ELSEVIER

Systems & Control Letters 30 (1997) 209–215

SYSTEMS & CONTROL LETTERS

# Computational complexity of a problem arising in fixed order output feedback design[1]

Minyue Fu[a,*], Zhi-Quan Luo[b]

[a] Department of Electrical and Computer Engineering, The University of Newcastle, Newcastle, NSW 2308, Australia
[b] Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ont. Canada, L8S 4K1

Received 1 July 1995; received in revised form 18 May 1996 and 12 February 1997; accepted 12 February 1997

**Abstract**

This paper is concerned with a matrix inequality problem which arises in fixed order output feedback control design. This problem involves finding two symmetric and positive definitive matrices $X$ and $Y$ such that each satisfies a linear matrix inequality and that $XY = I$. It is well-known that many control problems such as fixed order output feedback stabilization, $H_\infty$ control, guaranteed $H_2$ control, and mixed $H_2/H_\infty$ control can all be converted into the matrix inequality problem above, including static output feedback problems as a special case. We show, however, that this matrix inequality problem is NP-hard. © 1997 Elsevier Science B.V.

*Keywords:* Output feedback control; Static output feedback; Linear matrix inequality; $H_\infty$ control; Computational complexity

## 1. Main result

Denote by $\mathbb{R}_s^n$ the set of $n \times n$ symmetric and real matrices. The problem studied in this paper is stated as follows: *Given two affine mappings $L_1(\cdot), L_2(\cdot) : \mathbb{R}_s^n \to \mathbb{R}_s^n$, find positive definite $X, Y \in \mathbb{R}_s^n$ such that*

$$L_1(X) < 0, \quad L_2(Y) < 0, \quad XY = I. \tag{1}$$

The two inequalities above are called *linear matrix inequalities*, and the notation " $<$ " means the matrix is negative definite.

The motivation of the problem above stems from the fact that several important fixed order output feedback control problems, which include static output feedback control problems as a special case, can be converted into the above. Examples of these problem are such as fixed order output feedback stabilization, $H_\infty$ control, guaranteed $H_2$ control, and mixed $H_2/H_\infty$ control. See [3–10, 12] for references and Section 2 for discussions. It is generally believed that the problem in (1) is difficult to solve. Nevertheless, several recent approaches for solving the fixed order output feedback control problems rely on this conversion and certain iterative algorithms for solving (1). For example, the following iterative algorithm is proposed

* Corresponding author. E-mail: eemf@cc.newcastle.edu.au.

[1] The research of the first author is supported by Australian Research Council, while the research of the second author is supported by the Natural Science and Engineering Research Council of Canada under grant No. OPG0090301.


PII S0167-6911(97)00014-5

in [4, 6]:

$$X_k = \text{argmin}\{\alpha\colon L_1(X) < 0,\ I \leqslant Y_k^{1/2} X Y_k^{1/2} \leqslant \alpha I\},$$

$$Y_{k+1} = \text{argmax}\{\beta\colon L_2(Y) < 0,\ \beta I \leqslant X_k^{1/2} Y X_k^{1/2} \leqslant I\}, \quad k = 0, 1, \ldots$$

with some initial positive definite $Y_0$ satisfying $L_2(Y_0) < 0$. In [5], a different but equivalent formulation of the problem in (1) is analyzed and a cutting plane-like algorithm is provided. It is suggested in these references that these algorithms perform well in simulated examples.

On the other hand, the computational complexity issue for the static output stabilization problem is a unsolved theoretical problem which attracts many researchers. A recent survey paper [1] lists this problem as one of major open problems in systems and control. The most pertinent result is due to Blondel and Tsitsiklis [2] which shows that the problem of finding a static output feedback stabilizer from a given bounded set (a hypercube) is NP-complete. This result, however, does not show that the problem of static output feedback stabilization is NP-complete or NP-hard. Neither does it address the computational complexity issue for the problem in (1).

The result of this paper is simply stated as follows (see Section 3 for proof).

**Theorem 1.** *The matrix inequality problem in* (1) *is* NP-*hard.*

The implication of Theorem 1 is that any polynomial time algorithm used to solve (1) is expected to perform poorly in the worst case when the size of the problem, $n$, grows. Another consequence is that the algorithms proposed in [4–6] either have exponential running time or fail to solve (1) for some instances.

