $\hat{z}(t)$ is the estimate of $z(t)$ and K is the filter gain. For the Kalman filter design, the noise sources $w(t)$ and $v(t)$ were assumed to be uncorrelated, zero-mean, white signals with unit power spectrum density. The corresponding filter gain is given by

$$K = K_K = [0.447 \quad 0.909]^T.$$

For the "nominal" $\mathcal{H}_\infty$ filter design, we take $\gamma = 1.1 = 0.8$ dB and design the filter to achieve

$$\|e\|_2 < \gamma\left[\|w\|_2^2 + \|v\|_2^2\right]^{\frac{1}{2}}$$

where $e(t)$ denotes the estimation error $z(t) - \hat{z}(t)$. This yields

$$K = K_\infty = [1.035 \quad 2.181]^T.$$

In the above, $\|\cdot\|_2$ denotes the norm in $\mathcal{L}_2[0,\infty)$. We then apply the two filters to the perturbed plant of (2.1)–(2.3), with $\delta = -\bar{\delta}$ and $\delta = \bar{\delta}$. The frequency response magnitude of the transfer functions from $[w(t) \quad v(t)]^T$ and $r(t)$ to $e(t)$, denoted by $[G_{ew}(s) \quad G_{ev}(s)]$ and $G_{er}(s)$, respectively, are shown in Figs. 2 and 3 for both filters. From these figures, we make the following observations:

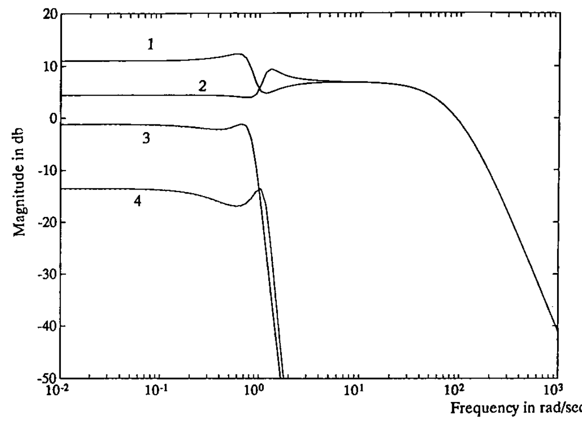

Fig. 2. Kalman filter for system (2.1)–(2.3). Curve 1: 10 $\log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$ for $\delta = 0.3$, curve 2: 10 $\log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$ for $\delta = -0.3$, curve 3: 20 $\log(\|G_{er} j\omega)\|)$ for $\delta = 0.3$, $g = 1$, curve 4: 20 $\log(\|G_{er}(j\omega)\|)$ for $\delta = -0.3$, $g = 1$.

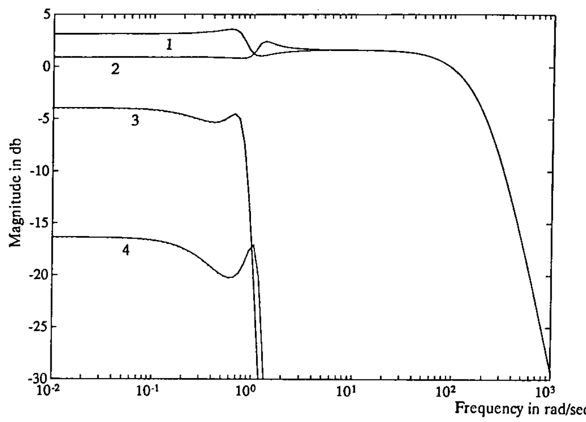

Fig. 3. Nominal $\mathcal{H}_\infty$ filter for system (2.1)–(2.3). Curve 1: 10 $\log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$ for $\delta = 0.3$, curve 2: 10 $\log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$ for $\delta = -0.3$, curve 3: 20 $\log(\|G_{er}(j\omega)\|)$ for $\delta = 0.3$, $g = 1$, curve 4: 20 $\log(\|G_{er}(j\omega)\|)$ for $\delta = -0.3$, $g = 1$.

1) The magnitude of $[G_{ew}(j\omega) \quad G_{ev}(j\omega)]$ and $G_{er}(j\omega)$ are worsened for both designs when the parameter uncertainty exists (note that $G_{er}(s) \equiv 0$ when there is no parameter uncertainty).
2) The magnitude of $G_{er}(j\omega)$, which is identically zero in the absence of parametric uncertainty, may be far more significant than that of $[G_{ew}(j\omega) \; G_{ev}(j\omega)]$ for both designs, even in the case of a moderate r.
3) The Kalman filter is more sensitive to changes in the parameter δ than the "nominal" $\mathcal{H}_\infty$ filter.

Based on the above observations, there is a need to consider the uncertainty in the design procedure, and a more robust filter design methodology needs to be developed.

For the case where there is no deterministic input signal, i.e., $r(t) \equiv 0$, a robust $\mathcal{H}_\infty$ filtering theory has been developed to cope with parameter uncertainty (see [6], [15], and [16]). We

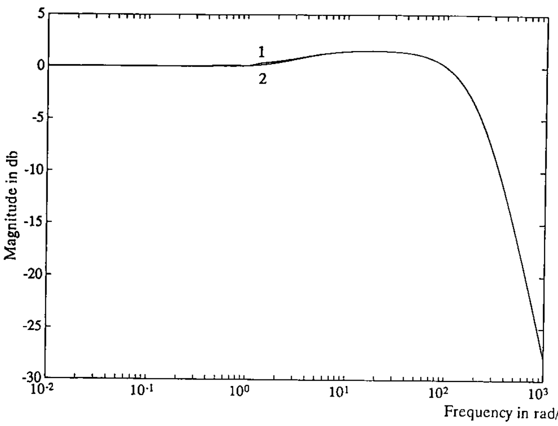

Fig. 4. Robust $\mathcal{H}_\infty$ filter of [6] for system (2.1)–(2.3). Curve 1: 10 $\log(\|G_{ew}(j\omega)\|^2+\|G_{ev}(j\omega)\|^2)$ for $\delta = 0.3$, curve 2: 10 $\log(\|G_{ew}(j\omega)\|^2 + \|G_{ev}(j\omega)\|^2)$ for δ = -0.3.

now show, by using the above example, that this theory yields signal estimates that are much more robust than those achieved by the Kalman and the "nominal" $\mathcal{H}_\infty$ filters. Applying the results of [6] to this example, the resulting robust $\mathcal{H}_\infty$ filter is given by

$$\dot{\hat{x}}(t) = \begin{bmatrix} -0.505 & -1.117 \\ -0.850 & -0.535 \end{bmatrix} \hat{x}(t) + K_r[y(t) - M\hat{x}(t)] \tag{2.4}$$

$$\hat{z}(t) = [0.077 \quad 100.018]\hat{x}(t) \tag{2.5}$$

$$K_r = [7.938 \quad 2.354]. \tag{2.6}$$

The corresponding magnitude of $[G_{ew}(j\omega)\ G_{ev}(j\omega)]$ is shown in Fig. 4 for $\delta = 0$, $-\bar{\delta}$, and $\bar{\delta}$. Note that this result is much more robust than those in Figs. 2 and 3.

The above results demonstrate that it is crucial to take the parameter uncertainty into account in the design of the filter. The methods in [6], [15], and [16], however, are not readily applicable to systems that have a known deterministic input signal. The focal point of this paper is to generalize the theory of [6], [15], and [16] to solve the robust $\mathcal{H}_\infty$ filtering problem with such a known input signal.

III. Problem Formulation and a Key Lemma

Consider the uncertain linear system, described by

$$(\Sigma): \quad \dot{x}(t) = [A(t) + \Delta A(t)]x(t) + B_1(t)w(t) + B_2(t)r(t), \quad x(0) = x_0 \tag{3.1}$$

$$y(t) = [C(t) + \Delta C(t)]x(t) + v(t) \tag{3.2}$$

$$z(t) = L(t)x(t) \tag{3.3}$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^p$ is the process noise, $r(t) \in \Re^r$ is a known deterministic input signal, $y(t) \in \Re^m$ is the measurement, $v(t) \in \Re^m$ is the measurement noise, and $z(t) \in \Re^q$ is the signal to be estimated. $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$, and $L(t)$ are known real bounded piecewise continuous matrix functions that describe the nominal system, and $\Delta A(t)$ and $\Delta C(t)$ represent parameter uncertainties in the matrices $A(t)$ and $C(t)$, respectively. The admissible uncertainties are assumed to be of the form

$$\Delta A(t) = H_1(t)F(t)E(t); \quad \Delta C(t) = H_2(t)F(t)E(t) \tag{3.4}$$

where $F(t) \in \Re^{i \times j}$ is an unknown matrix function with Lebesgue measurable elements satisfying

$$\|F(t)\| \le 1, \quad \forall t \tag{3.5}$$

where $\|X\|$ denotes, for a matrix X, the maximum singular value. $H_1(t)$, $H_2(t)$, and $E(t)$ are known bounded piecewise continuous matrix functions of appropriate dimensions that specify how the uncertain parameters in $F(t)$ enter the nominal matrices $A(t)$ and $C(t)$. Note that the uncertain matrix $F(t)$ is allowed to depend on the state, as long as (3.5) is satisfied. For the sake of notation simplification, we shall omit in the sequel the dependence on t in the matrices when there is no confusion.

