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Spatial $\mathcal{H}_2$ norm of flexible structures and its application in model order selection

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Abstract

This paper introduces a notion of Spatial $\mathcal{H}_2$ norm for flexible structures and studies its application in selecting the order of a flexible structure. The Spatial $\mathcal{H}_2$ norm differs from the standard $\mathcal{H}_2$ norm in that it is a measure of global performance in the spatial domain.

1 Introduction

Modelling of flexible structures could be a complicated procedure and often results in infinite-dimensional systems with lightly damped modes. This makes the control of such systems a very challenging problem [1, 2].

A problem that often occurs in the control of flexible structures is that, by nature, displacements over the entire structure are dynamically linked to displacements of every other point. This means that, a controller designed to minimize vibration at one point, could increase vibration somewhere else [3]. Hence, it is of interest to design controllers which result in vibration reduction in an average global sense. In [3] this problem is addressed using the idea of Spatial $\mathcal{H}_\infty$ norm minimization. Also, in [4], the idea of model reduction is extended to the case of flexible structures using Spatial $\mathcal{H}_\infty$ norm. In [5], the same ideas are applied to a piezoelectric-laminate beam to control the unwanted beam vibration using a Spatial LQG technique.

In this paper, we introduce the notion of Spatial $\mathcal{H}_2$ norm for flexible structures. We study the properties of this norm and one possible application, i.e., model order selection for flexible structures.

2 Spatial $\mathcal{H}_2$ norm

Dynamics of a flexible structure, such as a beam or a plate, can often be modelled via

$$G(s,r) = \sum_{i=0}^{\infty} \frac{F_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \tag{2.1}$$

where $r \in \mathcal{R}$, a bounded set. For a beam of length L, $\mathcal{R} = [0, L]$, for a plate of length L_1 and width L_2, $\mathcal{R} = \{(r_1, r_2) : r_1 \in [0, L_1],\ r_2 \in [0, L_2]\}$. The mode shapes satisfy the orthogonality condition:

$$\int_{\mathcal{R}} \phi_i(r)\phi_j(r)dr = \Phi_i^2 \delta_{ij} \tag{2.2}$$

where δ_{ij} is the Kronecker delta function.

The transfer functions of flexible structures consist of an infinite number of modes, hence they are infinite-dimensional systems. For controller design purposes, however, such a model is often approximated by a finite-order model. This is due to the fact that a linear controller of infinite bandwidth is not implementable in practice. Perhaps the easiest way of approximating an infinite-dimensional transfer function such as (2.1) with a finite-dimensional transfer function is to choose the first N modes of (2.1), i.e.,

$$G_N(s,r) = \sum_{i=0}^{N} \frac{F_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}. \tag{2.3}$$

The question here is how to choose N such that the effect of the neglected dynamics on the response of the system is negligible. Therefore, we need to have *a measure* to determine the contribution of each neglected mode to the response of the system. Two such measures are the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ norms. This paper deals with a particular form of the $\mathcal{H}_2$ norm which is used for this purpose.

Imagine for a moment that only the response of a particular point along the beam, such as $r = r_s$ is of interest to us. Let us call this transfer function $\tilde{G}_N(s)$, i.e., $\tilde{G}_N(s) = G_N(s, r_s)$. By definition, the $\mathcal{H}_2$ norm of $\tilde{G}_N$ is defined as the expected root-mean-square value of the output when the input is a unit variance white noise process, and is equivalent to:

$$\begin{aligned} \|\tilde{G}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\tilde{G}(j\omega)\tilde{G}(j\omega)^*\right\} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\sum_{i=0}^{N} \frac{F_i \Phi_i(r_s)}{-\omega^2 + j2\zeta_i\omega_i + \omega_i^2}\right|^2 d\omega. \end{aligned}$$

The above expression for the $\mathcal{H}_2$ norm of $\tilde{G}_N$ is computable, however, it does not give any insight as to how much each mode contributes to the total $\mathcal{H}_2$ norm of $\tilde{G}_N$. Having said this, the major problem with this choice of measure is that the $\mathcal{H}_2$ measure is totally ignorant of the response of other parts of the structure and only concentrates on the response at one particular point. Hence, if N is to be selected based on this measure, while it may be a suitable choice for a particular point, it may be too large or too small for the entire structure. This calls for a more suitable measure.