## 2. Output feedback control vs. the matrix inequality problem

The relationship between the fixed order output feedback control problems and the problem in (1) has been studied in a number of papers [5, 7, 10]. A brief summary is given below.

Consider the following linear time-invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ the exgoneous input, $u(t) \in \mathbb{R}^p$ the control input, $y(t) \in \mathbb{R}^q$ the measured output, $z(t) \in \mathbb{R}^r$ the controlled output, and $A$, $B_i$, $C_i$ and $D_{ij}$ are constant matrices of appropriate dimensions. A feedback controller of order $n_c$ is of the following form:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \qquad u(t) = C_c x_c(t) + D_c y(t), \tag{3}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the state of the controller, and

$$K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \tag{4}$$

is the matrix to be designed. Given the system (2) and the controller order $n_c$, the control problem is to find a $K$ such that certain design objective is met.

Let us define

$$\begin{aligned}
&\bar{x} = [x^{\mathrm{T}}\ x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}},\\
&\bar{A} = \begin{bmatrix} A & 0_{n\times n_{\mathrm{c}}} \\ 0_{n_{\mathrm{c}}\times n} & 0_{n_{\mathrm{c}}\times n_{\mathrm{c}}} \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_1 \\ 0_{n_{\mathrm{c}}\times m} \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} B_2 & 0_{n\times n_{\mathrm{c}}} \\ 0_{n_{\mathrm{c}}\times m} & I_{n_{\mathrm{c}}} \end{bmatrix},\\
&\bar{C}_1 = [C_1\ \ 0_{r\times n_{\mathrm{c}}}], \quad \bar{C}_2 = \begin{bmatrix} C_2 & 0_{r\times n_{\mathrm{c}}} \\ 0_{n_{\mathrm{c}}\times n} & I_{n_{\mathrm{c}}} \end{bmatrix},\\
&\bar{D}_{12} = [D_{12}\ \ 0_{r\times n_{\mathrm{c}}}], \quad \bar{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{r\times m} \end{bmatrix}, \quad \bar{D}_{11} = D_{11}.
\end{aligned} \tag{5}$$

The closed-loop system can be written as

$$\begin{aligned}
\dot{\bar{x}}(t) &= (\bar{A} + \bar{B}_2 K \bar{C}_2)\bar{x}(t) + (\bar{B}_1 + \bar{B}_2 K \bar{D}_{21})w(t),\\
z(t) &= (\bar{C}_1 + \bar{D}_{21} K \bar{C}_2)\bar{x}(t) + (\bar{D}_{11} + \bar{D}_{12} K \bar{D}_{21})w(t).
\end{aligned} \tag{6}$$

For the stabilization problem, the design objective is to have an asymptotically stable closed-loop matrix $\bar{A} + \bar{B}_2 K \bar{C}_2$. For the $H_\infty$ control problem, it is additionally required that the closed-loop transfer function of (6) has $H_\infty$ norm less than 1.

The notation $U_\perp$ for a given matrix $U$ means a(ny) matrix whose columns are the basis of the null space of $U$. In particular, $UU_\perp = 0$. The following results are known; see, e.g., [7, 10].

**Lemma 2.** *Given the system* (2), *there exists an output feedback stabilizer of order* $n_{\mathrm{c}}$ *if and only if there exist positive definite matrices* $X, Y \in \mathbb{R}_s^{n+n_{\mathrm{c}}}$ *such that*

$$\begin{aligned}
&L_1(X) := (\bar{C}_2)_\perp^{\mathrm{T}}(\bar{A}^{\mathrm{T}}X + X\bar{A})(\bar{C}_2)_\perp < 0,\\
&L_2(Y) := (\bar{B}_2^{\mathrm{T}})_\perp^{\mathrm{T}}(\bar{A}Y + Y\bar{A}^{\mathrm{T}})(\bar{B}_2^{\mathrm{T}})_\perp < 0,\\
&XY = I.
\end{aligned} \tag{7}$$