In this paper, we are concerned with obtaining an estimate $\hat{z}(t)$ of $z(t)$ over the horizon $[0, T]$, using the measurement history $\{y(\tau),\ 0 \le \tau \le t\}$ and the known input signal, $r(\cdot)$. The filter is required to provide a uniformly small filtering error, $e(t) = z(t) - \hat{z}(t)$, for any w and v in $\mathcal{L}_2\ [0, T]$ and $x_0 \in \Re^n$, and for all admissible uncertainties. We shall consider the following performance index

$$J(w, v, x_0, r, \hat{z}) = \|z - \hat{z}\|_2^2 - \gamma^2 [\|w\|_2^2 + \|v\|_2^2 + \|x_0 - \hat{x}_0\|_R^2] \tag{3.6}$$

where $\gamma > 0$ is a given scalar that indicates the level of "noise" attenuation, $\hat{x}_0$ is an *a priori* estimate of x_0, and $R = R^T > 0$ is a given weighting matrix that reflects the confidence in the estimate $\hat{x}_0$. In the above, $\|x\|_A^2$ denotes $x^T Ax$ and $\|v\|_2$ means the $\mathcal{L}_2[0, T]$-norm defined as $\|v\|_2^2 = \int_0^T v^T v dt$. Also, $M > N$ (respectively, $M \ge N$), where M and N are symmetric matrices, means that $M - N$ is a positive definite (respectively, semi-definite) matrix.

The admissible filter is assumed to be of the form

$$\hat{z} = \mathcal{G}_y y + \mathcal{G}_r r + \mathcal{G}_x \hat{x}_0$$

where $\mathcal{G}_y$, $\mathcal{G}_r$, and $\mathcal{G}_x$ are dynamic operators. The operator $\mathcal{G}_y$ is causal, whereas $\mathcal{G}_r$ can be either causal or noncausal, depending on whether the exogenous signal $r(\cdot)$ is, respectively, measured online or known *a priori*. It should be noted that we do not *a priori* restrict the admissible filter to be linear. Indeed, $\mathcal{G}_y$, $\mathcal{G}_r$, and $\mathcal{G}_x$ are allowed to be nonlinear and possibly time-varying operators.

The robust filtering problem for the system (Σ) consists of finding an estimate $\hat{z}(t)$, $\forall t \in [0, T]$, which minimizes

$$\sup_{w, v, x_0, F(t)} \{J(w, v, x_0, r, \hat{z})\}. \tag{3.7}$$

An optimal solution $\hat{z}^*(\cdot)$ will guarantee that for all admissible uncertainties

$$\|z - \hat{z}^*\|_2^2 \le \gamma^2 [\|w\|_2^2 + \|v\|_2^2 + \|x - \hat{x}_0\|_R^2] + J^*(r, \hat{x}_0), \quad \forall w, v \in \mathcal{L}_2[0, T], \quad \forall x_0 \in \Re^n$$

where $J^*(r, \hat{x}_0)$ denotes the minimum value of (3.7). The above filtering problem, which is indeed a worst-case filter design, will be referred to as *robust $\mathcal{H}_\infty$ filtering*.

We shall investigate the above filtering problem for three different patterns of information for $r(\cdot)$. Define the history up to time $\bar{t}$ of the measurement y and the signal r by

$$\mathcal{Y}_{\bar{t}} = \{y(\tau), \quad 0 \le \tau \le \bar{t}\}; \qquad \mathcal{R}_{\bar{t}} = \{r(\tau), \quad 0 \le \tau \le \bar{t}\}.$$

The problems we shall consider are as follows:

P.1) Filtering with a Noncausal Signal $r(\cdot)$: The signal $r(\cdot)$ is assumed to be known *a priori* for the whole horizon $[0, T]$. The problem is to find an estimate $\hat{z}(t)$, $\forall t \in [0, T]$, based on x_0, $\mathcal{Y}_t$, and $\mathcal{R}_T$, such that (3.7) is minimized.

P.2) Filtering with a Causal Signal $r(\cdot)$: The signal $r(\cdot)$ is measured online but cannot be predicted. The problem is to find an estimate $\hat{z}(t)$, $\forall t \in [0, T]$, based on x_0, $\mathcal{Y}_t$, and $\mathcal{R}_t$, such that (3.7) is minimized.

P.3) Filtering with a Causal Signal $r(\cdot)$ of Known Average: The signal $r(\cdot)$ is measured online and is of the form

$$r(t) = \bar{r}(t) + r_d(t), \quad \forall t \in [0, T] \tag{3.8}$$

where $\bar{r}(\cdot)$ is known *a priori* over the horizon $[0, T]$. The problem is to find an estimate $\hat{z}(t)$, $\forall t \in [0, T]$, based on $\hat{x}_0$, $\mathcal{Y}_t$, and $\{\mathcal{R}_t, \bar{\mathcal{R}}_T\}$, such that (3.7) is minimized where $\bar{\mathcal{R}}_T$ denotes the time history $\{\bar{r}(\tau), 0 \le \tau \le T\}$.

Remark 3.1: We observe that filtering problems with exogenous signals as in (P.1)–(P.3) arise in a number of practical situations. A solution to (P.1) and (P.2) resolves the problem of $\mathcal{H}_\infty$ filtering in the presence of, respectively, known and measured deterministic disturbances, and parametric uncertainty. The motivation for (P.3) stems from practical filtering problems with an exogenous signal r that is measured online and where one does not know the future of r but its average component $\bar{r}$ in the future is known.

Remark 3.2: Note that in the filtering problem (P.2) we have not assumed an *a priori* knowledge of a model that produces the signal r. We observe that when a latter model is known, it may be incorporated into the filter design by augmenting the system (Σ) to include this model. However, this *a priori* knowledge is, in many cases, inaccurate and hardly available. Moreover, it seems strange to assume anything on a signal that can be actually measured.

The key idea used here in order to guarantee robust performance is to convert the uncertainties to fictitious noise source and to solve an auxiliary filtering problem that does not involve parameter uncertainty. The performance index for this auxiliary problem, if solvable, yields an upper bound for the worst-case performance index of (3.7). Justification for this technique is provided in the sequel.

Introduce the following auxiliary system:

$$(\Sigma_a): \quad \dot{\eta}(t) = A\eta(t) + \begin{bmatrix} B_1 & \frac{\gamma}{\varepsilon} H_1 \end{bmatrix} \tilde{w}(t) + B_2(t) r(t), \quad \eta(0) = \eta_0 \tag{3.9}$$

$$y_a(t) = C_\eta(t) + \begin{bmatrix} 0 & \frac{\gamma}{\varepsilon} H_2 \end{bmatrix} \tilde{w}(t) + \tilde{v}(t) \tag{3.10}$$

$$z_a(t) = \begin{bmatrix} L^T & \varepsilon E^T \end{bmatrix}^T \eta(t) \tag{3.11}$$

where $\eta(t) \in \Re^n$ is the state, η_0 is an unknown initial state, $\tilde{w}(t) \in \Re^{p+i}$ and $\tilde{v}(t) \in \Re^m$ are noise signals, $y_a(t) \in \Re^m$ is the measurement, $z_a(t) \in \mathcal{R}^{q+j}$ is the signal to be estimated, r, A , B_1, B_2, C, E, H_1, H_2, and L are as in (3.1)–(3.4), and $\varepsilon > 0$ is a scaling parameter to be chosen. Associated with the system (Σ_a) we introduce an estimate for $z_a(t), \forall t \in [0, T]$, of the form

$$\hat{z}_a(t) = [\hat{z}^T(t) \quad 0]^T \tag{3.12}$$

where $\hat{z}(t)$ is an estimate of $L\eta(t)$ using the measurements $\{y_a(\tau), \quad 0 \le \tau \le t\}$ and the input signal r. Next, we define the following performance index for the estimate $\hat{z}_a(t)$:

$$J_a(\tilde{w}, \tilde{v}, \eta_0, r, \hat{z}, \varepsilon) = \|z_a - \hat{z}_a\|_2^2 - \gamma^2 \left[\|\tilde{w}\|_2^2 + \|\tilde{v}\|_2^2 + \|\eta_0 - \hat{x}_0\|_R^2\right] \tag{3.13}$$

where γ, $\hat{x}_0$, and R are as in (3.6).

