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Brief paper

Robust stabilization of nonlinear cascaded systems☆

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Abstract

This paper considers the problem of robust stabilization for an uncertain nonlinear system which is a cascaded interconnection of two subsystems. Both of the subsystems are allowed to be nonlinear, multi-variable, and containing uncertain parameters. We present a new approach to designing stabilizing controllers which assure both robust global asymptotic stability and local quadratic stability. Compared with existing results, the assumptions required for such robust stabilizing controllers to exist are significantly simplified.

Keywords: Nonlinear control; Cascaded system; Robust control; Quadratic stabilization

1. Introduction

In this paper, we will consider the robust stabilization problem for the following cascaded system:

$$\dot{x}_1 = A_{0,1}(q)x_1 + f_{12}(x_1, x_2, q) + g_1(x_1, x_2, u, q),$$

$$\dot{x}_2 = f_2(x_2, q) + g_2(x_2, u, q), \tag{1}$$

where $x_1 \in \mathbf{R}^{n_1}$ and $x_2 \in \mathbf{R}^{n_2}$ form the state of the system, $u \in \mathbf{R}^m$ is a control input, q is an uncertain parameter vector from a compact set $Q \subset \mathbf{R}^l$, $A_{0,1}(q)$ is a continuous matrix function of q, the vector functions $f_{12}(\cdot)$ and $f_2(\cdot)$ are smooth in x_1, x_2 and continuous in q with $f_{12}(x_1, 0, q) \equiv 0$ and $f_2(0, q) \equiv 0$, the vector functions $g_1(\cdot)$ and $g_2(\cdot)$ are smooth in x_1, x_2 and u, and continuous in q with $g_1(x_1, x_2, 0, q) \equiv 0$ and $g_2(x_2, 0, q) \equiv 0$. We denote $n = n_1 + n_2$. Eq. (1) models a cascaded system in which the cascading is realized by injecting the state x_2 and the input u of the x_2-subsystem into the x_1-subsystem.

The notion of *input to state stability* (ISS), initially proposed by Sontag (1989), has been commonly used to study stabilization of nonlinear cascaded systems (e.g., see Arcak, Angeli, & Sontag, 2002; Jiang, Teel, & Praly, 1994). Using Lyapunov method, Jankovic, Sepulchre, and Kokotovic (1996) and Mazenc and Praly (1996) relaxed the ISS condition for stabilization of nonlinear cascaded systems. For nonlinear systems cascaded by a chain of integrators, which is often called *upper-triangle form* or *feedforward system* in the literature, a recursive design method based on the use of saturation functions was proposed by Teel (1992) for global asymptotic stabilization. Many generalizations of this method can be found (e.g., see Arcak, Teel, & Kokotovic, 2001; Kaliora & Astolfi, 2001; Marconi & Isidori, 2000). The use of saturation functions can be viewed to generate a *multi-step controller* in the sense that different controllers are applied depending on the state of the system.

In all the references cited above, it is assumed that either the cascaded nonlinear system (1) has no parametric uncertainties or such uncertainties do not appear in the linear part of the system. Note that this is a serious restriction because stabilizing controllers designed based on such an assumption do not possess any robustness when the system model is slightly deviated from the actual system. Unfortunately, the design techniques in the aforementioned papers do not apply directly to systems with parametric uncertainties in the linearized model (Marconi & Isidori, 2000). Approaches which allow parametric uncertainties in the linearized model include Su and Fu (1998, 1999), and Marconi and Isidori (2000).

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In this paper, we consider a class of cascaded systems with parametric uncertainties which are much more general than the upper-triangular structure. We study the problem of designing a robust stabilizing controller for such a system. More precisely, we consider the cascaded nonlinear system (1) in which the driven subsystem is a multi-variable system while the driving subsystem is not assumed in the upper-triangular form. If the system is void of parametric uncertainties or the uncertainties are restricted by a very special structure, this system can usually be transformed into a feedforward system. However, we do not impose such a restriction. Consequently, the recursive design method based on saturation functions is not applicable in this case. Due to this reason, we present a new, two-step controller design method for the stabilization of the system. In the first step, a global stabilizing controller of the driving subsystem is used to steer the state x_2 of the subsystem into a small region around the origin while the state x_1 of the driven subsystem is bounded. In the second step, a non-quadratic Lyapunov function is constructed and a nonlinear controller based on this Lyapunov function keeps the state x_2 in the small region and steers the state (x_1, x_2) to zero. Our approach differs from other multi-step approaches such as in Teel (1992) and its generalizations as mentioned earlier is that our approach does not require a precise system model in the controller design. This significantly improves the robustness of the resultant system. In particular, for a feedforward system, our approach allows any bounded uncertain parameters as long as the bounds are known. Compared with the existing results, the admissible uncertain parameter set is notably expanded. Furthermore, the common assumption in the existing works that the order of x_2 in the interconnection term $f_{12}(x_1, x_2, q)$ is greater than one is removed. For a feedforward system, our approach not only allows a linear term of the first element of x_2 to enter this interconnection term but also allows the system to be in a more general upper-triangular structure.