**Lemma 3.** *Given the system* (2), *there exists an output feedback* $H_\infty$ *controller of order* $n_{\mathrm{c}}$ *which renders the* $H_\infty$ *norm of the closed-loop transfer function to be less than* 1 *if and only if there exist positive definite matrices* $X, Y \in \mathbb{R}_s^{n+n_{\mathrm{c}}}$ *such that*

$$\begin{aligned}
&L_1(X) := \begin{bmatrix} \bar{C}_2 \\ \bar{D}_{21} \\ 0 \end{bmatrix}_\perp^{\mathrm{T}} \begin{bmatrix} \bar{A}^{\mathrm{T}}X + X\bar{A} & X\bar{B}_1 & \bar{C}_1^{\mathrm{T}} \\ \bar{B}_1^{\mathrm{T}}X & -I & 0 \\ \bar{C}_1 & 0 & -I \end{bmatrix} [\bar{C}_2\ \bar{D}_{21}\ 0]_\perp < 0,\\
&L_2(Y) := [\bar{B}_2^{\mathrm{T}}\ 0\ \bar{D}_{21}^{\mathrm{T}}]_\perp^{\mathrm{T}} \begin{bmatrix} Y\bar{A}^{\mathrm{T}} + \bar{A}Y & \bar{B}_1 & Y\bar{C}_1^{\mathrm{T}} \\ \bar{B}_1^{\mathrm{T}} & -I & 0 \\ \bar{C}_1 Y & 0 & -I \end{bmatrix} [\bar{B}_2^{\mathrm{T}}\ 0\ \bar{D}_{21}^{\mathrm{T}}]_\perp < 0,\\
&XY = I.
\end{aligned} \tag{8}$$

Lemmas 2 and 3 show that the matrix inequality problem in (1) arises in both cases. The guaranteed $H_2$ control problem and mixed $H_2/H_\infty$ problems have the same feature; see [4, 5]. Although we have only discussed continuous-time systems, fixed order output feedback control problems for discrete-time systems are analogous.

## 3. Proof of Theorem 1

As in almost all NP-hardness analysis cases, our basic idea is to polynomially transform a known NP-complete problem to the problem in (1). Polynomial transformation means that the resulting problem is obtained in polynomial time and the size of the resulting problem is polynomial of the size of the original problem. We will use the following 3-SAT problem which is known to be NP-complete [11]:

### *3.1. 3-SAT problem*

The instance of the problem involves an integer $n$, which is the number of Boolean variables $z=(z_1,z_2,\dots,z_n)$, and a CNF formula $F(z)=C_1\wedge C_2\wedge\cdots\wedge C_m$, where each clause $C_i$ is the disjunction of three literals which are either a $z_i$ or its negation $\bar{z}_i$. For example, $C_2=z_3\vee\bar{z}_6\vee z_8$. The problem is to determine whether or not there exists a Boolean truth assignment for $z$ such that $F(z)$ is satisfied, i.e., $F(z)$ is true. Since each clause contains three variables (or their negations), this problem is called 3-SATISFIABILITY problem, or 3-SAT, for short. Note that the number of clauses $m$ is at most $C_{2n}^3$ (i.e., $2n$-choose-3), thus a polynomial in $n$.

Now we show how the 3-SAT problem can be polynomially transformed into the problem in (1).

**Proof of Theorem 1.** Consider any instance of the 3-SAT problem with $m$ clauses $C_1,\dots,C_m$ defined over $n$ Boolean variables. We construct an equivalent instance of the problem (1) as follows. Let $z=(z_1,\dots,z_n)$ denote the Boolean variables of the 3-SAT problem. Then the corresponding matrix problem (1) will have dimension equal to $2n$. Let $X$ and $Y$ be two matrices of size $2n\times 2n$. We associate the entries $X_{2i-1,2i-1}$, $X_{2i,2i}$, $Y_{2i-1,2i-1}$ and $Y_{2i,2i}$ to the variable $z_i$. To enforce the Boolean condition $z_i=0$ or $z_i=1$, we introduce the following conditions:

$$\begin{aligned}
&XY=I,\\
&X_{i,j}=Y_{i,j}=0, && \forall i\neq j,\ i,j=1,\dots,2n,\\
&X_{2i-1,2i-1}+X_{2i,2i}=\tfrac{5}{2}, && \forall i=1,\dots,n,\\
&Y_{2i-1,2i-1}+Y_{2i,2i}=\tfrac{5}{2}, && \forall i=1,\dots,n.
\end{aligned}\tag{9}$$

In view of the second condition above, the first condition above is equivalent to

$$\begin{aligned}
&X_{2i-1,2i-1}Y_{2i-1,2i-1}=1,\\
&X_{2i,2i}Y_{2i,2i}=1.
\end{aligned}\tag{10}$$

