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Minimum switching control for adaptive tracking

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SUMMARY

The switching adaptive control method has been used for quite a few years to solve the adaptive stabilization and model reference adaptive control problems. However, a serious problem with the switching control method is that the number of 'candidate' controllers can potentially be very large, especially for multi-input–multi-output systems. In this paper, we consider a class of minimum-phase multi-input–multi-output plants with some mild compactness assumptions. Given any polynomial reference input, we provide a switching control law which guarantees exponentially stability of the closed-loop system with exponential tracking performance. The main contribution of the paper is that we give the minimum number of candidate controllers required for switching. In particular, the number is equal to 2 for single-input–single-output plants (one for each sign of the high-frequency gain), and is equal to 2^m for m-input–m-output plants. That is, the number is independent of the degree and the relative degree of the plant. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: switching adaptive control; adaptive tracking; supervisory control; multi-input–multi-output adaptive control

1. INTRODUCTION

Most of the classical model reference adaptive control methods (works *priori* to 1980) are based on the following set of basic assumptions:

- The plant is of minimum phase;
- An upper bound of the plant degree is known;
- Its relative degree is known;
- Its sign of the high frequency gain is known; and
- The reference model has the same relative degree as the plant.

See, for example, References [1, 2] for overviews. It is well recognized that this set of assumptions are often unrealistic in practical applications where the plant may be difficult to

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model. Adaptive controllers designed based on these assumptions may become non-robust, as shown by well-known examples of Rohrs *et al.* [3].

One of the important lines of adaptive control research in recent years is to investigate the minimal set of assumptions needed for the plant so that it can be adaptively stabilized. This line of research can be traced back to a paper by Morse [4] which raised a number of open questions regarding the classical assumptions. The first breakthrough was given in a paper by Nussbaum [5] which provides a new adaptation method (called Nussbaum gain later) for treating the case where the sign of the high frequency gain of the plant is unknown. Nussbaum's result was generalized by Martensson [6] which shows surprisingly that asymptotic stabilization of a minimal plant can be achieved with a rather weak assumption, i.e. one only needs to know the degree of a stabilizing controller. In fact, even this condition can be relaxed. Because Martensson's approach involves an exhaustive on-line search over the space of candidate gain matrices before 'latching on' to an appropriate stabilizer, two serious problems arise: (1) Lyapunov stability cannot be guaranteed and, consequently, an excessive overshoot may occur; (2) the output must be free of (even arbitrarily small) persistent measurement noises to avoid possible destabilization. These problems have been reported and carefully analysed in a paper by Fu and Barmish [7].

An alternative approach to adaptive stabilization, called switching adaptive control, was proposed by Fu and Barmish [7] to assure Lyapunov stability (in fact, exponential stability) and to permit small measurement noises. More explicitly, Fu and Barmish show that adaptive stabilization of a family of unknown multi-input–multi-output (MIMO) plants Σ can be achieved if the following mild assumptions are satisfied:

- The upper bound $n_{\max}$ of the degree of the plant family is known;
- Every member of Σ is stabilizable and detectable;
- For each $n \leqslant n_{\max}$, the set of state-space realization matrices of the subfamily Σ_n of plants with degree n is compact (i.e. bounded and closed), and the compact set is known;

Indeed, it is shown in Reference [7] that there exists a finite number of fixed linear time-invariant controllers (which will be called candidate controllers in the sequel) such that every member of Σ will be stabilized by at least one of them. In other words, Σ has a finite partition $\Sigma = \bigcup_{i=1}^{N} \Sigma_i$ for a finite N such that each Σ_i admits a single robust stabilizer. Consequently, a switching mechanism is applied on-line to search for a correct controller for an arbitrary unknown member of Σ. The resulting controller is piecewise linear time-invariant with at most a finite number of switchings. The closed-loop system is guaranteed to be exponentially stable, and robust with respect to small measurement noises. Further, an extension of this result is given in Reference [8] to treat the case where singular perturbations to the plants exist, i.e. the compactness assumption above is violated.

A slightly different switching control approach, called hysteresis switching, is also reported in a series of papers by Middleton *et al.* [9], Morse *et al.* [10], and Weller and Goodwin [11] to solve the problem of model reference adaptive control. No compactness assumption is required for this approach. However, the family of plants Σ to be dealt with need to satisfy the following assumptions:

- Every member of Σ is of minimum phase;
- Every member of Σ is stabilizable and detectable; and
- An upper bound $n_{\max}$ of the plant degree is known.