Hence, we have the following result:

Lemma 3.1: Consider the systems (Σ) and (Σ_a) together with the performance indices (3.6) and (3.13), respectively. Then we have that for any $\varepsilon > 0$

$$\sup_{w,v,x_0,F(t)} \{J(w, v, x_0, r, \hat{z})\} \le \sup_{\tilde{w},\tilde{v},\eta_0} \{J_a(\tilde{w}, \tilde{v}, \eta_0, r, \hat{z}, \varepsilon)\}.$$

Proof: For any given x_0, F, w, v, r, and $\hat{z}$ for the system (Σ) and any $\varepsilon > 0$, take

$$\eta_0 = x_0, \quad \tilde{w}(t) = \begin{bmatrix} w(t) \\ \varepsilon\gamma^{-1} F E x(t) \end{bmatrix}, \quad \tilde{v}(t) = v(t). \tag{3.14}$$

Then, for all $t \in [0, T]$, we have

$$\eta(t) = x(t), \quad y_a(t) = y(t), \quad z_a(t) = \begin{bmatrix} z(t) \\ \varepsilon E x(t) \end{bmatrix}$$

which implies

$$J_a(\tilde{w}, \tilde{v}, \eta_0, r, \hat{z}, \varepsilon) = \|z - \hat{z}\|_2^2 - \gamma^2 \left[\|w\|_2^2 + \|v\|_2^2 + \|x_0 - \hat{x}_0\|_R^2\right] + \varepsilon^2 \left[\|Ex\|_2^2 - \|FEx\|_2^2\right].$$

Now, considering (3.5) with η_0, $\tilde{w}$, and $\tilde{v}$, as in (3.14), we obtain that

$$J_a(\tilde{w}, \tilde{v}, \eta_0, r, \hat{z}, \varepsilon) \ge J(w, v, x_0, r, \hat{z})$$

and the result follows immediately. □

In view of Lemma 3.1, our approach for solving the robust $\mathcal{H}_\infty$ filtering problem involves consideration of the worst-case of the performance bound (3.13) in *lieu* of the worst-case performance (3.7). This leads to the following problem:

Find an estimate $\hat{z}_a(t), \forall t \in [0, T]$, of the form (3.12) using the measurements $\{y_a(\tau), \quad 0 \le \tau \le t\}$ and the known input signal, r, that solves the auxiliary problem

$$\min_{\hat{z}} \left\{ \sup_{\tilde{w},\tilde{v},\eta_0} [J_a(\tilde{w}, \tilde{v}, \eta_0, r, \hat{z}, \varepsilon)] \right\} \tag{3.15}$$

subject to (3.9)–(3.12). Note that the system (Σ_a) is parameterized by ε, which is a scaling parameter to be searched in order that an estimate, $\hat{z}$, solving (3.15) be found.

Remark 3.3: The above estimate $\hat{z}$, with y_a replaced by y, will provide an estimate of z for the robust $\mathcal{H}_\infty$ filtering problem. Note that the case of $r(t) \equiv 0$ has been analyzed in [6], [15], and [16], in both the continuous and the discrete-time contexts, and it has been shown there that the above estimate $\hat{z}$ guarantees the following $\mathcal{H}_\infty$ performance

$$\|z-\hat{z}\|_2^2 < \gamma^2\left[\|w\|_2^2 + \|v\|_2^2 + \|x - \hat{x}_0\|_R^2\right]$$

for any w and v in $\mathcal{L}_2\,[0,T]$ and $x_0 \in \Re^n$ and for all admissible uncertainties.

Remark 3.4: Although the auxiliary problem (3.15) does not have any parameter uncertainty, it cannot be treated via the standard $\mathcal{H}_\infty$ estimation techniques. This is because the estimate $\hat{z}_a(t)$ is restricted by (3.12). Therefore, an alternative solution is required.

IV. Main Results

In this section, a solution to the auxiliary filtering problem introduced in Section III will be developed using a game theoretic approach, where the estimator plays against adversaries composed of the noise sources and the initial state.

A. Game Theoretic Solution to the Auxiliary Problem

The deterministic linear-quadratic game problem is to find $\hat{z} \in \mathcal{L}_2t[0,T]$, worst-case noise signals, $\tilde{w}(\cdot)$, $\tilde{v}(\cdot) \in \mathcal{L}_2t[0,T]$ and worst-case initial state $\eta_0 \in \Re^n$ to achieve

$$\min_{\hat{z}} \max_{\tilde{w},\tilde{v},\eta_0} J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon) \tag{4.1}$$

subject to (3.9)–(3.11). As in the problems P.1)–P.3), the estimate $\hat{z}(t)$, $\forall t \in [0,T]$ is based on $\hat{x}_0$, $\mathcal{Y}_t$, and either $\mathcal{R}_T$, $\mathcal{R}_t$, or $\{\mathcal{R}_t, \mathcal{R}_T\}$, depending on the available information on the exogenous signal $r(\cdot)$ at time t.

In view of (3.10)–(3.13), the optimization problem (4.1) can be recast into the form

$$\min_{\hat{z}} \max_{\tilde{w},\tilde{v},\eta_0} \{\hat{J}_a = \|L\eta - \hat{z}\|_2^2 + \|\varepsilon E\eta\|_2^2 - \gamma^2[\|\tilde{w}\|_2^2 + \|y_a - C\eta - D_w\tilde{w}\|_2^2 + \|\eta_0 - \hat{x}_0\|_2^2]\} \tag{4.2}$$

where

$$B_w = \begin{bmatrix} 0 & \frac{\gamma}{\varepsilon}H_2\end{bmatrix}.$$

Inspired by [2], the above game will be solved in two stages. We consider first the maximization of J_a with respect to $\tilde{w}$ and η_0 for given $\hat{z}$ and y_a. Then, a min–max optimization of the resulting cost function will be performed with respect to $\hat{z}$ and y_a, respectively.

We first find the necessary conditions for optimality of η_0 and $\tilde{w}$, for given $\hat{z}$ and y_a. We shall later provide a sufficiency proof. To begin, we adjoin to the performance index J_a the constraint (3.9) using a Lagrange multiplier $2\gamma^2\lambda$, i.e., we consider the modified performance index

$$\begin{aligned}\hat{J}_a &= \|L\eta - \hat{z}\|_2^2 + \|\varepsilon E\eta\|_2^2 \\ &\quad - \gamma^2\left[\|\tilde{w}\|_2^2 + \|\eta_0 - \hat{x}_0\|_R^2 + \|y_a - C\eta - D_w\tilde{w}\|_2^2\right] \\ &\quad + 2\gamma^2\lambda^T\int_0^T(-\dot{\eta} + A\eta + B_w\tilde{w} + B_2r)dt\end{aligned}$$

where

$$B_w = \begin{bmatrix} B_1 & \frac{\gamma}{\varepsilon}H_1\end{bmatrix}.$$

By using standard optimization results, the maximizing strategies η_0^* and $\tilde{w}^*$ must satisfy the following:

$$\eta_0^* = \hat{x}_0 + R^{-1}\lambda(0) \tag{4.3}$$

$$\tilde{w}^* = \bar{D}\left[B_w^T\lambda + D_w^T(y_a - C\eta^*)\right] \tag{4.4}$$

$$\begin{aligned}\dot{\lambda} &= -A^T\lambda - \gamma^{-2}C_1^TC_1\eta^* - C^T(y_a - C\eta^*) \\ &\quad + C^TD_w\tilde{w}^* + \gamma^{-2}L^T\hat{z}, \quad \lambda(T) = 0\end{aligned} \tag{4.5}$$

where η^* is the solution of (3.9) with $\eta_0 = \eta_0^*$ and $w = \tilde{w}^*$, and

$$C_1^T = \begin{bmatrix} L^T & \varepsilon E^T\end{bmatrix}, \qquad \bar{D} = \left(I + D_w^TD_w\right)^{-1}. \tag{4.6}$$

Note that (3.9) together with (4.3)–(4.5) give rise to a linear two-point boundary value problem whose solution can be assumed in the following form:

$$\eta^*(t) = \hat{\eta}(t) + Q(t)\lambda(t) \tag{4.7}$$

where $\hat{\eta}$ and Q are to be determined.