Substituting (10) into the fourth condition in (9) yields

$$\frac{1}{X_{2i-1,2i-1}}+\frac{1}{X_{2i,2i}}=\frac{5}{2}.$$

This together with the third condition of (9) shows that $X_{2i-1,2i-1}$ and $X_{2i,2i}$ can only take one of the following two sets of values:

$$\begin{aligned}X_{2i-1,2i-1}&=2,\\ X_{2i,2i}&=\tfrac{1}{2},\end{aligned}\quad\text{or}\quad\begin{aligned}X_{2i-1,2i-1}&=\tfrac{1}{2},\\ X_{2i,2i}&=2.\end{aligned}\tag{11}$$

We interpret the first solution as $z_i=1$ and the second solution as $z_i=0$. In other words, $z_i=1$ if and only if $X_{2i-1,2i-1}=2$, and $z_i=0$ if and only if $X_{2i,2i}=2$. In this way, we have enforced the Boolean constraint $z_i=0$ or 1. Also, the conditions $X>0$ and $Y>0$ are satisfied automatically by virtue of (11).

We next construct a linear inequality for each clause in the 3-SAT problem instance. Suppose, for example, the $j$th clause is given by $C_j=z_{j_1}\vee\bar{z}_{j_2}\vee z_{j_3}$, where $1\leqslant j_1,j_2,j_3\leqslant n$. We introduce a linear inequality of the following form:

$$X_{2j_1-1,2j_1-1} + X_{2j_2,2j_2} + X_{2j_3-1,2j_3-1} > 2.25. \tag{12}$$

In general, if $C_j$ involves 3 literals $u_{j_1}$, $u_{j_2}$ and $u_{j_3}$, where $u_{j_1}$ is equal to either $z_{j_1}$ or its negation $\bar{z}_{j_1}$, and $u_{j_2}$, $u_{j_3}$ are defined similarly, then we introduce a linear inequality of the form similar to (12) where the left hand side consist of three terms and the right hand side is equal to 2.25. The first term on the left is equal to $X_{2j_1-1,2j_1-1}$ if $u_{j_1} = z_{j_1}$, or is equal to $X_{2j_1,2j_1}$ if $u_{j_1} = \bar{z}_{j_1}$. The other two terms are defined similarly. In this way, we have introduced a total of $m$ linear inequalities of $X$, each for a clause in the 3-SAT problem.

We claim that the 3-SAT problem is satisfied if and only if there exist diagonal matrices $X$ and $Y$ satisfying $X > 0$, $Y > 0$, $XY = I$ and the conditions (9) and (12). Indeed, as we have argued above, the conditions $X > 0$, $Y > 0$ and $XY = I$ can always be satisfied, provided that the diagonal entries of $X$ and $Y$ are given by (11). The remaining linear inequalities are of the form (12). Consider any truth assignment for the Boolean variables $z_1, \ldots, z_n$. Let us choose

$$\begin{aligned} &X_{2i-1,2i-1} = 2, \; X_{2i,2i} = \tfrac{1}{2}, \quad \text{if } z_i = 1, \\ &X_{2i-1,2i-1} = \tfrac{1}{2}, \; X_{2i,2i} = 2, \quad \text{if } z_i = 0. \end{aligned} \tag{13}$$

Consider a clause $C_j$ in the 3-SAT problem. By definition, $C_j$ is not satisfied if and only if all of its three literals have a value of zero. By (13), this happens exactly when each of the three terms in the left hand side of the linear inequality for $C_j$ is equal to $\frac{1}{2}$. This means the inequality for $C_j$ is not satisfied since its left hand side has a value of $\frac{3}{2}$ which is smaller than 2. In the other case when $C_j$ is satisfied, at least one of the three literals has a value of 1. As a result, the corresponding term in the linear inequality has a value of 2. Since the other two terms in the left hand side of the linear inequality for $C_j$ has a value of at least $\frac{1}{2}$, the left hand side is greater than or equal to 3. In other words, the linear inequality is satisfied when $C_j$ is satisfied. This shows that the 3-SAT problem is satisfiable if and only if there exist matrices $X$, $Y$ satisfying the equations of the form (9) and the linear inequalities of the form (12).