The basic idea in Morse *et al.* [10] and Weller and Goodwin [11] involves two steps. The first step is to construct an estimator for each subfamily of plants with the same McMillian degree, the same relative degree, the same high-frequency gain sign, and the same 'permutation' of outputs. Subsequently, a classical model reference adaptive controller is designed for each such subfamily. The second step is to use a so-called hysteresis switching algorithm based on Middleton *et al.* to adaptively select a correct controller.

The number of candidate controllers for an uncertain plant family can potentially be very large. For example, the method in Reference [11] requires $2^m \times m! \times mn_{\max}$ candidate controllers for an m-input–m-output plant. For example, even for $m = 5$ and $n_{\max} = 10$ which are very moderate, this number is equal to 192 000. The excessively large number of candidate controllers mean that it may potentially take an extremely long dwell time before a correct controller can be found, assuming a 'pre-routed' switching law is used. To alleviate this problem, several methods have been reported recently. The so-called *supervisory control approach* is proposed by Morse [12, 13] to improve the transient response. The main idea of the supervisory control approach is to apply an 'optimal' candidate controller based on certain on-line estimation rather than sequentially eliminating invalid controllers. This approach has also been studied by Hocherman-Frommer *et al.* [14, 15], Narendra and Balakrishnan [16, 17], Narendra and Xiang [18], Hespanha [19], and Hespanha *et al.* [20]. Nice simulation results have been demonstrated in these papers. A different approach to speeding up the switching process is the *localization approach* proposed by Zhivoglyadov *et al.* [21–25] where a fast algorithm is introduced to prune 'bad' candidate controllers, and therefore a 'good' set of candidate controllers is quickly localized. Another method which uses a falsification idea is introduced by Safonov and his co-workers [26, 27] in the context of model reference adaptive control. More references on switching adaptive control can be found in Reference [28].

Regardless of the switching method, one of the key problems in switching adaptive control is how to partition a given uncertain plant family Σ. Recall that the partitioning needs to be done such that each subfamily Σ_i admits a single robust stabilizer. Typically, assumptions are imposed to guarantee the existence of a finite partition for the given Σ. But the partitioning process is usually done by trial and error, which often leads to a large partition cardinality N. Having a large N requires a lot of design and storage of candidate controllers because the design is typically done off-line. Furthermore, a large N tends to complicate the pruning process in the switching process.

For a general uncertain plant family, the partitioning problem can be quite difficult. In this paper, we study a particular class of uncertain plant families for which the partitioning problem can be solved. More precisely, we consider an adaptive tracking problem for a family of uncertain plants. That is, given a family of uncertain plants and a reference signal, we want to design an output feedback controller such that the closed-loop system corresponding to any plant in the family is exponentially stable and its output exponentially approaches the reference signal. The family of plants we consider in this paper have m inputs and m outputs and are assumed to be minimum phase invariant and to satisfy some boundedness assumptions. The reference signal is assumed to be a polynomial function. We show that the minimum partition cardinality is given by 2^m. Furthermore, the corresponding minimum partition has the property that any other partition must be a refinement of this minimum partition in the sense that each partitioned subfamily must be obtained by further dividing the minimum partition.

Our approach involves two key ideas: (1) We divide the family of plants into 2^m subfamilies, each robustly stabilizable by a single linear time-invariant controller. This step is based on an important paper on robust stabilization by Wei and Barmish [29]. By modifying their controller and relaxing the assumptions they use for the controller design, we show that each subfamily can be controlled by a single linear time-invariant controller such that the closed-loop system associated with any member of the subfamily is exponentially stable and its output exponentially tracks the reference signal. (2) Once these 2^m candidate controllers have been determined, any switching adaptive control algorithm can be applied. In particular, a simple switching algorithm similar to the one in Fu and Barmish [7] can be employed on-line to search for a correct controller. After at most $2^m - 1$ switchings, a correct controller will be found for an arbitrary member of the plant family.

The rest of this paper is organized as follows: Section 2 formalizes the adaptive tracking problem and assumptions; Section 3 considers the design of candidate controllers; Section 4 provides the switching algorithm and the main result on exponential stability and tracking; and Section 5 concludes with some remarks.