Differentiating (4.7) and considering (3.9), (4.4), and (4.5), we obtain that $\hat{\eta}$ and Q satisfy

$$\begin{aligned}\dot{\hat{\eta}} &= (A + \gamma^{-2}QC_1^TC_1)\hat{\eta} + \bar{B}_1(y_a - C\hat{\eta}) \\ &\quad - \bar{B}_2\hat{z} + B_2r, \qquad \hat{\eta}(0) = \hat{x}_0\end{aligned} \tag{4.8}$$

$$\begin{aligned}\dot{Q} &= \tilde{A}Q + Q\tilde{A}^T + Q\left(\gamma^{-2}C_1^TC_1 - C^T\tilde{D}C\right)Q \\ &\quad + B_w\bar{D}B_w^T, \qquad Q(0) = R^{-1}\end{aligned} \tag{4.9}$$

where

$$\tilde{D} = I - D_w\bar{D}D_w^T, \qquad \tilde{A} = A - B_wD_w^T\tilde{D}C, \tag{4.10}$$

$$\bar{B}_1 = (QC^T + B_wD_w^T)\tilde{D}, \qquad \bar{B}_2 = \gamma^{-2}QL^T. \tag{4.11}$$

We note that $\tilde{D} = (I + D_wD_w^T)^{-1}$ and $\bar{D}D_w^T = D_w^T\tilde{D}$. Also observe that since $Q(0) > 0$, it follows by well-known results on Riccati differential equations (see, e.g., [10]) that if the solution $Q(t)$ to (4.9) exists over $[0,T]$, then $Q(t)$ is symmetric positive definite over $[0,T]$.

Substituting (4.3) and (4.4) in J_a, it is easy to show that J_a becomes

$$\begin{aligned}J_a &= \|L(\hat{\eta} + Q\lambda) - \hat{z}\|_2^2 \\ &\quad + \|\varepsilon E(\hat{\eta} + Q\lambda)\|_2^2 - \gamma^2\Big[\|\bar{D}^{-\frac{1}{2}}B_w^T\lambda\|_2^2 \\ &\quad + \|\tilde{D}^{\frac{1}{2}}[y_a - C(\hat{\eta} + Q\lambda)]\|_2^2 + \|\lambda(0)\|_{R^{-1}}^2\Big].\end{aligned} \tag{4.12}$$

In the sequel, we shall perform the min–max optimization of J_a with respect to $\hat{z}$ and y_a, respectively. Adding to (4.12) the identically zero term

$$\begin{aligned}0 &= \gamma^2\int_0^T\frac{d}{dt}[\lambda^TQ\lambda]dt + \gamma^2\|\lambda(0)\|_{R^{-1}}^2 \\ &= \gamma^2\Big\{-\gamma^{-2}\|C_1Q\lambda\|_2^2 + \|\tilde{D}^{\frac{1}{2}}CQ\lambda\|_2^2 + \|\bar{D}^{\frac{1}{2}}B_w^T\lambda\|_2^2 \\ &\quad + \int_0^T 2\lambda^TQ\left[\gamma^{-2}(L^T\hat{z} - C_1^TC_1\hat{\eta}) - C^T\tilde{D}(y_a - C\hat{\eta})\right]dt \\ &\quad + \|\lambda(0)\|_{R^{-1}}^2\Big\}\end{aligned}$$

it can be easily derived that

$$J_a = \|L\hat{\eta} - \hat{z}\|_2^2 + \|\varepsilon E\hat{\eta}\|_2^2 - \gamma^2\|\tilde{D}^{\frac{1}{2}}(y_a - C\hat{\eta})\|_2^2.$$

Next, introducing the changes of variables

$$\bar{z} = L\hat{\eta} - \hat{z} \tag{4.13}$$

$$\bar{v} = y_a - C\hat{\eta} \tag{4.14}$$

the min–max optimization of J_a with respect to $\hat{z}$ and y_a, respectively, results in the following minimax problem

$$\min_{\bar{z}} \max_{\bar{v}} \left\{ J_a = \|\varepsilon E\hat{\eta}\|_2^2 + \|\bar{z}\|_2^2 - \gamma^2\|\tilde{D}^{\frac{1}{2}}\bar{v}\|_2^2 \right\} \tag{4.15}$$

subject to (4.9) and

$$\dot{\hat{\eta}} = \bar{A}\hat{\eta} + \bar{B}_1\bar{v} + \bar{B}_2\bar{z} + B_2 r, \quad \hat{\eta}(0) = \hat{x}_0 \tag{4.16}$$

where

$$\bar{A} = A + \varepsilon^2\gamma^{-2}QE^TE.$$

Consider the following Riccati equation

$$-\dot{X} = \bar{A}^TX + X\bar{A} + X\left(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T\right)X + \varepsilon^2E^TE, \quad X(T) = 0 \tag{4.17}$$

where the existence of $X(t)$ over $[0,T]$ is assumed. Note that $X(t)$ is symmetric positive semi-definite over $[0,T]$. From (4.16) and (4.17), by completion to squares we get

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}(\hat{\eta}^TX\hat{\eta})dt + \hat{x}_0^TX(0)\hat{x}_0 \\ &= \|\bar{z} + \bar{B}_2^TX\eta\|_2^2 - \|\bar{z}\|_2^2 - \gamma^2\|\tilde{D}^{\frac{1}{2}}(\bar{v} - \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\hat{\eta})\|_2^2 \\ &\quad + \gamma^2\|\tilde{D}^{\frac{1}{2}}\bar{v}\|_2^2 - \|\epsilon E\hat{\eta}\|_2^2 \\ &\quad + 2\int_0^T \hat{\eta}^TXB_2rdt + \hat{x}_0^TX(0)\hat{x}_0. \end{aligned} \tag{4.18}$$

Adding the above zero quantity to J_a of (4.15), we obtain

$$J_a = \|\bar{z} + \bar{B}_2^TX\hat{\eta}\|_2^2 - \gamma^2\|\tilde{D}^{\frac{1}{2}}(\bar{v} - \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\hat{\eta})\|_2^2 + 2\int_0^T \hat{\eta}^TXB_2rdt + \|\hat{x}_0\|_{X(0)}^2.$$

Hence, by defining

$$\zeta_z = \bar{z} + \bar{B}_2^TX\hat{\eta}, \quad \zeta_v = \bar{v} - \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\hat{\eta} \tag{4.19}$$

the minimax problem of (4.15) can be changed to

$$\min_{\zeta_z} \max_{\zeta_v} \left\{ J_a = \|\zeta_z\|_2^2 - \gamma^2\|\tilde{D}^{\frac{1}{2}}\zeta_v\|_2^2 + 2\int_0^T \hat{\eta}^TXB_2rdt + \|\hat{x}_0\|_{X(0)}^2 \right\} \tag{4.20}$$

subject to (4.9), (4.17), and

$$\dot{\hat{\eta}} = \hat{A}\hat{\eta} + \bar{B}_1\zeta_v + \bar{B}_2\zeta_z + B_2r, \quad \hat{\eta}(0) = \hat{x}_0 \tag{4.21}$$

where

$$\hat{A} = A + \varepsilon^2\gamma^{-2}QE^TE + \left(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T\right)X.$$

Now, by using standard optimization techniques, we can easily show that the maximizing strategy ζ_v^* satisfies

$$\zeta_v^* = \gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T\theta \tag{4.22}$$

where

$$\dot{\theta} = -\hat{A}^T\theta - XB_2r, \quad \theta(T) = 0. \tag{4.23}$$

Note that, since the worst-case $\tilde{w}$ and $\tilde{v}$ may not be bound by causality constraints, the optimal strategy of ζ_v should not necessarily be causal with respect to the signal r.

Adding to (4.20) the identically zero term

$$0 = 2\int_0^T \frac{d}{dt}(\theta^T\hat{\eta})dt + 2\theta^T(0)\hat{x}_0$$

and substituting (4.22) in (4.20) and (4.21), it follows that

$$J_a = \|(L + \bar{B}_2^TX)\hat{\eta} + \bar{B}_2\theta - \hat{z}\|_2^2 + \bar{J}(r, \hat{x}_0) \tag{4.24}$$

where

$$\bar{J}(r, \hat{x}_0) = \gamma^{-2}\|\tilde{D}^{-\frac{1}{2}}\bar{B}_1^T\theta\|_2^2 - \|\bar{B}_2^T\theta\|_2^2 + 2\int_0^T \theta^TB_2rdt + 2\theta^T(0)\hat{x}_0 + \|\hat{x}_0\|_{X(0)}^2. \tag{4.25}$$

Hence, the minimax problem reduces to minimizing (4.24) with respect to $\hat{z}$ subject to (4.9), (4.17), and

$$\dot{\hat{\eta}} = \left[\hat{A} + \bar{B}_2(L + \bar{B}_2^TX)\right]\hat{\eta} + \gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T\theta - \bar{B}_2^T\hat{z} + B_2r, \quad \hat{\eta}(0) = \hat{x}_0. \tag{4.26}$$

The solution of the above minimization problem will depend on the available information on the input signal $r(\cdot)$. In the next three theorems, we provide a solution to the minimax problem of (4.1) for each of the information patterns of $r(\cdot)$ discussed in Section III. First, we introduce a condition that is fundamental to the results in this paper.