It can be seen that the constraints (9) and (12) are of the form

$$X > 0, \; Y > 0, \; XY = I, \; L_1(X) \geqslant 0, \; L_2(Y) \geqslant 0, \quad X, Y \text{ diagonal} \tag{14}$$

for some appropriate choice of $L_1$ and $L_2$. More specifically, $L_1(X)$ consists of

$$X_{i,j} = 0, \quad \forall i \neq j, \; i,j = 1, \ldots, 2n,$$

$$X_{2i-1,2i-1} + X_{2i,2i} = \tfrac{5}{2}, \quad \forall i = 1, \ldots, n$$

and (12) for each clause $C_j$. Similarly, $L_2(Y)$ consists of

$$Y_{i,j} = 0, \quad \forall i \neq j, \; i,j = 1, \ldots, 2n,$$

$$Y_{2i-1,2i-1} + Y_{2i,2i} = \tfrac{5}{2}, \quad \forall i = 1, \ldots, n.$$

To cast the problem (14) into the form of (1), we need to replace the equality constraints in $L_1(X) \geqslant 0$ and $L_2(Y) \geqslant 0$ with some strict inequality constraints. This is accomplished as follows. First, the diagonal conditions on $X$ and $Y$ can be replaced by

$$-\varepsilon < X_{i,j} < \varepsilon, \quad -\varepsilon < Y_{i,j} < \varepsilon, \quad \forall i \neq j, \; 1 \leqslant i,j \leqslant 2n, \tag{15}$$

where $\varepsilon$ is some small constant. Next we need to replace

$$X_{2i-1,2i-1} + X_{2i,2i} = \tfrac{5}{2}$$

with the following two strict inequalities:

$$X_{2i-1,2i-1} + X_{2i,2i} > \tfrac{5}{2} - \varepsilon, \quad X_{2i-1,2i-1} + X_{2i,2i} < \tfrac{5}{2} + \varepsilon. \tag{16}$$

Similarly, the equality

$$Y_{2i-1,2i-1} + Y_{2i,2i} = \tfrac{5}{2}$$

is replaced with

$$Y_{2i-1,2i-1} + Y_{2i,2i} > \tfrac{5}{2} - \varepsilon, \quad Y_{2i-1,2i-1} + Y_{2i,2i} < \tfrac{5}{2} + \varepsilon. \tag{17}$$

With $\varepsilon$ sufficiently small, we see that the condition $XY = I$ together with (15)–(17) imply that the equalities (11) hold approximately. Consequently, it remains true that a clause $C_j$ is satisfied if and only if the corresponding linear inequality of the form (12) holds. Indeed, with the correspondence (13) the left hand side of this inequality will be approximately equal to $\frac{3}{2}$ when $C_j$ is not satisfied, and to 3 when $C_j$ is satisfied.

To formalize the argument above, we claim that the choice of $\varepsilon = 0.01/n$ will suffice. Indeed, for this $\varepsilon$, (15) and $XY = I$ imply that

$$0.9999 < X_{i,i}Y_{i,i} < 1.0001, \quad \forall i = 1,\ldots,2n.$$

Let us denote

$$\begin{aligned} &X_{ii}Y_{ii} = 1 + \delta_i, && i = 1,\ldots,2n,\\ &X_{2i-1,2i-1} + X_{2i,2i} = \tfrac{5}{2} + \eta_i, && i = 1,\ldots,n,\\ &Y_{2i-1,2i-1} + Y_{2i,2i} = \tfrac{5}{2} + \phi_i, && i = 1,\ldots,n. \end{aligned} \tag{18}$$

By the relations (16) and (17), we have

$$\begin{aligned} &|\eta_i| \leqslant \varepsilon \leqslant 0.01, && i = 1,\ldots,n,\\ &|\phi_i| \leqslant \varepsilon \leqslant 0.01, && i = 1,\ldots,n,\\ &|\delta_i| \leqslant 0.0001, && i = 1,\ldots,2n. \end{aligned} \tag{19}$$

Manipulating the equations in (18) yields

$$\frac{1+\delta_{2i-1}}{X_{2i-1,2i-1}} + \frac{1+\delta_{2i}}{X_{2i,2i}} = \frac{5}{2} + \phi_i.$$

Eliminating the variable $X_{2i-1,2i-1}$ and rearranging the terms gives

$$\left(\tfrac{5}{2} + \phi_i\right) X_{2i,2i}^2 - \left(\left(\tfrac{5}{2} + \phi_i\right)\left(\tfrac{5}{2} + \eta_i\right) + \delta_{2i-1} - \delta_{2i}\right) X_{2i,2i} + (1 + \delta_{2i})\left(\tfrac{5}{2} + \eta_i\right) = 0.$$