2. PROBLEM FORMULATION AND ASSUMPTIONS

Let $\mathscr{R}^{m\times m}[s]$ denote the set of all $m \times m$ rational matrices. Given a set of rational matrices $\Sigma \subset \mathscr{R}^{m\times m}[s]$, representing a family of uncertain plants, and a polynomial time function $r(\cdot) : \mathscr{R} \to \mathscr{R}^m$, the adaptive tracking problem considered in this paper is as follows: *Find an adaptive controller C as depicted in Figure 1 such that for any $G_P(s) \in \Sigma$, the closed-loop system is exponentially stable and its output $y(t)$ will exponentially approach $r(t)$, i.e.*

$$\|y(t) - r(t)\| \leqslant M\mathrm{e}^{-\lambda t} \tag{1}$$

for some $M > 0$ and $\lambda > 0$.

Before we introduce the assumptions on the plant family we need to be involved with some notational matters.

Definition 1 (*Wei and Barmish [29]*)
Given $G(s) \in \mathscr{R}^{m\times m}[s]$ and two $m \times m$ polynomial matrices $N(s)$ and $D(s)$, the pair $(N(s), D(s))$ is called a row *Hermite factorization* if the following conditions hold:

1. $D(s)$ is invertible and $G(s) = N(s)D^{-1}(s)$;
2. $N(s)$ and $D(s)$ are coprime in the closed right-half plane;
3. $D_{ii}(s)$ is a monic polynomial for $i = 1, \ldots, m$;
4. $D_{ij}(s) = 0$ for all $i < j$;
5. $\deg D_{ij}(s) < \deg D_{ii}(s)$ for all $i > j$,

where $D_{ij}(s)$ is the ijth element of $D(s)$.

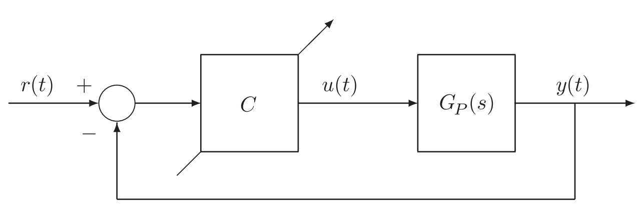

Figure 1. Adaptive tracking problem.

Remark 1
It is known that there always exists a row Hermite factorization for any rational matrix, as pointed out in Reference [29]. This factorization is actually unique if the coprimeness condition above is strengthened to include the open left-half plane. The reason we use a weaker coprimeness condition is to allow a simpler factorization for parameterized rational matrices. For example, a row Hermite factorization of

$$G(s,q) = \frac{s+1}{s+q}, \quad q \in [1/2, 2]$$

is given by

$$N(s,q) = s+1; \quad D(s,q) = s+q$$

when the weaker version of coprimeness condition is used. For the stronger version of coprimeness condition, the row Hermite factorization of $G(s,q)$ at $q=1$ must be given by

$$N(s) = 1, \quad D(s) = 1$$

which causes discontinuity.

Based on the remark above, we can express Σ in an equivalent form

$$(\mathcal{N}_P, \mathcal{D}_P) = \{(\mathcal{N}_P(s), \mathcal{D}_P(s)) : \text{ a row Hermite factorization for } G_P(s) \in \Sigma\} \tag{2}$$

However, for notational simplicity, we will also denote $(\mathcal{N}_P, \mathcal{D}_P)$ by Σ unless confusion arises.

Remark 2
Using duality, we can define the so-called column Hermite factorization $(\tilde{D}(s), \tilde{N}(s))$ for every $G(s)$, i.e. $G(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$. All the properties above about the row Hermite factorization also apply using duality to the column Hermite factorization. In this paper, row Hermite factorization will be used for the plant and column Hermite factorization, for the controller.

Definition 2
A given family of polynomials $\mathcal{P}$ is said to be of *degree d* if every polynomial $p(s) \in \mathcal{P}$ is of degree d. $\mathcal{P}$ (possibly with different degrees) is called *spectrally bounded* if the set of zeros of $\mathcal{P}$ is a bounded set. The *closure* of $\mathcal{P}$ is defined to be the set of all limiting polynomials convergent from a sequence of polynomials of the same degree in $\mathcal{P}$. A family of polynomial matrices is called spectrally bounded if every matrix element family is spectrally bounded.