The organization of this paper is as follows. In Section 2, the formulation and assumptions about the robust stabilization problem are discussed for the uncertain nonlinear cascaded system (1). Then we will introduce a new design method for the system (1) in Section 3. A design example and simulation results will be presented in Section 4. Finally, conclusions and discussion will be given in Section 5. The work in this paper is based on our previous work (Su & Fu, 1998, 1999) on the upper-triangular structure.

2. Formulation and assumptions

In this section, the problem under consideration is formulated and the assumptions are discussed.

Denote the locally linearized model of (1) at the origin in $\mathbf{R}^n$ by

$$\dot{x} = A_0(q)x + B_0(q)u, \quad x \in \mathbf{R}^n, \tag{2}$$

where $x = [x_1', x_2']'$ and, for $\forall q \in Q$,

$$A_0(q) = \begin{bmatrix} A_{0,1}(q) & A_{0,12}(q) \\ 0 & A_{0,2}(q) \end{bmatrix}, \quad B_0(q) = \begin{bmatrix} B_{0,1}(q) \\ B_{0,2}(q) \end{bmatrix}. \tag{3}$$

For simplicity, we first consider the case where

$$g_1(x_1, x_2, u, q) \equiv 0 \quad \text{and} \quad g_2(x_2, u, q) = B_2(x_2, q)u. \tag{4}$$

The discussion will be extended to a general case in the next section.

Under the structure (4), we have $B_{0,1}(q) = 0$ and $B_{0,2}(q) = B_2(0, q)$. Due to the assumption in the last section that $f_{12}(x_1, x_2, q)$ is smooth in x_1, x_2 and continuous in q and $f_{12}(x_1, 0, q) \equiv 0$ for $\forall x_1 \in \mathbf{R}^{n_1}$ and $q \in Q$, there exist matrix functions $F_{11}(x_1, x_2, q)$ and $F_{12}(x_2, q)$, smooth in (x_1, x_2) and continuous in q, such that for all $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$ and $q \in Q$,

$$f_{12}(x_1, x_2, q) = F_{11}(x_1, x_2, q)x_1 + [A_{0,12}(q) + F_{12}(x_2, q)]x_2 \tag{5}$$

and

$$F_{11}(x_1, 0, q) \equiv 0, \quad F_{12}(0, q) \equiv 0.$$

Suppose that system (1) also satisfies the following assumptions:

Assumption 1 (*Critical stability of the driven subsystem*). There exist a small neighborhood $\Omega \subset \mathbf{R}^{n_2}$ around the origin in $\mathbf{R}^{n_2}$ and a positive definite matrix $P_1 \in \mathbf{R}^{n_1 \times n_1}$ such that, for $\forall x_1 \in \mathbf{R}^{n_1}$, $\forall x_2 \in \Omega$ and $\forall q \in Q$,

$$P_1[A_{0,1}(q) + F_{11}(x, q)] + [A_{0,1}(q) + F_{11}(x, q)]' P_1 \leqslant 0. \tag{6}$$

Assumption 2 (*Global asymptotic stabilizability of the driving system*). There exists a smooth controller $u_0(x_2)$ such that the system

$$\dot{x}_2 = f_2(x_2, q) + g_2(x_2, u_0(x_2), q) \tag{7}$$

is globally asymptotically stable.

Assumption 3 (*Growth constraint*). There exist smooth and positive definite functions $\gamma_1(x_2)$ and $\gamma_2(x_2)$ such that

$$\|F_{11}(x_1, x_2, q)\| \leqslant \gamma_1(x_2), \quad \|F_{12}(x_2, q)\| \leqslant \gamma_2(x_2). \tag{8}$$

Assumption 4 (*Local quadratic stabilizability*). There exist a linear state feedback matrix $K = [K_1 \ \ K_2]$ and a symmetric and positive-definite matrix P_0

$$P_0 = \begin{bmatrix} P_1 & -P_1 W \\ -W' P_1 & P_2 + W' P_1 W \end{bmatrix} \tag{9}$$

such that, for $\forall q \in Q$,

$$P_0[A_0(q) + B_0(q)K] + [A_0(q) + B_0(q)K]' P_0 < 0. \tag{10}$$

That is, a locally quadratic Lyapunov function of the system is given by

$$V_0(x_1, x_2) = (x_1 - W x_2)' P_1 (x_1 - W x_2) + x_2' P_2 x_2.$$

In the rest of this section, we provide some justifications for the assumptions above. We first note that critical stability of the driven system is commonly assumed in the work of cascaded nonlinear systems or feedforward nonlinear systems (see, for example, Jankovic et al., 1996; Marconi & Isidori, 2000; Teel, 1992). Assumption 1 is the robust version of this common assumption. To justify this assumption, we note several points:

(1) When the state x_1 is a scalar and $F_{11}(x_1, x_2, q) \equiv 0$, this condition merely requires $A_{0,1}(q)$ to be non-positive. In fact, $A_{0,1}(q) \equiv 0$ and $F_{11}(x_1, x_2, q) \equiv 0$ are required in the most of works for feedforward systems (see for example, Marconi & Isidori, 2000; Teel, 1992).