Condition 1: There exists an $\varepsilon > 0$ such that the following holds:

a) There exists a solution $Q(t)$ to (4.9) over $[0,T]$.
b) There exists a solution $X(t)$ to (4.17) over $[0,T]$.

The first result provides a solution to the minimax problem of (4.1) with a noncausal exogenous signal $r(\cdot)$.

Theorem 4.1: Consider the system (Σ_a), where the input signal $r(\cdot)$ is known a priori over the horizon $[0,T]$. Then the minimax problem of (4.1) has a solution if Condition 1 is satisfied. An optimal solution for $\eta_0, \tilde{w}$, and ζ_v is given by (4.3), (4.4), and (4.22), respectively, whereas a minimizing strategy for $\hat{z}$ is

$$\hat{z}^* = (L + \bar{B}_2^TX)\hat{\eta}^* + \bar{B}_2^T\theta \tag{4.27}$$

where $\theta(t)$ is as in (4.23), and $\hat{\eta}^*(t), \forall t \in [0,T]$, satisfies

$$\dot{\hat{\eta}}^* = \hat{A}\hat{\eta}^* + \left(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T\right)\theta + B_2r, \quad \hat{\eta}^*(0) = \hat{x}_0. \tag{4.28}$$

Moreover, the optimal value of the performance index J_a is $\bar{J}(r, \hat{x}_0)$ of (4.25).

Proof: See the Appendix. □

We note that in view of (4.23), the above minimizing strategy $\hat{z}^*$ is noncausal with respect to the signal $r(\cdot)$. Since $r(\cdot)$ is known in advance over $[0,T]$, the above noncausality is not a problem for the implementation of $\hat{z}^*$.

The next theorem deals with the case where the input signal $r(\cdot)$ is measured online but cannot be predicted. In this situation, the minimizing strategy for $\hat{z}(\cdot)$ is required to be causal with respect to $r(\cdot)$. Firstly, we decompose $\hat{\eta}$ of (4.26) as follows:

$$\hat{\eta} = \hat{\eta}_c + \hat{\eta}_1 \tag{4.29}$$

where $\hat{\eta}_c(t)$ and $\hat{\eta}_1(t)$ are, respectively, the "causal" and "noncausal" parts of $\hat{\eta}$ at time t, which are given by

$$\dot{\hat{\eta}}_c = \left[\hat{A} + \bar{B}_2(L + \bar{B}_2^T X)\right]\hat{\eta}_c - \bar{B}_2^T \hat{z} + B_2 r, \qquad \hat{\eta}_c(0) = \hat{x}_0. \tag{4.30}$$

$$\dot{\hat{\eta}}_1 = \left[\hat{A} + \bar{B}_2(L + \bar{B}_2^T X)\right]\hat{\eta}_1 + \gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T\theta, \qquad \hat{\eta}_1(0) = 0. \tag{4.31}$$

Theorem 4.2: Consider the system (Σ_a) where the input signal $r(\cdot)$ is given online. Then the minimax problem of (4.1) has a solution if Condition 1 is satisfied. An optimal solution for $\eta_0, \tilde{w}$, and ζ_v is given by (4.3), (4.4), and (4.22), respectively, whereas a minimizing strategy for $(\hat{z})$ is

$$\hat{z}^* = (L + \bar{B}_2^T X)\hat{\eta}_c^* \tag{4.32}$$

where $\hat{\eta}_c^*(t), \forall t \in [0,T]$, is the "causal" part of $\hat{\eta}^*(\cdot)$ of (4.28) at time t, given by

$$\dot{\hat{\eta}}_c^* = \hat{A}\hat{\eta}_c^* + B_2 r, \qquad \hat{\eta}_c^*(0) = \hat{x}_0.$$

Moreover, the value of the performance index is

$$J_a(\tilde{w}^*, \zeta_v^*, \eta_0^*, r, \hat{z}^*, \varepsilon) = \|(L + \bar{B}_2^T X)\hat{\eta}_1 + \bar{B}_2^T\theta\|_2^2 + \bar{J}(r, \hat{x}_0)$$

with $\theta(\cdot), \bar{J}(r, \hat{x}_0)$, and $\hat{\eta}_1(\cdot)$ being given by (4.23), (4.25), and (4.31), respectively.

Proof: See the Appendix. □

We now provide a solution to the game problem of (4.1) in the case of a "causal" input signal with a known average, i.e., $r(\cdot)$ is measured online and is of the form (3.8) with $\bar{r}(\cdot)$ being known *a priori* over [0, T]. We begin by introducing the following decomposition of θ and $\hat{\eta}$

$$\theta = \theta_a + \theta_d \tag{4.33}$$

$$\hat{\eta} = \hat{\eta}_a + \hat{\eta}_d \tag{4.34}$$

where $\theta_a(t)$ and $\hat{\eta}_a(t)$ are the "causal" parts at time t of $\theta(\cdot)$ and $\hat{\eta}(\cdot)$ respectively, whereas $\theta_d(t)$ and $\hat{\eta}_d(t)$ are the corresponding "noncausal" parts, which are given by

$$\dot{\theta}_a = -\hat{A}^T\theta_a - XB_2\bar{r}, \qquad \theta_a(T) = 0 \tag{4.35}$$

$$\dot{\theta}_d = -\hat{A}^T\theta_d - XB_2 r_d, \qquad \theta_d(T) = 0 \tag{4.36}$$

$$\dot{\hat{\eta}}_a = \left[\hat{A} + \bar{B}_2(L + \bar{B}_2^T X)\right]\hat{\eta}_a + \gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T\theta_a - B_2^T\hat{z} + B_2 r, \qquad \hat{\eta}_a(0) = \hat{x}_0 \tag{4.37}$$

$$\dot{\hat{\eta}}_d = \left[\hat{A} + \bar{B}_2(L + \bar{B}_2^T X)\right]\hat{\eta}_d + \gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T\theta_d + B_2 r_d, \qquad \hat{\eta}_d(0) = 0 \tag{4.38}$$

where $\bar{r}(\cdot)$ and $r_d(\cdot)$ satisfy (3.8).

Theorem 4.3: Consider the system (Σ_a) where the input signal $r(\cdot)$ is given online and is of the form (3.8) with $\bar{r}(\cdot)$ being known *a priori* over [0, T]. Then the minimax problem of (4.1) has a solution if Condition 1 is satisfied. An optimal solution for η_0, $\tilde{w}$, and ζ_v is given by (4.3), (4.4), and (4.22), respectively, and a minimizing strategy for $\hat{z}$ is

$$\hat{z}^* = (L + \bar{B}_2^T X)\hat{\eta}_a^* + \bar{B}_2^T\theta_a \tag{4.39}$$

where $\theta_a(\cdot)$ is as in (4.35) and $\hat{\eta}_a^*(t), \forall t \in [0,T]$ is the "causal" part of $\hat{\eta}^*(\cdot)$ of (4.28) at time t, given by

$$\dot{\hat{\eta}}_a^* = \hat{A}\hat{\eta}_a^* + \left(\gamma^{-2}\bar{B}_1\tilde{D}^{-1}\bar{B}_1^T - \bar{B}_2\bar{B}_2^T\right)\theta_a + B_2\bar{r}, \quad \hat{\eta}_a^*(0) = \hat{x}_0.$$

Moreover, the value of the performance index is

$$J_a(\tilde{w}^*, \zeta_v^*, \eta_0^*, r, \hat{z}^*, \varepsilon) = \|(L + \bar{B}_2^T X)\hat{\eta}_d + \bar{B}_2^T\theta_d\|_2^2 + \bar{J}(r, \hat{x}_0)$$

where $\bar{J}(r, \hat{x}_0), \theta_d(\cdot)$ and $\hat{\eta}_d(\cdot)$ satisfy (4.25), (4.36), and (4.38), respectively.

Proof: See the Appendix. □

Remark 4.1: We observe from the results of Theorems 1–3 that a linear estimate for z happens to give an optimal solution to the minimax auxiliary problem of (4.1) for each of the three patterns for the input signal $r(\cdot)$.

B. The Robust $\mathcal{H}_\infty$ Filter

Similar to the case of standard $\mathcal{H}_\infty$ filtering, a solution to each of the robust $\mathcal{H}_\infty$ filtering problems of Section III is obtained from $\hat{z}^*$ of Theorems 4.1–4.3, with the adversaries $\tilde{w}$, $\tilde{v}$,and η_0 not necessarily playing their optimal strategies $\tilde{w}^*$, $\tilde{v}^*$, and η_0^*, respectively.

In view of Lemma 3.1 and Theorems 4.1–4.3, we can easily derive the following corollaries.