Remark 3
A few comments on the boundedness condition are in order. If a family of polynomials $\mathcal{P}$ contains a zero polynomial (which is identically equal to zero), then our definition of spectral boundedness implies that $\mathcal{P}$ is not spectrally bounded because the zero polynomial has zeros everywhere. In fact, $\mathcal{P}$ with maximum degree d is spectrally bounded if and only if the following conditions hold:

- It does not contain a zero polynomial;
- For every $1 \leqslant d < \bar{d}$, the subfamily of polynomials in $\mathcal{P}$ with degree d has the following property:
  - The set of polynomial coefficients is a bounded set in $\mathcal{R}^{d+1}$;

— There exists some $\delta > 0$ such that the absolute value of the leading coefficient of every polynomial in the subfamily is no less than δ.

Denote by $z_i(s)$ the ith lower principal minor of $N_P(s)$, i.e. $z_i(s)$ is the determinant of the part of $N_P(s)$ with the first $(i-1)$ rows and columns deleted. Further denote the families of polynomials

$$\mathscr{Z}_i = \{z_i(s) : N_P(s) \in \mathscr{N}_P\}; \quad i = 1, \ldots, m \tag{3}$$

We will adopt the following set of assumptions in the rest of the paper.

Assumption A1 (*Minimum phase invariance*)
det $N_P(s)$ is Hurwitz for every member $N_P(s)$ in the closure of $\mathscr{N}_P$.

Assumption A2 (*Upper bound of degree*)
The upper bound of the degree $d_{\max}$ of $D_{ij}(s)$ over $\mathscr{D}_P$ is known.

Assumption A3 (*Spectral boundedness of numerator*)
Z_i is spectrally bounded for every $i = 1, \ldots, m$.

Assumption A4 (*Spectral boundedness of denominator*)
$\mathscr{D}_P$ is spectrally bounded.

Remark 4
The Assumptions A1–A2 are similar to those used in References [10, 11]. The boundedness assumptions will enable us to significantly reduce the number of candidate controllers and to guarantee many other nice properties such as exponential stability and linear piecewise-invariant control. Note that the spectral boundedness assumptions are rather weak in view of Remark 3.

Finally we define the *maximum degree* of the reference signal $r(t)$ to be

$$n_r = \max\{n_1, \ldots, n_m\} \tag{4}$$

where n_i the polynomial degree of the ith component of $r(t), i = 1, \ldots, m$.

3. DESIGN OF CANDIDATE CONTROLLERS

Our method for designing candidate controllers is motivated by a robust stabilization approach of Wei and Barmish [29]. These authors consider a family of uncertain plants satisfying assumptions similar to A1–A4 (slightly stronger though) and the following additional one.

Assumption A5
The leading coefficient of every principal minor $z_i(s)$ of $N_P(s)$ is either positive invariant or negative invariant over $\mathscr{N}_P$.

With this additional assumption, it is shown in Reference [29] that there exists a single linear time-invariant controller to robustly stabilize the whole family of plants.

Our first result, Lemma 1 shows that a given family of plants satisfying Assumptions A1–A4 can be decomposed into 2^m subfamilies such that each subfamily will satisfy not only Assumptions A1–A4 but also Assumption A5. Using the design approach of Wei and Barmish [29], we can find a linear time-invariant stabilizer for each of the 2^m subfamilies of plants. Consequently, 2^m candidate controllers can be designed to cover the whole family Σ.

Lemma 1

Given a family of transfer matrices Σ in (2) satisfying Assumptions A1–A4, let $Z_i, i = 1, \ldots, m$ be given by (3) and define

$$\mathscr{Z}_i^+ = \{z_i(s) : z_i(s) \in \mathscr{Z}_i \text{ with positive leading coefficient}\} \tag{5}$$

$$\mathscr{Z}_i^- = \{z_i(s) : z_i(s) \in \mathscr{Z}_i \text{ with negative leading coefficient}\} \tag{6}$$

Given any sign vector

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad \alpha_i \in \{-, +\}, \quad i = 1, \ldots, m \tag{7}$$

define

$$\Sigma_\alpha = \{(N_P(s), D_P(s)) \in \Sigma, z_i(s) \in \mathscr{Z}_i^{\alpha_i}\} \tag{8}$$

Then, we have the following properties:

(i)
$$\Sigma = \bigcup_\alpha \Sigma_\alpha \tag{9}$$

(ii) Each subfamily Σ_α satisfies Assumptions A1–A5.