(2) When x_1 is not a scalar, a similar condition was discussed by Jankovic et al. (1996). It is assumed in A2, Lemma 1 of the above paper that the driven subsystem has the form

$$\dot{x}_1 = \begin{bmatrix} \tilde{f}_{11}(x_{11}) \\ F_{12}x_{12} + \tilde{f}_{12}(x_{11}, x_{12}) \end{bmatrix}, \quad x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \tag{11}$$

where $\dot{x}_{11} = \tilde{f}_{11}(x_{11})$ is globally asymptotically stable and (11) is Lyapunov stable. In our case, we do not consider $\tilde{f}_{11}(x_{11})$. It can be shown that requiring (11) to be Lyapunov stable is similar to Assumption 1.

(3) To show that stabilizability may be impossible without (6), we consider the following example:

$$\dot{x}_1 = \varepsilon x_1 + 2x_2 + x_2^2, \quad \varepsilon > 0,$$

$$\dot{x}_2 = u.$$

It is easy to verify that its local linearized model is stabilizable. However, $2x_2 + x_2^2 \geqslant -1$, implying that $x_1(t)$ will diverge if $x_1(0) > 1/\varepsilon$ regardless what control is used.

Assumption 3 implies that the interconnection term $f_{12}(x_1, x_2, q)$ has a linear growth with respect to x_1. This is also a common assumption in stabilization of feedforward nonlinear systems; see, e.g., Jankovic et al. (1996), and Mazenc and Praly (1996). An important property of this assumption is given below:

Lemma 1. *Suppose system* (1) *satisfies Assumptions* 1–3. *Then* x_1 *in system* (1) *is bounded within any finite time interval*, *i.e.*, *given any* $x(0)$ *and* $T > 0$, *there exists* $M(x(0), T)$ *such that* $\|x_1(t)\| \leqslant M(x(0), T)$ *for all* $0 \leqslant t \leqslant T$.

Proof. See Jankovic et al. (1996). □

Assumption 4 is naturally required because we want the controlled system to be locally quadratically stable. The problem of designing local quadratic stabilizers and their associated Lyapunov functions is not the focus of this paper. We only note that many design approaches are available. For example, Wei (1990) provides a systematic design method for quadratic stabilization of uncertain feedforward linear systems.

3. Multi-step design for nonlinear cascaded systems

In this section, the design method of two-step controller is presented for the case where $g_1(x_1, x_2, u, q) \equiv 0$ and $g_2(x_2, u, q) = B_2(x_2, q)u$ in system (1). The result will be extended to a general case later.

We first design a robust controller for system (1) when x_2 is sufficiently small. This design is based on the use of a non-quadratic Lyapunov function. Suppose system (1) satisfies Assumptions 1–4. Define

$$V_2(x_2) = x_2' P_2 x_2, \quad V_2^{-1}(\mu) := \{x_2 | V_2(x_2) < \mu\}, \tag{12}$$

where $\mu > 0$ is to be specified. Then choose

$$V(x_1, x_2) = (x_1 - Wx_2)' P_1 (x_1 - Wx_2) + \int_0^{V_2(x_2)} s(w)\,\mathrm{d}w \tag{13}$$

as a non-quadratic Lyapunov function candidate for system (1) in the region $R^{n_1} \times V_2^{-1}(\mu)$, where $s(w)$ is a positive, smooth, and monotonically non-decreasing function for $w \in [0, \mu)$, with $s(0) = 1$ and

$$\int_0^{V_2} s(w)\,\mathrm{d}w \to \infty, \quad \text{as } V_2 \to \mu. \tag{14}$$

Remark 1. A particular choice of $s(\cdot)$ is given by

$$s(w) = \left(\frac{\mu}{\mu - w}\right)^k, \quad k \geqslant 1.$$

For the case of $k = 1$, this corresponds to a Lyapunov function used in Teel and Praly (1995). In general, the Lyapunov function (13) is non-quadratic. However it is a locally quadratic Lyapunov function around the origin because $s(\cdot)$ is a smooth function around the origin. We also note that the function $\int_0^{V_2(x_2)} s(w)\,\mathrm{d}w$ resembles a "potential barrier" and the Lyapunov function (13) is valid only for $(x_1, x_2) \in R^{n_1} \times V_2^{-1}(\mu)$, i.e.,

$$V(x_1, x_2) \to \infty \quad \text{as } V_2(x_2) \to \mu.$$

This implies that, for system (1), if $x_2(0) \in V_2^{-1}(\mu)$ then $x_2(t) \in V_2^{-1}(\mu)$ for all $t > 0$ as long as that $\dot{V}(x_1, x_2) \leqslant 0$. In other words, the region $V_2^{-1}(\mu)$ is forward invariant for the driving subsystem if $\dot{V}(x_1, x_2) < 0$.