Corollary 4.1: Consider the system (Σ) with a noncausal input signal $r(\cdot)$, and let $\gamma > 0$ be a given scalar. Then, if Condition 1 is satisfied, the following filter

$$\hat{z} = (L + \bar{B}_2^T X)\hat{x} + \bar{B}_2^T\theta \tag{4.40}$$

$$\dot{\theta} = -\hat{A}^T\theta - XB_2 r, \qquad \theta(T) = 0 \tag{4.41}$$

$$\dot{\hat{x}} = A_e\hat{x} + \bar{B}_1(y - C\hat{x}) - \bar{B}_2\bar{B}_2^T\theta + B_2 r, \quad \hat{x}(0) = \hat{x}_0 \tag{4.42}$$

where

$$A_e = \bar{A} - \bar{B}\bar{B}_2^T X \tag{4.43}$$

will guarantee the performance

$$\|x - \hat{z}\|_2^2 \le \gamma^2\left[\|w\|_2^2 + \|v\|_2^2 + \|x_0 - \hat{x}_0\|_R^2\right] + \bar{J}(r, \hat{x}_0) \tag{4.44}$$

for any w and v in $L_2[0,T]$ and $x_0 \in \Re^n$ and for all admissible uncertainties, where $\bar{J}(r, \hat{x}_0)$ satisfies (4.25).

Remark 4.2: The estimate $\hat{z}$ depends causally on the measurements, $y(\cdot)$, but is noncausal with respect to the input signal, $r(\cdot)$. Note that first the signal $\theta(\cdot)$ needs to be computed by backward integration of (4.41). Then, the estimate $\hat{z}(\cdot)$ is obtained causally from (4.40) and (4.42).

Observe that when (3.1) has no input, signal $r(\cdot)$, $\theta(\cdot)$ will be identically zero over [0, T]. In this case, it is easy to see that the filter (4.40)–(4.42) recovers the robust $\mathcal{H}_\infty$ filter of [6], and provides the robust performance

$$\|z-\hat{z}\|_2^2 \le \gamma^2\left[\|w\|_2^2+\|v\|_2^2+\|x_0-\hat{x}_0\|_R^2\right], \quad \forall x_0 \in \Re^n, \quad \forall w, v \in \mathcal{L}_2[0,T]. \tag{4.45}$$

Also note that in the case where there are no parameter uncertainties, i.e., $E=0$, it is easy to see from (4.17) that $X(t)=0, \forall t \in [0,T]$. This implies that $\theta(t)=0, \forall t \in [0,T]$, and the filter (4.40)–(4.42) recovers the standard $\mathcal{H}_\infty$ filter with a known deterministic input $r(\cdot)$. Moreover, the performance of (4.44) reduces to (4.45).

We now present a robust filter that is causal with respect to both the measurements, $y(\cdot)$, and the deterministic input signal $r(\cdot)$.

Corollary 4.2: Consider the system (Σ) where the input signal $r(\cdot)$ is causally measured, and let $\gamma>0$ be a given scalar. Then, if Condition 1 is satisfied, the following filter

$$\hat{z}=(L+\bar{B}_2^T X)\hat{x}_c \tag{4.46}$$

$$\dot{\hat{x}}_c = A_e\hat{x}_c+\bar{B}_1(y-C\hat{x}_c)+B_2 r, \quad \hat{x}(0)=\hat{x}_0 \tag{4.47}$$

will guarantee the performance

$$\|z-\hat{z}\|_2^2 \le \gamma^2\left[\|w\|_2^2+\|v\|_2^2+\|x_0-\hat{x}_0\|_R^2\right] + \|(L+\bar{B}_2^T X)\hat{\eta}_1+\bar{B}_2^T\theta\|_2^2+\bar{J}(r,\hat{x}_0) \tag{4.48}$$

for any w and v in $\mathcal{L}_2[0,T]$ and $x_0 \in \Re^n$, and for all admissible uncertainties, where $\theta(\cdot)$, $\bar{J}(r,\hat{x}_0)$, and $\hat{\eta}_1(\cdot)$ are as in (4.23), (4.25), (4.31), respectively.

Remark 4.3: As in the case of a noncausal input signal $r(\cdot)$, when there is no parameter uncertainty in system (Σ) $X(t)$ is identically zero on $[0,T]$. In this situation, the filter (4.46)–(4.47) reduces to a standard $\mathcal{H}_\infty$ filter with a known deterministic input, and (4.48) becomes (4.45).

Corollary 4.3: Consider the system (Σ) where the input signal $r(\cdot)$ is measured online and its average component $\bar{r}(\cdot)$ is known *a priori* over $[0,T]$. Given a scalar $\gamma>0$, then, if Condition 1 is satisfied, the following filter

$$\hat{z}=(L+\bar{B}_2^T X)\hat{x}_a+\bar{B}_2^T\theta_a \tag{4.49}$$

$$\dot{\theta}_a=-\hat{A}^T\theta_a-XB_2\bar{r}, \quad \theta_a(T)=0 \tag{4.50}$$

$$\dot{\hat{x}}_a=A_e\hat{x}_a+\bar{B}_1(y_a-C\hat{x}_a)-\bar{B}_2\bar{B}_2^T\theta_a+B_2 r, \quad \hat{x}_a(0)=\hat{x}_0 \tag{4.51}$$

will guarantee the performance

$$\|z-\hat{z}\|_2^2 \le \gamma^2\left[\|w\|_2^2+\|v\|_2^2+\|x_0-\hat{x}_0\|_R^2\right] + \|(L+\bar{B}_2^T X)\hat{\eta}_d+\bar{B}_2^T\theta_d\|_2^2+\bar{J}(r,\hat{x}_0) \tag{4.52}$$

for any w and v in $\mathcal{L}_2[0,T]$ and $x_0 \in \Re^n$, and for all admissible uncertainties, where $\bar{J}(r,\hat{x}_0)$, $\theta_d(\cdot)$, and $\hat{\eta}(\cdot)$ are given by (4.25), (4.36), and (4.38), respectively.

Remark 4.4: The component of $\hat{z}$ due to the measurement y in Corollaries 4.1–4.3 turns out to be identical to the estimate of z in [6] which considered the case without a known input signal.

In Theorems 4.1–4.3 and Corollaries 4.1–4.3, the Riccati equation for $X(t)$ depends on the one for $Q(t)$; see (4.17). This may be inconvenient for searching a suitable parameter ε. If desired, the Riccati equation for $X(t)$ can be replaced by one that is decoupled from $Q(t)$ but with a spectral radius constraint. This is represented in the next lemma which can be proved similarly as Theorem 3.7 in [10]. First, introduce the Riccati differential equation

$$-\dot{P}=A^TP+PA+PB_wB_w^TP+\varepsilon^2E^TE, \quad P(T)=0. \tag{4.53}$$

Lemma 4.1: Suppose that (4.9) has a solution $Q(t)$ on $[0,T]$. Then (4.17) has a bounded solution $X(t)$ on $[0,T]$ if and only if (4.53) has a bounded solution $P(t)$ on $[0,T]$ and $\rho[P(t)Q(t)]<\gamma^2, \quad \forall t \in [0,T]$. Furthermore, $X(t)=[I-\gamma^{-2}P(t)Q(t)]^{-1}P(t)$.

In the above, $\rho(\cdot)$ denotes the spectral radius of a matrix. In view of Lemma 4.1, the results of Theorems 4.1–4.3 and Corollaries 4.1–4.3 also hold with Condition 1 replaced by the following:

Condition 2: There exists an $\varepsilon>0$ such that the following holds:

a) There exists a solution $P(t)$ to (4.53) over $[0,T]$.
b) There exists a solution $Q(t)$ to (4.9) over $[0,T]$.
c) $\rho[P(t)Q(t)]<\gamma^2, \quad \forall t \in [0,T]$.

V. An Example

We consider the example in Section II and will show that a filter designed by using the proposed robust $\mathcal{H}_\infty$ filtering method will yield much more satisfactory signal estimates, i.e., improved robustness properties, compared to the Kalman and standard $\mathcal{H}_\infty$ filters based on the nominal model.

The uncertainty in this example is of the form (3.4)–(3.5) with

$$H_1=[0.3 \quad 0]^T, \qquad H_2=0, \qquad E=[0 \quad 1].$$

Since the simulation of Section II is for the infinite horizon case, we apply below the theory of Sections III and IV for a very large T. For simplicity, we assume that $x_0=0$ and that $r(t)$ is a unit step input that is known *a priori*, and we take R very large, $\hat{x}_0=0$, $g=10$, and $\gamma=1.1$.