Proof

By Assumption A3 and Remark 3, $\mathscr{Z}_i$ does not contain a zero polynomial for every $i = 1, \ldots, m$. This implies

$$\mathscr{Z}_i = \mathscr{Z}_i^+ \cup \mathscr{Z}_i^-, \quad i = 1, \ldots, m \tag{10}$$

which in turn implies Condition (i). Furthermore, the sets $\mathscr{Z}_i^+$ and $\mathscr{Z}_i^-$ are spectrally bounded because $\mathscr{Z}_i$ is spectrally bounded. This implies that Assumptions A1–A4 are also obvious for each Σ_α. □

Remark 5

Since each $\mathscr{Z}_i^+$ and $\mathscr{Z}_i^-$ have opposite signs for the leading coefficient, it is clear that plants in different Σ_α in (8), if unstable, do not share a common linear time-invariant stabilizer. Therefore, any other partition of Σ must be a refinement of the partition in (9) in the sense that it must be obtained by further dividing the partition in (9), provided that the plants in Σ are all unstable. This is an important property of the partition in (9).

Our next step is to show that each Σ_α satisfying Assumptions A1–A5 admits a linear time-invariant robust stabilizer. For this, we refer to Reference [29] and note that the assumptions required there for robust stabilization can be restated as Assumptions A1, A2, A5 and the following:

Assumption (a)
The uncertain polynomials are parameterized using some uncertain parameter vector q, i.e. $N_P(s)$ and $D_P(s)$ are expressed as $N_P(s,q)$ and $D_P(s,q)$, respectively. In addition, $q \in Q$ for some compact set Q.

Assumption (b)
$N_P(s,q)$ and $D_P(s,q)$ are continuous in q, $q \in Q$.

Assumption (c)
For every $d \in d_{\max}$ and $1 \leqslant i, j \leqslant m$, the subset of the (i,j)th element of $N_P(s,q)$ (resp. $D_P(s,q)$) with degree d, when q ranges in Q, is a compact set.

It is not difficult to see that Assumptions (a), (b) and (c) imply Assumptions A3–A4. A careful examination of the proof in Reference [29] shows that the properties in Assumptions (a)–(c) used in the proof are only those in Assumptions A3–A4. Hence, the robust stabilization result in Reference [24] still applies when the assumptions are relaxed. The details are given below.

Theorem 1
Consider a family of uncertain plants $\Sigma_\alpha \subset \mathscr{R}^{m\times m}[s]$ satisfying Assumptions A1–A5. Then, for any reference signal polynomial reference signal $r(\cdot) : \mathscr{R} \to \mathscr{R}^m$ of degree $n = (n_1, \ldots, n_m)$, there exists an $m \times m$ rational matrices $C(s)$ such that the closed-loop system associated with any uncertain plant $G_P(s) \in \Sigma_\alpha$, as depicted in Figure 2, is Hurwitz stable and its output $y(t)$ exponentially tracks $r(t)$.

The rest of this section is devoted to the proof of the theorem above. A controller design procedure will be introduced at the same time. The proof and design procedure are modified from References [29, 30]. Two steps are involved: The first step is to cascade the plant by an integrator matrix so that the tracking problem becomes a robust stabilization problem. The second step is to apply a design procedure similar to Wei and Barmish [29] to achieve robust stabilization.

3.1. Step 1: Conversion of robust tracking to robust stabilization

Recall that the maximum degree of the reference signal is given by n_r. Define the integrator matrix

$$I(s) = \mathrm{diag}\{s^{-n_r}, \ldots, s^{-n_r}\} \in \mathscr{R}^{m\times m}[s] \tag{11}$$

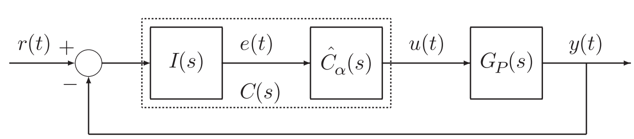

Figure 2. Tracking problem.