Because $g_2(x_2, u, q)$ and $f_2(x_2, q)$ are smooth in x_2 and continuous in q (hence so is $B_2(x_2, q)$), there exists $F_2(x_2, q)$ with $F_2(0, q) \equiv 0$, $\forall q \in Q$ such that

$$f_2(x_2, q) = A_{0,2}(q)x_2 + F_2(x_2, q)x_2.$$

And there exist also positive definite functions $\gamma_3(x_2)$ and $\gamma_4(x_2)$ such that $\|B_2(x_2, q) - B_{0,2}(q)\| \leqslant \gamma_3(x_2)$ and $\|F_2(x_2, q)\| \leqslant \gamma_4(x_2)$. For $\forall x \in R^{n_1} \times \Omega, q \in Q$, denote

$$A(x, q) = \begin{bmatrix} A_{0,1}(q) + F_{11}(x, q) & A_{0,12}(q) + F_{12}(x_2, q) \\ 0 & A_{0,2}(q) + F_2(x_2, q) \end{bmatrix}.$$

Lemma 2. *Suppose system* (1) *satisfies Assumptions* 3 *and* 4. *Then there exist sufficiently small* $\mu > 0$ *and* $\varepsilon > 0$ *such that*

$$V_2^{-1}(\mu) \subset \Omega$$

and it holds for $\forall x \in R^{n_1} \times V_2^{-1}(\mu)$ *and* $\forall q \in Q$:

$$P_0[A(x,q)+B(x_2,q)K]+[A(x,q)+B(x_2,q)K]'P_0 < -\varepsilon I, \tag{15}$$

where $B(x_2,q)=[0\ \ B_2'(x_2,q)]'$.

Proof. From the structure of the matrix $A(x,q)$, we have

$$P_0[A(x,q)+B(x_2,q)K]+[A(x,q)+B(x_2,q)K]'P_0 = P_0[A_0(q)+B_0(q)K]+[A_0(q)+B_0(q)K]'P_0+\Delta(x,q),$$

where

$$\Delta(x,q) = P_0\begin{bmatrix} F_{11}(x,q) & F_{12}(x_2,q) \\ 0 & F_2(x_2,q)\end{bmatrix} + \begin{bmatrix} F_{11}(x,q) & F_{12}(x_2,q) \\ 0 & F_2(x_2,q)\end{bmatrix}' P_0 + P_0[B(x_2,q)-B_0(q)]K + K'[B(x_2,q)-B_0(q)]'P_0$$

and $\Delta(x_1,0,q)\equiv 0$ for $\forall x_1 \in R^{n_1}$ and $q \in Q$.

Following Assumption 4, there exists an $\varepsilon_1 > 0$ such that

$$P_0[A(x,q)+B(x_2,q)K]+[A(x,q)+B(x_2,q)K]'P_0 \leqslant -\varepsilon_1 I + \Delta(x,q).$$

In addition, following Assumption 3 and the properties of $F_2(x_2,q)$ and $B(x_2,q)-B_0(q)$, we have

$$\Delta(x_1,x_2,q) \leqslant \gamma(x_2) I \tag{16}$$

for some smooth and positive definite $\gamma(x_2)$. Hence,

$$P_0[A(x,q)+B(x_2,q)K]+[A(x,q)+B(x_2,q)K]'P_0 \leqslant -\varepsilon I \quad \forall \varepsilon \in (0,\varepsilon_1),$$

when μ is sufficiently small. □

The derivative of $V(x_1,x_2)$ along the trajectory of system (1) is given by

$$\dot V(x_1,x_2) = x'[PA(x,q)+A'(x,q)P]x + 2x'PB(x_2,q)u, \tag{17}$$

where

$$P = \begin{bmatrix} P_1 & -P_1 W \\ -W'P_1 & s(V_2)P_2 + W'P_1W \end{bmatrix}$$

with its inverse given by

$$S = \begin{bmatrix} P_1^{-1} + (s(V_2))^{-1} W P_2^{-1} W' & (s(V_2))^{-1} W P_2^{-1} \\ (s(V_2))^{-1} P_2^{-1} W' & (s(V_2))^{-1} P_2^{-1} \end{bmatrix}.$$

Consider the following state feedback controller for $\forall x_1 \in \mathbf{R}^{n_1}$, $x_2 \in V_2^{-1}(\mu)$:

$$u_l(x_1,x_2) = (s(V_2(x_2)))^{-1} K_1 x_1 + [K_2 + (1-(s(V_2(x_2)))^{-1})K_1 W]x_2$$

and denote that

$$\bar V_2^{-1}(\rho\mu) := \{x_2 | V_2(x_2) \leqslant \rho\mu\}, \tag{18}$$

where $\rho \in (0,1)$ is a given constant. We have the following main result:

Theorem 1. *Suppose that system* (1) *satisfies Assumptions* 1–4. *Then the closed-loop system controlled by the following controller*

$$u = \begin{cases} u_0(x_2), & 0 \leqslant t < T, \\ u_l(x_1,x_2), & T \leqslant t < \infty \end{cases} \tag{19}$$

is robustly globally asymptotically stable and locally quadratically stable where $T=0$ *if* $x_2(0) \in \bar V_2^{-1}(\rho\mu)$, *otherwise*, T *is the first moment when* $V_2(x_2)=\rho\mu$.