For $\varepsilon=0.1$, both (4.9) and (4.17) have a bounded solution over $[0,T]$. As T and R approach infinity, these solutions converge (numerically) to

$$Q=\begin{bmatrix}3.7197 & 0.0794\\ 0.0794 & 0.0235\end{bmatrix}; \quad X=\begin{bmatrix}0.0119 & 0.0006\\ 0.0006 & 0.0113\end{bmatrix}.$$

Since r is a constant and we are considering $T\to\infty$, (4.41) simplifies to

$$\theta=-(\hat{A}^T)^{-1}XB_2 r. \tag{5.1}$$

In view of the above, the following stationary form of the filter (4.40)–(4.42) is used:

$$\dot{\hat{x}} = A_e\hat{x} + \bar{B}_1(y - C\hat{x}) + B_r r \tag{5.2}$$
$$\hat{z} = (L + \bar{B}_2^T X)\hat{x} + D_r r \tag{5.3}$$

where

$$B_r = [I + \bar{B}_2\bar{B}_2^T(A^T)^{-1}X]B_2 \tag{5.4}$$

and

$$D_r = -\bar{B}_2^T(\hat{A}^T)^{-1}XB_2. \tag{5.5}$$

Computation of the matrices above yields

$$\dot{\hat{x}}(t) = \begin{bmatrix} -0.505 & -1.117 \\ 0.850 & -0.535 \end{bmatrix}\hat{x}(t) + \begin{bmatrix} 0.918 \\ -0.241 \end{bmatrix} r + \begin{bmatrix} 7.939 \\ 2.354 \end{bmatrix}\{y(t) - [0 \quad 100]\hat{x}(t)\} \tag{5.6}$$
$$\hat{z}(t) = [0.077 \quad 100.018]\hat{x}(t) + 0.1242r(t) \tag{5.7}$$

or equivalently

$$\hat{z}(s) = G_y(s)y(s) + G_r(s)r(s) \tag{5.8}$$

where

$$G_y(s) = \frac{236.05s + 794.138}{s^2 + 236.436s + 795.091} \tag{5.9}$$
$$G_r(s) = \frac{0.124s^2 + 5.905s + 1049.27}{s^2 + 236.436s + 795.091}. \tag{5.10}$$

Assuming that the parameter δ is constant, then $y(s)$ and $z(s)$ can be written as

$$y(s) = G_{yw}(s)w(s) + v(s) + G_{yr}(s)r(s)$$
$$z(s) = G_{zw}(s)w(s) + G_{zr}(s)r(s)$$

where all the transfer functions depend on δ. The estimation error, $e(s) = z(s) - \hat{z}(s)$, is then given by

$$e(s) = G_{ew}(s)w(s) + G_{ev}(s)v(s) + G_{er}(s)r(s) \tag{5.11}$$

where

$$G_{ew}(s) = G_{zw}(s) - G_y(s)G_{yw}(s) \tag{5.12}$$
$$G_{ev}(s) = -G_y(s) \tag{5.13}$$
$$G_{er}(s) = G_{zr}(s) - G_y(s)G_{yr}(s) - G_r(s). \tag{5.14}$$

The plots of 20 $\log(|G_{er}(j\omega)|)$ and 10 $\log(|G_{ew}(j\omega)|^2 + |G_{ev}(j\omega)|^2)$ are shown in Fig. 5 for $g = 10$ and different values of δ. Obviously, this filter performs far better than the Kalman filter and the "nominal" $\mathcal{H}_\infty$ filter in Section II. It is observed that the improvement for $G_{er}(s)$ is more significant in the low-frequency range than in the mid frequencies. This is because our design was done for constant r and steady-state.

It is worth noting that the function $G_{ew}(s)$ for the filter (5.6)–(5.7) is actually identical to that of the robust filter (2.4)–(2.6) in Section II where r is not considered. This is natural because the auxiliary filtering problem (3.14) is identical to the one in [6] when $r \equiv 0$, $\hat{x}_0 = 0$, and $R \to \infty$.

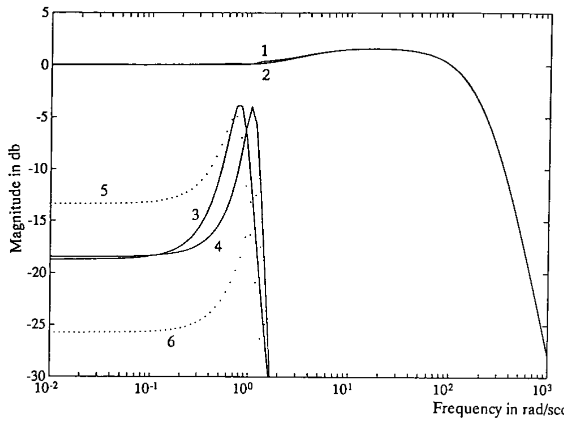

Fig. 5. New robust $\mathcal{H}_\infty$ filter for system (2.1)–(2.3). Curve 1: 10 $\log(\|G_{\epsilon w}(j\omega)\|^2 + \|G_{\epsilon v}(j\omega)\|^2)$ for $\delta = 0.3$, curve 2: 10 $\log(\|G_{\epsilon w}(j\omega)\|^2 + \|G_{\epsilon v}(j\omega)\|^2)$ for δ = -0.3, curve 3: 20 $\log(\|G_{\epsilon r}(j\omega)\|)$ for $\delta = 0.3$, $g = 10$, curve 4: 20 $\log(\|G_{\epsilon r}(j\omega)\|)$ for δ = -0.3, $g = 10$, curve 5: 20 $\log(\|G_{\epsilon r}(j\omega)\|)$ for $\delta = 0.3$, $g = 10$ (*cancellation method*), curve 6: 20 $\log(\|G_{\epsilon r}(j\omega)\|)$ for δ = -0.3, $g = 10$ (*cancellation method*).

For comparison purposes, we finally consider an alternative method for designing $G_{er}(s)$, which we call the *cancellation method*. The idea is simply as follows: Because $G_r(s)$ only effects $G_{er}(s)$, it is therefore obvious that the role of $G_r(s)$ is to minimize $G_{er}(s)$ in a certain sense. One possibility is to choose $G_r(s)$ such that $G_{er}(s)$ is completely cancelled, i.e., $G_{er}(s) \equiv 0$ for the nominal model. However, we show using the above example that such a method may not give an optimal worst-case solution. This observation is made from the dotted curves in Fig. 5 that correspond to the magnitude spectrum of $G_{er}(s)$, resulting from the cancellation method for different values of δ.

VI. Conclusion

A new robust $\mathcal{H}_\infty$ filtering method has been developed via a game theoretic approach for systems with both parametric uncertainty and a known input signal in the finite time horizon setting. The solution to the robust $\mathcal{H}_\infty$ filtering involves two Riccati differential equations with a scaling parameter. The robust $\mathcal{H}_\infty$ filter contains two components, one for process and measurement noise attenuation and the other for the attenuation of the known input signal. The former turns out to be the same as in [6], which treated the case without known input signal, but the latter depends on the *a priori* information on the external signal.

The simulation of the results in an example has demonstrated that the proposed robust filtering method offers far better robustness, and hence performance, than the conventional $\mathcal{L}_2$ and $\mathcal{H}_\infty$ filtering, for both noise and bias attenuation.

We expect that this new method can also be applied to the following problems:

i) robust $\mathcal{L}_2$ filtering with parametric uncertainty and known input signal
ii) robust tracking for systems with parametric uncertainty.

APPENDIX
PROOFS OF THEOREMS 4.1–4.3

Proof of Theorem 4.1: Defining $e=\eta-\hat{\eta}$ with $\hat{\eta}$ satisfying (4.16), it follows from (3.9) that

$$\dot{e}=\left(\tilde{A}+B_w\bar{D}B_w^TQ^{-1}\right)e-\gamma^{-2}QC_1^TC_1\hat{\eta}+B_w(\tilde{w}-w)-QC^T\tilde{D}(y_a-C\hat{\eta})+\bar{B}_2\hat{z},\quad e(0)=\eta_0-\hat{x}_0 \tag{A.1}$$

where

$$\bar{w}=\bar{D}\left[B_w^TQ^{-1}e+D_w^T(y_a-C\eta)\right]. \tag{A.2}$$

Now, considering (A.1) and (4.9) it is easy to obtain that

$$\frac{d}{dt}(e^TQ^{-1}e)=-\gamma^{-2}\|C_1e\|^2+\|Ce\|_{\tilde{D}}^2+\|B_w^TQ^{-1}e\|_{\bar{D}}^2+2\gamma^{-2}e^T\left(L^T\hat{z}-C_1^TC_1\hat{\eta}\right)-2e^TC^T\tilde{D}(y_a-C\hat{\eta})+2e^TQ^{-1}B_w(\tilde{w}-w).$$