the family of cascaded plants

$$\hat{\Sigma} = \{\hat{G}_P(s) = G_P(s)I(s) : G_P(s) \in \Sigma\}$$
$$= \{(N_P(s), D_P(s)I^{-1}(s)) : (N_P(s), D_P(s)) \in \Sigma\} \tag{12}$$

and their subfamilies $\hat{\Sigma}_\alpha$. We will denote

$$\hat{D}_P(s) = D_P(s)I^{-1}(s)$$

Since Σ_α satisfies Assumptions A1–A5, it is easy to verify that $\hat{\Sigma}_\alpha$ still satisfies Assumptions A1–A5. Using Part (ii) of Lemma 1, we know that there exists a robust stabilizer for $\hat{\Sigma}_\alpha$. Further, the following fact is well-known: If there exists a controller $\hat{C}_\alpha(s) \in \mathscr{R}^{m\times m}[s]$ which robustly exponentially stabilizes $\hat{\Sigma}_\alpha$, then the following controller:

$$C_\alpha(s) = I(s)\hat{C}_\alpha(s) = \hat{C}_\alpha(s)I(s) \tag{13}$$

will robustly exponentially stabilize Σ_α and guarantee the exponential tracking requirement for any reference signal $r(t)$ with maximum degree n_r. This is illustrated in Figure 2.

3.2. *Step 2: Robust stabilization of* $\hat{\Sigma}_\alpha$

This step is constructive. That is, a design procedure is given for $\hat{C}_\alpha(s)$. This procedure is identical to a robust stabilization procedure in Reference [29].

Step 2.1: Choose

$$T_m = \mathrm{diag}\{\pm 1, \pm 1, \ldots, \pm 1\}$$

such that all the lower principal minors $\hat{z}_i(s)$, $i = 1, 2, \ldots, m$, of

$$\hat{N}_P(s) = T_m N_P(s)$$

have positively invariant leading coefficients. We also define $\hat{z}_{m+1}(s) = 1$.

Step 2.2: The controller will be of the form

$$\hat{C}_\alpha(s) = A_m^{-1}(s)B_m(s)T_m \tag{14}$$

where $A_m(s)$ and $B_m(s)$ are determined recursively, i.e. for $i = 1, 2, \ldots, m$

$$A_i(s) = \mathrm{diag}\{a_1(s), a_2(s), \ldots, a_i(s), 0, 0, \ldots, 0\}$$
$$B_i(s) = \mathrm{diag}\{b_1(s), b_2(s), \ldots, b_i(s), 1, 1, \ldots, 1\}$$

where the polynomials $a_i(s)$ and $b_i(s)$ are Hurwitz (but specified later). In addition, for $i = 1, 2, \ldots, m$, we define

$$V_i(s) = \mathrm{diag}\{a_1(s), a_2(s), \ldots, a_i(s), 1, 0, \ldots, 0\}$$
$$W_i(s) = \mathrm{diag}\{b_1(s), b_2(s), \ldots, b_i(s), 0, 1, \ldots, 1\}$$
$$\Delta_i(s) = \det(A_i(s)\hat{D}_P(s) + B_i(s)\hat{N}_P(s))$$
$$\Delta_i'(s) = \det(V_i(s)\hat{D}_P(s) + W_i(s)\hat{N}_P(s))$$

and will denote by $\pi_i(s)$ the determinant of the top left $i \times i$ submatrix of $A_i(s)\hat{D}_P(s) + B_i(s)\hat{N}_P(s)$.

Step 2.3 (*Initialization*): Take

$$A_0(s) = 0$$
$$B_0(s) = I$$
$$\pi_0(s) = 1$$
$$V_0(s) = \mathrm{diag}\{1, 0, \ldots, 0\}$$
$$W_0(s) = \mathrm{diag}\{0, 1, \ldots, 1\}$$

which gives

$$\Delta_0(s) = \det(A_0(s)\hat{D}_P(s) + B_0(s)\hat{N}_P(s)) = \det \hat{N}_P(s)$$
$$\Delta_0'(s) = \det(V_0(s)\hat{D}_P(s) + W_0(s)\hat{N}_P(s)) = \hat{d}_{11}(s)\hat{z}_2(s)$$

where $\hat{d}_{11}(s)$ is the (1,1) entry of $\hat{D}_P(s)$.