Proof. First, we consider the case where $x_2(0) \in \bar V_2^{-1}(\rho\mu)$ and $u_l(x_1,x_2)$ is applied. It will be shown that the state x can be steered by the controller $u_l(x_1,x_2)$ to zero while the state x_2 is kept within $V_2^{-1}(\mu)$. Denoting $z=[z_1'\ \ z_2']'=Px$, Eq. (17) is written as

$$\dot V(x_1,x_2) = z'[A(x,q)S + SA'(x,q)]z + 2z'B(x_2,q)u.$$

Denote $(s(V_2(x_2)))^{-1}$ by s^{-1} and let

$$u = K_1 x_1 + K_2 x_2 + v.$$

Then,

$$\dot V(x_1,x_2) = z'\{[A(x,q)+B(x_2,q)K]S + S[A(x,q)+B(x_2,q)K]'\}z + 2z'B(x_2,q)v. \tag{20}$$

It follows from Lemma 2 that the matrices $A(x,q)$ and $B(x_2,q)$ satisfy inequality (15). Moreover, rewrite (15) and S as

$$[A(x,q)+B(x_2,q)K]S_0 + S_0[A(x,q)+B(x_2,q)K]' < -\varepsilon S_0^2 \quad \forall x \in \mathbf{R}^{n_1} \times V_2^{-1}(\mu)\ \ \forall q \in Q \tag{21}$$

and

$$S = s^{-1}S_0 + (1-s^{-1})\begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{22}$$

respectively, where $S_0=P_0^{-1}$. Note the fact that $B_1(x_1,x_2,q)\equiv 0$ in the case under consideration. Substituting (21) and (22) into (20) leads to

$$\dot V(x_1,x_2) = 2s^{-1}z'(A(x,q)+B(x_2,q)K)S_0 z + 2z'B(x_2,q)v + 2(1-s^{-1})z'\left\{A(x,q)+\begin{bmatrix} 0 \\ B_2(x_2,q)\end{bmatrix}K\right\} \times \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} z.$$

Considering the structure of the matrix $A(x,q)$, this equality is rewritten

$$\dot V(x_1,x_2) = 2s^{-1}z'(A(x,q)+B(x_2,q)K)S_0 z + 2(1-s^{-1})z_2'B_2(x_2,q)(K_1x_1 - K_1Wx_2) + 2(1-s^{-1})z_1'[A_1(x,q)P_1^{-1}]z_1 + 2z_2'B_2(x_2,q)v.$$

Let $v=-(1-s^{-1})(K_1x_1-K_1Wx_2)$. We have

$$\dot V(x_1,x_2) = 2s^{-1}z'(A(x,q)+B(x_2,q)K)S_0 z + 2(1-s^{-1})z_1'[A_1(x,q)P_1^{-1}]z_1. \tag{23}$$

Applying (21) and Assumption 1 into (23), we get,

$$\dot{V}(x_1, x_2) \leqslant -\varepsilon(s(V_2))^{-1}\lambda_{\min}(PS_0^2P)x'x. \tag{24}$$

In particular, we have

$$\dot{V}(x_1, x_2) < 0 \quad \forall x \in \mathbf{R}^{n_1} \times V_2^{-1}(\mu) - \{0, 0\}.$$

Hence, it holds

$$V(x_1(t), x_2(t)) \leqslant V(x_1(0), x_2(0)) \quad \forall t \geqslant 0.$$

Using (13), the above inequality gives

$$\int_0^{V_2(x_2(t))} s(w)\,\mathrm{d}w \leqslant V(x_1(0), x_2(0)) \quad \forall t \geqslant 0.$$

From property (14) of the function $s(\cdot)$, the above inequality implies $x_2(t) \in V_2^{-1}(\mu)$, $\forall t > 0$, i.e., $V_2^{-1}(\mu)$ is forward invariant for the driving subsystem with the controller $u_l(x_1, x_2)$. Therefore, (24) further implies that the state x is asymptotically steered to zero by the controller u_l while the state x_2 remains in $V_2^{-1}(\mu)$. From (24), we have

$$\dot{V}(x_1, x_2) \leqslant -\bar{\varepsilon}x'x \quad \forall x \in \mathbf{R}^{n_1} \times \bar{V}_2^{-1}(\rho\mu), \tag{25}$$

where

$$\bar{\varepsilon} = \min_{x_2 \in \bar{V}_2^{-1}(\rho\mu)} \varepsilon(s(V_2))^{-1}\lambda_{\min}(PS_0^2P).$$

The robust local quadratic stability follows from (25) and the fact that $V(x_1, x_2)$ is locally quadratic.

Next, the case where $x_2(0) \notin \bar{V}_2^{-1}(\rho\mu)$ is considered. It follows from Assumption 2 and Lemma 1 that x_2 is steered to the origin by $u_0(x_2)$ asymptotically while x_1 is bounded, thus there exists a finite time T such that $V_2(x_2(T)) = \rho\mu$. At T, the controller is switched to $u_l(x_1, x_2)$. It is shown in the first part of the proof that the state x is then steered to zero asymptotically by the controller. Therefore, this theorem is proven. □

Remark 2. The controller (19) has the following special property: $u(t)$ is a function of the state $x(t)$ and initial state $x_2(0)$. This is because the switching time T depends on $x_2(0)$. However, u is time-invariant in the sense that the function does not change when the initial time is shifted. This is because system (1) and the two controllers $u_0(x_2)$ and $u_l(x_1, x_2)$ are all time-invariant, which imply that the switching time T is also time-invariant. Since controller (19) is time-invariant, the stability result in Theorem 1 is also uniform with respect to the initial time (see notions of uniform stability in Khalil, 1996, pp. 134–135).