To overcome this difficulty, we propose the following spatial $\mathcal{H}_2$ norm for $G(s,r)$ and we will show how this measure could be used to select a suitable order for the system.

$$\ll G \gg_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathcal{R}} tr\left\{G_N(j\omega,r)G_N(j\omega,r)^*\right\} dr d\omega. \tag{2.4}$$

The notation $\ll G \gg_2$ is used here to emphasize the fact that we are dealing with a system whose output lives in the finite dimensional subspace of an infinite-dimensional space. At the first glance, it may seem to be very difficult to find an analytic expression for the above norm. However, thanks to the orthogonality of the mode shapes, this could be considerably simplified. In particular, we have the following Lemma.

Lemma 2.1 *Consider $G(s,r)$ as defined in (2.3) and suppose that the mode shapes satisfy the orthogonality condition (2.2). Then, $\ll G \gg_2^2 = \sum_{i=1}^{N} \|\tilde{G}_i\|_2^2$ where $\tilde{G}_i(s) = \frac{F_i\Phi_i}{s^2+2\zeta_i\omega_i s+\omega_i^2}$.*

Proof: A proof will be given in a more complete version of this paper.

Remark: It is possible to give an alternative proof of Lemma 2.1 based on state-space ideas. Notice that a state space realization of (2.3) is as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t,r) &= C(r)x(t) \end{aligned} \tag{2.5}$$

It can be shown that $\ll G \gg_2^2 = \|\hat{G}\|_2^2$ where $\Gamma'\Gamma = \int_{\mathcal{R}} C(r)'C(r)dr$ and $\hat{G}(s) = \Gamma(sI-A)^{-1}B$. To this end, it should be pointed out that the above interpretation of the Spatial $\mathcal{H}_2$ norm makes it possible for the standard control design techniques to be applied. In other words, it is possible to reduce the problem of designing a controller to minimize the closed-loop Spatial $\mathcal{H}_2$ norm of the system into a standard $\mathcal{H}_2$ controller design for the system $\hat{G}(s)$.

At this stage, we introduce a more general measure of performance, i.e., the weighted Spatial $\mathcal{H}_2$ norm. This norm is defined as follows:

$$\ll G \gg_{2,\Pi}^2 = \int_0^{\infty}\int_{\mathcal{R}} tr\left\{G(t,r)\Pi(r)G(t,r)'\right\} dr dt.$$

In this case it can be shown that $\ll G \gg_{2,\Pi}^2 = \|\hat{G}_\Pi\|_2^2$ where $\Gamma_\Pi'\Gamma_\Pi = \int_0^L C(r)'\Pi(r)C(r)dr$. Notice that $\Pi(r) \geq 0$ acts as a weighting function to emphasize the region of interest. In case of a flexible structure where $G(s,r) = C(r)(sI-A)^{-1}B$, it can be shown that a choice of $\Pi(r) = \delta(r-r_s)$ results in a standard 2-norm and $\Pi(r) = 1$ results in the Spatial $\mathcal{H}_2$ norm as discussed above. If $\Pi(r)$ is a more complicated function of r, then it may not be possible to directly use the orthogonality conditions. In that case, it may be necessary to use numerical integration techniques to reduce the weighted Spatial $\mathcal{H}_2$ norm to an equivalent $\mathcal{H}_2$ norm, which is solvable using standard techniques. However, in that case, the result of Lemma 2.1 may not apply.

3 Application in model order selection

Earlier, it was argued that the the $\mathcal{H}_2$ norm of a transfer function such as $G(s,r_s)$ is not a good measure to determine the order of a flexible structure. The main reason is that $\|G(s,r_s)\|_2$ does not convey any spatial information about $G(s,r)$. Furthermore, it is not clear how much each mode contributes to $\|G(s,r_s)\|_2$. The Spatial $\mathcal{H}_2$ norm, however, seems to be a more suitable measure for this purpose since it determines the global $\mathcal{H}_2$ norm of the system in an average sense. Moreover, it is clear from Lemma 2.1, how much each mode contributes to the Spatial $\mathcal{H}_2$ norm of the system. Hence, to choose a suitable order for a flexible structure, we could start with the first mode, then increase the number of modes and at each stage determine the Spatial $\mathcal{H}_2$ norm of the next mode and decide whether its contribution is significant or not.

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