Step 2.4 (*Recursion*): For any $0 \leqslant i \leqslant m-1$, select an arbitrary Hurwitz polynomial $b_{i+1}(s)$ with a positive leading coefficient and degree k_{i+1} satisfying

$$k_{i+1} \geqslant \deg \Delta_i'(s) - \deg \Delta_i(s) - 1 \quad \forall \hat{N}_P(s), \hat{D}_P(s) \tag{15}$$

To explain the design procedure for $a_{i+1}(s)$, we observe that

$$\Delta_{i+1}(s) = a_{i+1}(s)\Delta_i'(s) + b_{i+1}(s)\Delta_i(s)$$
$$\Delta_{i+1}'(s) = \pi_i(s)d_{(i+1)(i+1)}(s)\hat{z}_{i+1}(s)$$

Denote

$$a_{i+1}(s) = \sum_{j=0}^{l_{i+1}} \alpha_{i+1,j} s^j$$

where l_{j+1} is the order of $a_{i+1}(s)$, and denote

$$\Delta_{i+1,k}(s) = \left(\sum_{j=0}^{k} \alpha_{i+1,j} s^j\right)\Delta_i'(s) + b_{i+1}(s)\Delta_i(s), \quad k = 0, 1, \ldots, l_{i+1}$$

We have

$$\Delta_{i+1,k+1}(s) = \alpha_{i+1,k+1} s^{k+1}\Delta_i'(s) + \Delta_{i+1,k}(s), \quad k = 0, 1, \ldots, l_{i+1} - 1 \tag{16}$$

and $\Delta_{i+1}(s) = \Delta_{i+1,l_{i+1}}(s)$. The design procedure for $a_{i+1}(s)$ is given below.

Step 2.4.1 (*Recursion*): For $k = 0, 1, 2, \ldots,$ choose $\alpha_{i+1,k} > 0$ to guarantee that $\Delta_{i+1,k}(s)$ is Hurwitz invariant and spectrally bounded. If required, $\alpha_{i+1,k}$ can be chosen such that $a_{i+1,k}(s)$ is also Hurwitz.

Step 2.4.2 (*Termination*): Terminate the recursion when $\pi_{i+1}(s)$ has a positively invariant leading coefficient. Set $l_{i+1} = k$ at the termination.

3.3. *Step 3: Proof of robust stabilization*

Suppose the recursion in Step 2.4 succeeds. We will get $\hat{C}_\alpha(s)$ in (14) such that

$$\Delta_m(s) = \det(A_m(s)\hat{D}_P(s) + B_m(s)T_m N_P(s))$$

is Hurwitz invariant. This will imply that $\hat{C}_\alpha(s)$ is a robust stabilizer for $\hat{\Sigma}_\alpha$. Hence, it remains to show that Step 2.4 indeed succeeds.

By Assumptions A1 and A3, $\Delta_0(s)$ is Hurwitz invariant and spectrally bounded. Now we can start the recursion. Take any $0 \leqslant i \leqslant m-1$, by Assumption A2, k_{i+1} in (15) is finite. Also, the choice of k_{i+1} means that

$$\deg \Delta_i'(s) \leqslant \deg b_{i+1}(s) + \deg \Delta_i(s) + 1$$

To see the success of Step 2.4.2 when l_{i+1} is sufficiently large, an inductive argument is used. It is straightforward to check that

$$\pi_1(s) = a_1(s)\hat{d}_{11}(s) + b_1(s)\hat{n}_{11}(s)$$

where $\hat{n}_{ij}(s)$ is the (i,j) entry of $\hat{N}_P(s)$. Hence, it suffices to have l_1 such that

$$\deg a_1(s)\hat{d}_{11}(s) > \deg b_1(s)\hat{n}_{11}(s)$$

for all $\hat{N}_P(s)$ and $\hat{D}_P(s)$. Similarly,

$$\pi_2(s) = \det \begin{bmatrix} a_1\hat{d}_{11} + b_1\hat{n}_{11} & b_1\hat{n}_{12} \\ a_2\hat{d}_{21} + b_2\hat{n}_{21} & a_2\hat{d}_{22} + b_2\hat{n}_{22} \end{bmatrix}$$

Again, if l_2 is such that

$$\deg a_2(s)\hat{d}_{22}(s) > \deg b_2(s)\hat{n}_{22}(s)$$

for all $\hat{N}_P(s)$ and $\hat{D}_P(s)$, the leading coefficient of $\pi_2(s)$ is positively invariant. The argument above can be easily generalized to $\pi_3(s), \pi_4(s), \ldots$.