Remark 3. Although the switching decision in (19) is time based, this controller is not open loop because the switching time is a function of the state x_2. This point is important because the feedback nature of the controller gives some inherent robustness against possible small feedback noises. More precisely, the role of the parameter ρ is tolerate "small" feedback noises. The smaller the value of ρ, the larger the error margin is for the switching time T.

Remark 4. We comment that system (1) is assumed to have no additive input disturbances. If such a disturbance exists, the controller in Theorem 1 may experience a potential problem that state x_2 of the driving subsystem may leave $V_2^{-1}(\mu)$. One simple remedy is to reset the controller back to $u_0(x_2)$ whenever x_2 leaves $V_2^{-1}(\mu)$. If the disturbance has a finite L_2 norm, it can be argued that the result in Theorem 1 will still be valid.

Now, we extend the result in Theorem 1 to the case where $g_1(x_1, x_2, u, q) \neq 0$ and $g_2(x_2, u, q)$ is not linear in u.

To deal with this case, we introduce a new state vector $x_3 = u \in \mathbf{R}^m$ and a new control input $v = \dot{x}_3$. Then system (1) can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, q) + f_{12}(x_1, x_2, q) + g_1(x_1, x_2, x_3, q),\\
\dot{x}_2 &= f_2(x_2, q) + g_2(x_2, x_3, q),\\
\dot{x}_3 &= v.
\end{aligned} \tag{26}$$

Effectively, this allows the dynamics of x_2 and x_3 to form a new driving subsystem, and the new control input v does not appear the driven subsystem. Notice the fact that the function $g_1(x_1, x_2, x_3, q)$ is smooth in x_1, x_2, x_3 and continuous in q while $g_1(x_1, x_2, 0, q) \equiv 0$ for $\forall x_1 \in \mathbf{R}^{n_1}$, $\forall x_2 \in \mathbf{R}^{n_2}$ and $\forall q \in Q$. We have

$$g_1(x_1, x_2, x_3, q) = [B_{0,1}(q) + F_{13}(x_1, x_2, x_3, q)]x_3,$$

where $F_{13}(x_1, x_2, x_3, q)$ is smooth in x_1, x_2, x_3 and continuous in q, $B_{0,1}(q)$ is continuous in q. Moreover, there exists a smooth and positive definite function $\gamma_5(x_1, x_2, x_3)$ such that

$$\|F_{13}(x_1, x_2, x_3, q)\| \leqslant \gamma_5(x_1, x_2, x_3).$$

To solve the robust stabilization problem for system (1), a linear growth of $F_{13}(x_1, x_2, x_3, q)$ on x_1 is only required for the matrix function. But, for the sake of simplicity, we assume that:

Assumption 5 (*Extension growth constraint*). The function $\gamma_5(x_1, x_2, x_3)$ is independent from x_1, hence will be denoted by $\gamma_5(x_2, x_3)$.

The local linearized model of the system is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} A_{0,1}(q) & A_{0,12}(q) & B_{0,1}(q)\\ 0 & A_{0,2}(q) & B_{0,2}(q)\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ I\end{bmatrix}v, \tag{27}$$

in the region $R^{n_1} \times V_2^{-1}(\mu) \times [-\alpha, \alpha]^m$ where $[-\alpha, \alpha]^m \subset \mathbf{R}^m$, every element of a vector from $[-\alpha, \alpha]^m$ belongs to $[-\alpha, \alpha]$ and α is a sufficiently small positive constant.

Lemma 3. *Suppose that Assumption* 4 *holds. Then, the local linearized model* (27) *is quadratically stabilizable by*

$$v = -\gamma(x_3 - Kx)$$

for a sufficiently large $\gamma > 0$. *Furthermore,* P_1, *the* (1, 1)*-block of the Lyapunov matrix* (9), *can still be the* (1, 1)*-block of the Lyapunov matrix for* (27).

Proof. From Assumption 4, we know that

$$[A_0(q)+B_0(q)K]'(P_0+\delta P_0) + (P_0+\delta P_0)[A_0(q)+B_0(q)K]<0$$

for all $q \in Q$, provided that δP_0 is sufficiently small. Choose a Lyapunov function for (27) as follows:

$$V_0(x_1,x_2,x_3)=x'(P_0+\delta P_0)x+\varepsilon(x_3-Kx)'(x_3-Kx)$$

for some $\varepsilon>0$. Its derivative along the trajectory of (27) is given by

$$\dot V_0 = 2x'(P_0+\delta P_0)(A_0(q)x+B_0(q)x_3) + 2\varepsilon(x_3-Kx)'(v-K(A_0(q)x+B_0(q)x_3)).$$