Hence, we have

$$\begin{aligned}0&=\gamma^2\int_0^T\frac{d}{dt}(e^TQ^{-1}e)dt-\gamma^2\times\left[\|e(T)\|_{Q^{-1}(T)}^2+\|\eta_0-\hat{x}_0\|_R^2\right]\\&=V(\bar{w})+2\gamma^2\int_0^Te^TQ^{-1}B_w(\tilde{w}-w)dt-\gamma^2\|e(T)\|_{Q^{-1}(T)}^2\end{aligned} \tag{A.3}$$

where

$$V(\bar{w})=-\|C_1e\|_2^2+\gamma^2\|\tilde{D}^{\frac{1}{2}}Ce\|_2^2+\gamma^2\|\bar{D}^{\frac{1}{2}}B_w^tQ^{-1}e\|_2^2+2\int_0^Te^T[L^T\hat{z}-C_1^TC_1\hat{\eta}-\gamma^2C^T(y_a-C\hat{\eta})]dt+\gamma^2\|\eta_0-\hat{x}\|_R^2. \tag{A.4}$$

Using (A.2), we can rewrite $J_a(\tilde{w},\tilde{v},\eta_0,r,\hat{z},\varepsilon)$ as

$$J_a=\bar{J}(\bar{w})-2\gamma^2\int_0^T\left(B_w^TQ^{-1}e\right)^T(\tilde{w}-\bar{w})dr-\gamma^2\|\bar{D}^{-\frac{1}{2}}(\tilde{w}-\bar{w})\|_2^2 \tag{A.5}$$

where

$$\bar{J}_a(\bar{w})=\|L\eta-\hat{z}\|_2^2+\|\varepsilon E\eta\|_2^2-\gamma^2\left[\|\bar{w}\|_2^2+\|y_a-C_\eta-D_w\bar{w}\|_2^2+\|\eta_0-\hat{x}_0\|_R^2\right]. \tag{A.6}$$

Next, considering (A.2) and (A.6), and in view of the fact that $\eta=\hat{\eta}+e$, it follows that

$$\begin{aligned}\bar{J}_a(\bar{w})&=\|L\hat{\eta}-\hat{z}\|_2^2+\|Le\|_2^2+\|\varepsilon E\hat{\eta}\|_2^2+\|\varepsilon Ee\|_2^2\\&-\gamma^2\left[\|\bar{D}^{\frac{1}{2}}B_w^TQ^{-1}e\|_2^2+\|\tilde{D}^{\frac{1}{2}}(y_a-C\hat{\eta})\|_2^2+\|\tilde{D}^{\frac{1}{2}}Ce\|_2^2+\|\eta_0-\hat{x}_0\|_R^2\right]\\&+2\int_0^Te^T\left[L^T(L\hat{\eta}-\hat{z})+\varepsilon^2E^TE\right]\hat{\eta}dt\\&+2\gamma^2\int_0^Te^TC^T\tilde{D}(y_a-C\hat{\eta})dt.\end{aligned} \tag{A.7}$$

Adding to (A.5) the zero quantity of (A.3) and taking into account (A.4) and (A.7), it is easy to obtain that

$$J_a=\|\bar{z}\|_2^2+\|\varepsilon E\hat{\eta}\|_2^2-\gamma^2\left[\|\tilde{D}^{\frac{1}{2}}\bar{v}\|_2^2+\bar{D}^{-\frac{1}{2}}(\tilde{w}-w)\|_2^2+\|e(T)\|_{Q^{-1}(T)}^2\right] \tag{A.8}$$

where $\bar{z}$ and $\bar{v}$ are given by (4.13) and (4.14), respectively.

Similarly to the analysis in Section IV, adding to (A.8) the identically zero term of (4.18) we obtain

$$J_a=\|\bar{z}+\bar{B}_2^TX\hat{\eta}\|_2^2-\gamma^2\left[\|\tilde{D}^{\frac{1}{2}}\zeta_v\|_2^2+\|\bar{D}^{-\frac{1}{2}}(\tilde{w}-\bar{w})\|_2^2+\|e(T)\|_{Q^{-1}(T)}^2\right]+\|\hat{x}_0\|_{X(0)}^2+2\int_0^T\hat{\eta}^TX\bar{B}_2rdt \tag{A.9}$$

where, as in Section IV, $\zeta_v=\bar{v}-\gamma^{-2}\tilde{D}^{-1}\bar{B}_1^TX\hat{\eta}$. Next, adding to (A.9) the zero quantity

$$0=\int_0^T\frac{d}{dt}(\theta^T\hat{\eta})dt+2\theta^T(0)\hat{x}_0$$

where $\theta(\cdot)$ satisfies (4.23), it can be derived by completing the squares that

$$J_a=\|\left(L+\bar{B}_2^TX\right)\hat{\eta}+\bar{B}_2^T\theta-\hat{z}\|_2^2-\gamma^2\times\left[\|\bar{D}^{-\frac{1}{2}}(\tilde{w}-\bar{w})\|_2^2+\|\tilde{D}^{\frac{1}{2}}(\zeta_v-\gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T\theta)\|_2^2+\|e(T)\|_{Q^{-1}(T)}^2\right]+\bar{J}(r,\hat{x}_0) \tag{A.10}$$

where $\bar{J}(r,\hat{x}_0)$ is given by (4.25) and is independent of $\eta_0,\zeta_v,\tilde{w}$, and $\hat{z}$.

Hence, it follows from (A.10) that a minimax strategy is obtained by choosing

$$\zeta_v^*=\gamma^{-2}\tilde{D}^{-1}\bar{B}_1^T\theta;\quad\tilde{w}^*=\bar{w};\quad\eta_0^*=\bar{\eta}_0 \tag{A.11}$$

$$\hat{z}^*=\left(L+\bar{B}_2^TX\right)\hat{\eta}+\bar{B}_2^T\theta \tag{A.12}$$

where $\bar{\eta}_0$ is such that $e(T)=0$. Note that in view of the definition of λ in Section IV, with the above $\tilde{w}$ and η_0, we have that $e=Q^{-1}\lambda$. Then, by considering (A.2) we conclude that the optimal strategy of (A.11) and (A.12) is identical to (4.3), (4.4), (4.22), and (4.27). □

Proof of Theorem 4.2: The proof is similar to that of Theorem 4.1. The main difference is that the minimizing strategy for $\hat{z}$ is required to be causal with respect to both y and r. Since the worst-case choice for $\tilde{w}$ and ζ_v is not bound by causality constraints, as in the proof of Theorem 4.1, we obtain that (A.11) constitutes a maximizing strategy for ζ_v, $\tilde{w}$, and η_0.

Now substituting (A.11) in (A.10) and (4.16) and considering the decomposition of $\hat{\eta}$ in (4.29), we obtain that

$$J_a(\tilde{w}^*,\zeta_v^*,\eta_0^*,\hat{z})=\|\left(L+\bar{B}_2^TX\right)\hat{\eta}_c-\hat{z}+\left(L+\bar{B}_2^TX\right)\hat{\eta}_1+\bar{B}_2^T\theta\|_2^2+\bar{J}(r,\hat{x}_0)$$

where $\hat{\eta}_c$ and $\hat{\eta}_1$ are as in (4.30) and (4.31), respectively. Finally, since both $\hat{\eta}_1$ and θ are noncausal with respect to r, it follows that $\hat{z}^*=(L+\bar{B}_2^TX)\hat{\eta}_c$ is a causal minimizing strategy for $\hat{z}$, and the optimal value of J_a is $\|(L+\bar{B}_2^TX)\hat{\eta}_1+\bar{B}_2^T\theta\|_2^2+\bar{J}(r,\hat{x}_0)$, which concludes the proof. □

Proof of Theorem 4.3: As in the proof of Theorems 4.1 and 4.2, (A.11) is again the worst-case strategy for ζ_v, $\tilde{w}$, and η_0. Using the latter strategy and taking into account the decomposing of θ and $\hat{\eta}$ in (4.33) and (4.34), respectively, it follows from (A.10) that

$$J_a(\tilde{w}^*, \zeta_v^*, \eta_0^*, \hat{z}) = \|(L + \bar{B}_2^T X)\hat{\eta}_a + \bar{B}_2^T\theta_a - \hat{z} + \bar{B}_2^T\theta_d\|_2^2 + (L + \bar{B}_2^T X)\hat{\eta}_d + \bar{J}(r, \hat{x}_0)$$

where θ_a, θ_d, $\hat{\eta}_a$, and $\hat{\eta}_d$ satisfy (4.35) to (4.38), respectively. Finally, recalling that $\theta_a(t)$ and $\hat{\eta}_a(t)$ are based on the available information up to time t, namely, $\mathcal{R}_t$ and $\bar{\mathcal{R}}_T$, whereas $\theta_d(t)$ and $\hat{\eta}_d(t)$ are noncausal with respect to the available information on r, the desired result follows immediately, similar to the proof of Theorem 4.2. □

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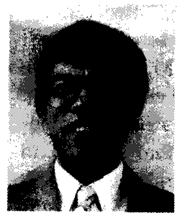

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