Notice that when $\eta_i = \phi_i = \delta_{2i} = \delta_{2i-1} = 0$ the above equation implies $X_{2i} = 2$ or $\frac{1}{2}$. When $\eta_i$, $\phi_i$, $\delta_{2i-1}$ and $\delta_{2i}$ are nonzero but small, as are specified by (19), the above equation implies

$$\text{either} \quad X_{2i,2i} \in (0.25, 0.75) \quad \text{or} \quad X_{2i,2i} \in (1.75, 2.25).$$

Similarly, we can show that

$$\text{either} \quad X_{2i-1,2i-1} \in (0.25, 0.75) \quad \text{or} \quad X_{2i-1,2i-1} \in (1.75, 2.25).$$

Moreover, relation (16) implies that each of the two intervals (0.25, 0.75) and (1.75, 2.25) must contain exactly one of the two variables $X_{2i,2i}$, $X_{2i-1,2i-1}$. Consequently, by choosing $\varepsilon = 0.01/n$ and enforcing (15)–(17) in our construction of $L_1(X)$ and $L_2(Y)$, we can ensure that a clause of a 3-SAT problem is satisfied exactly when the corresponding linear inequality in $L_1(X)$ holds. For example, for the linear inequality (12) corresponding to the clause $C_j = z_{j_1} \vee \bar{z}_{j_2} \vee z_{j_3}$, we have

$$X_{2j_1-1,2j_1-1} + X_{2j_2,2j_2} + X_{2j_3-1,2j_3-1} \begin{cases} > 2.25 & \text{if } C_j \text{ is satisfiable,} \\ < 2.25 & \text{otherwise.} \end{cases}$$

That is, the chosen $\varepsilon$ is sufficient for determining the satisfiability of the 3-SAT problem.

Since the conversion from the 3-SAT problem into the problem (1) is clearly polynomial time, we complete the proof of Theorem 1. □

## 4. Conclusion

We have shown that the matrix inequality problem (1), which arises in the fixed order feedback control problems, is NP-hard. This result suggests that algorithms for fixed order feedback control which aim at solving (1) may perform poorly in the worst case when the system order becomes large. However, we note that the result above does not imply that the fixed order output feedback control problems are NP-hard because not every instance of the problem in (1) corresponds to a control problem. Further research is needed in this direction.

## References

[1] V. Blondel, M. Gevers, A. Lindquist, Survey on the state of systems and control, European J. Control 1 (1995) 5–23.

[2] V. Blondel, J.N. Tsitsiklis, NP-hardness of some linear control design problems, in: Proc. 3rd European Control Conf., Rome, 1995, pp. 2066–2071.

[3] L. El Ghaoui, F. Oustry, M. AitRami, An LMI-based linearization algorithm for static output-feedback and related problems, IEEE Trans. Automat. Control, to appear.

[4] J.C. Geromel, P.L.D. Peres, R.E. Skelton, LMI numerical solution for output feedback stabilization, Proc. American Control Conf., Baltimore, MD, 1994, pp. 40–44.

[5] J.C. Geromel, P.L.D. Peres, S.R. Souza, Convex analysis of output feedback control problems: Robust stability and performance, IEEE Trans. Automat. Control, to appear.

[6] J.C. Geromel, S.R. Souza, R.E. Skelton, Static output feedback controllers stability and convexity, preprint.

[7] P. Gahinet, P. Apkarian, A linear matrix inequality approach to $H_\infty$ control, Internat. J. Robust Nonlinear Control 4 (1992) 421–448.

[8] K.M. Grigoriadis, R.E. Skelton, Low order control design for LMI problems using alternating projection methods, Automatica 32 (1996) 1117–1125.

[9] T. Iwasaki, R.E. Skelton, The XY-centering algorithm for the dual LMI problem: A new approach to fixed order control design, Internat. J. Control 62 (1995) 1257–1272.

[10] T. Iwasaki, R.E. Skelton, All controllers for the general $\mathscr{H}_\infty$ control problem: LMI existence conditions and state space formulas, Automatica 30 (1994) 1307–1317.

[11] C.H. Papadimitriou, K. Steglitz, Combinatorial Optimization: Algorithms and Complexity, Prentice-Hall, Englewood Cliffs, NJ, 1982.

[12] V.L. Syrmos, C. Abdallah, P. Dorato, K. Grigoriadis, Static output feedback: A survey, Automatica, to appear.