Letting $\bar x_3 = x_3 - Kx$, we have

$$\dot V_0 = 2x'(P_0+\delta P_0)(A_0(q)+B_0(q)K)x + 2x'(P_0+\delta P_0)B_0(q)\bar x_3 - 2\varepsilon\bar x_3'K(A_0(q)+B_0(q)K)x + 2\varepsilon\bar x_3'(v-KB_0(q)\bar x_3).$$

Take $v=-\gamma\bar x_3=-\gamma(x_3-Kx)$. It is easy to verify that $\dot V_0<0$, $\forall q\in Q$ when $\gamma>0$ is sufficiently large. Hence, system (27) is quadratically stabilizable. To verify that P_1 can still be the (1, 1)-block of the Lyapunov matrix for (27), we simply take $\delta P_0 = -\varepsilon K'K$ and ε sufficiently small. This gives $V_0(x_1,0,0)=x_1'P_1x_1$. □

The stabilizing controller for (1) again consists of two parts: first, the global controller $u_0(x_2)$ is applied to (1) until x_2 is sufficiently small in some finite time T. Due to Assumption 2 that $u_0(x_2)$ is a smooth function, $u_0(x_2(T))=x_3(T)$ also belongs to a sufficiently small neighborhood at the origin in $\mathbf{R}^m$. Then treat the subsystem associated with x_2 and x_3 as a new driving system. Now apply Theorem 1 to come up with a controller $v_l(x_1,x_2,x_3)$ for $v(t)$. The resulting controller

$$u(t)=x_3(t)=\int_0^t v_l(x_1,x_2,x_3)\,\mathrm{d}t$$

will robustly stabilize system (1). This is summarized below:

Theorem 2. *Suppose that system* (1) *satisfies Assumptions* 1–4 *and* 5. *Then the system is robustly globally asymptotically stabilizable and locally quadratically stabilizable.*

Proof. The proof simply follows from Theorem 1 and Lemma 3. □

4. Simulation

We now illustrate the design procedure in Section 3 by the following example:

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \begin{bmatrix}-x_1+x_2+q_{13}x_3+0.5x_1x_4^2+q_{14}x_4\\ x_3-q_{24}(x_4-x_4^2)\\ x_4\\ -2x_3-2x_4+u\end{bmatrix}, \tag{28}$$

where $q=[q_{13},\ q_{14},\ q_{24}]\in[-0.1,\ \ 0.1]^3$. System (28) is the cascade interconnection of two subsystems

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix}=\begin{bmatrix}-1&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}q_{13}x_3+0.5x_1x_4^2+q_{14}x_4\\ x_3-q_{24}(x_4-x_4^2)\end{bmatrix} \tag{29}$$

and

$$\begin{bmatrix}\dot x_3\\ \dot x_4\end{bmatrix}=\begin{bmatrix}0&1\\-2&-2\end{bmatrix}\begin{bmatrix}x_3\\x_4\end{bmatrix}+\begin{bmatrix}0\\u\end{bmatrix}. \tag{30}$$

Select

$$V_1(x_1,x_2)=[x_1\ \ x_2]P_1\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad P_1=\begin{bmatrix}5&-5\\-5&7\end{bmatrix},$$

$$V_2(x_3,x_4)=[x_3\ \ x_4]P_2\begin{bmatrix}x_3\\x_4\end{bmatrix},\quad P_2=\begin{bmatrix}6&3\\3&3\end{bmatrix}$$

as Lyapunov functions of subsystems (29) and (30), respectively. Select $\mu=0.3$ to give

$$V_2^{-1}(0.3)=\{(x_3,\ x_4)'|V_2(x_3,x_4)<0.3\}.$$

The matrices $A(x,q)$ and $B(x,q)$ are given by

$$A(x,q)=\begin{bmatrix}-1+0.5x_4^2&1&q_{13}&q_{14}\\0&0&1&-q_{24}(1-x_4)\\0&0&0&1\\0&0&-2&-2\end{bmatrix},$$

$$B(x,q)=[0\ \ 0\ \ 0\ \ 1]'.$$

Choose

$$P_0=\begin{bmatrix}5&-5&-3&-1\\-5&7&5&2\\-3&5&12&3\\-1&2&3&3\end{bmatrix},\quad K=[2\ \ -4\ \ -6\ \ -6].$$

It is easily verified that (15) holds. Following Theorem 1, we obtain the robust controller as follows:

$$u=\begin{cases}0, & 0\leqslant t\leqslant T,\\ (1-0.3333V_2)(2x_1-4x_2) & \\ \quad +(10.6667V_2-6)x_3 & t>T,\\ \quad +(4.6667V_2-6)x_4, & \end{cases} \tag{31}$$

where T is the first time when the state $(x_3,\ x_4)'$ of subsystem (30) enters the region $\bar V_2^{-1}(\rho\mu)$ and $\rho=0.8$. It is verified that this controller is robustly globally asymptotically stabilizing. Fig. 1 shows the time response of the state (x_1,x_2,x_3,x_4) and the control input u of system (28) with initial state $(x_1(0),x_2(0),x_3(0),x_4(0))=(1,1,1,1)$ and $q_{13}=q_{14}=q_{24}=-0.1$. We can see from Fig. 1 that the controller switches at about $t=2.2$ s.

5. Conclusion

In this paper, we have studied the robust stabilization problem for a class of uncertain cascaded nonlinear systems. A two-step control design method is developed for the uncertain nonlinear systems. Although this method is conceptually

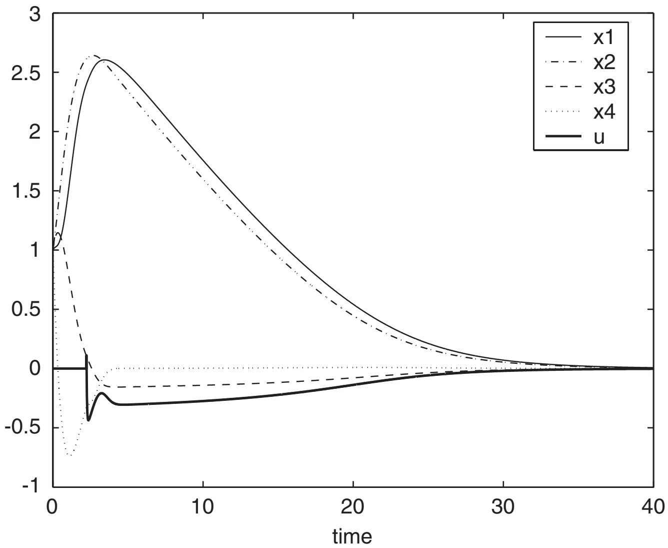

Fig. 1. State and control trajectories.

simple, the key technical difficulty is how to 'fuse' two controllers together, where each controller works for a given state region. It is generally inappropriate to simply switch from one controller to another when the state crosses from one region to another, even when the two regions overlap. Appropriate modifications are usually necessary for the controllers and the associated Lyapunov functions. We stress that the existing techniques for combining a global controller with a local controller, as in Teel and Kapoor (1997), and Prieur and Praly (1999), do not apply in our case because our second controller is not a local controller, i.e., it does not operate only in a neighborhood around the origin. In our case, a given local controller and its associated quadratic Lyapunov function are modified as in (19) and (13) to ensure the required stabilization properties, which turns out to be a major technical task.

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References

Arcak, M., Angeli, D., & Sontag, E. (2002). A unifying integral ISS framework for stability of nonlinear cascades. *SIAM Journal on Control and Optimization*, *40*(6), 1888–1904.

Arcak, M., Teel, A., & Kokotovic, P. (2001). Robust nonlinear control of feedforward systems with unmodeled dynamics. *Automatica*, *37*(2), 265–272.

Jankovic, M., Sepulchre, R., & Kokotovic, P. V. (1996). Constructive Lyapunov stabilization of nonlinear cascade systems. *IEEE Transactions on Automatic Control*, *41*(12), 1723–1735.

Jiang, Z. P., Teel, A. R., & Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Mathematics of Control Signals and Systems*, *7*, 95–120.

Kaliora, G., & Astolfi, A. (2001). A simple design for the stabilization of cascaded nonlinear systems with bounded control. In *Proceedings of 40th conference on decision and control*, Orlando, FL. Also appeared as Nonlinear control of feedforward systems with bounded signals. *IEEE Transactions on Automatic Control*, *49*(11), 1975–1990, 2004.

Khalil, H. K. (1996). *Nonlinear systems* (2nd ed.), Englewood Cliffs, NJ: Prentice-Hall.

Marconi, L., & Isidori, A. (2000). Robust global stabilization of a class of uncertain feedforward nonlinear systems. *Systems and Control Letters*, *41*(4), 281–290.

Mazenc, F., & Praly, L. (1996). Adding integrations saturated controls and stabilization for feedforwarding systems. *IEEE Transactions on Automatic Control*, *41*(11), 1559–1578.

Prieur, C., & Praly, L. (1999). Uniting local and global controllers. *Proceedings of IEEE conference on decision and control* (pp. 1214–1219). Also in *Mathematics of Control*, *Signals and Systems*, *14*, 143–172, 2001.

Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, *34*(4), 435–443.

Su, W., & Fu, M. (1998). Robust stabilization of a class of nonlinear systems upper-triangular structure. In *Proceedings of IEEE conference on decision and control* (pp. 1983–1988).

Su, W., & Fu, M. (1999). Robust nonlinear forwarding with smooth state feedback control. In *Proceedings of IFAC'99 world congress*, F (pp. 435–440), Beijing, China.

Teel, A. (1992). *Feedback stabilization*: *Nonlinear solutions to inherently nonlinear problems*. Memorandum no. UCB/ERL M92/65.

Teel, A., & Kapoor, N. (1997). Uniting local and global controllers. In *Proceedings of European control conference*, Brussels.

Teel, A., & Praly, L. (1995). Tools for semiglobal stabilization by partial state and output-feedback. *SIAM Journal on Control and Optimization*, *33*(5), 1443–1488.

Wei, K. (1990). Quadratic stabilizability of linear systems with structural independent time-varying uncertainties. *IEEE Transactions on Automatic Control*, *35*(3), 268–277.

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