The success of Step 2.4.1 can also be shown by induction. Firstly, we can guarantee that $\Delta_{i+1,0}(s)$ is Hurwitz invariant and spectrally bounded when $\alpha_{i+1,0} > 0$ is sufficiently small. This is because $b_{i+1}(s)\Delta_i(s)$ is Hurwitz invariant and spectrally bounded and $\pi_i(s)$ is spectrally bounded with a positive leading coefficient. Indeed, all the zeros of $\Delta_{i+1,0}(s)$ will approach those of $b_{i+1}(s)\Delta_i(s)$ when $\alpha_{i+1,0} \to 0$ with the possible exception of one zero approaching $-\infty$ if

$$\deg \Delta_i'(s) = \deg b_{i+1}(s) + \deg \Delta_i(s) + 1$$

The existence of the required $\alpha_{i+1,k}$ for $k > 0$ is similarly guaranteed if we note (16), the Hurwitz invariance and spectral boundedness of $\Delta_{i+1,k}(s)$ and that

$$\deg s^{k+1}\Delta_i'(s) \leqslant \deg \Delta_{i+1,k}(s) + 1$$

Using a similar argument as above, when $\alpha_{i+1,k}(s)$ is Hurwitz, $a_{i+1,k+1}(s)$ can always be made Hurwitz by choosing $\alpha_{i+1,k+1} > 0$ small enough. Hence, Step 2.4.1 always succeeds. This ends our proof.

Remark 6

Given a family of plants Σ satisfying Assumptions A1–A4, Theorem 1 implies that at most 2^m candidate controllers are sufficient to cover the whole Σ. We point out that this number is also minimal if there are no further assumptions available. To see this, consider the following family of plants:

$$\Sigma = \left\{ \operatorname{diag}\left\{ \alpha_1 \frac{1}{s-1}, \ldots, \alpha_m \frac{1}{s-1} \right\}, \alpha_i \in \{-1, 1\}, i = 1, \ldots, m \right\} \tag{17}$$

Obviously, it is necessary to have 2^m linear time-invariant stabilizers to cover the whole family, one for each combinations of $\{\alpha_i\}$. Also in view of Remark 5, we call the partition (9) the minimum partition.

Remark 7

We emphasize an important property of the proposed controller, i.e. $\hat{C}_\alpha(s)$ is diagonal, minimum phase and stable. Also, the degree of the controller is typically low.

4. SWITCHING ALGORITHM

Once the 2^m candidate controllers $\{C_\alpha(s)\}$ are designed, the next step is to specify a switching algorithm which, when applied on-line, is able to adaptively find a correct controller for any given plant $G_P(s) \in \Sigma$. To this end, many suitable switching algorithms (e.g. those in [12–17]) can be considered. Since in our case the number of candidate controllers is typically small (unless the number of inputs/outputs is large), the design of the switching algorithm may not be critical. For illustrative purposes, we adopt an algorithm similar to Fu and Barmish [7].

We index all the subfamilies of plants Σ_α and the controllers $C_\alpha(s)$ by Σ_i and $C_i(s)$, respectively, $i = 1, \ldots, 2^m$. We use the plant output to generate the following signal:

$$\dot{\phi}(t) = \|e(t)\|^2 \tag{18}$$

where

$$e(s) = I(s)(y(s) - r(s)) \tag{19}$$

(see Figure 2). Define a test function

$$\forall(t, \tau) = \phi(t) - \phi(t - \tau) = \int_{t-\tau}^{t} \|e(t)\|^2 \, \mathrm{d}t \tag{20}$$

for $t \geqslant 0$ and $\tau \in [0, t]$.

Given any plant $G_P(s) \in \Sigma$, suppose the controller $C_i(s)$ is applied at some time t_{i-1}. If $G_P(s) \in \Sigma_i$, then a nice property of $e(t)$ is that it converges to zero exponentially. It follows that there exists a *dwell time* $\tau_i > 0$ such that $V(t, \tau_i)$ has the following monotonic decreasing property:

$$V(t, \tau_i) \leqslant \rho V(t - \tau_i, \tau_i), \quad \forall t \geqslant t_{i-1} + 2\tau_i, \;\; G_P(s) \in \Sigma_i \tag{21}$$

for any prescribed $\rho \in (0, 1)$; see more details in Reference [7].

On the other hand, if $G_P(s) \notin \Sigma_i$, one of the three cases will happen:

(1) Property (21) fails at $t = t_{i-1} + 2\tau_i$ immediately;
(2) (21) holds for a little while after $t = t_{i-1} + 2\tau_i$ and then fails at, say t_i; and
(3) (21) holds forever.

In the first case, we will know immediately (at $t_{i-1} + 2\tau_i$) that $G_P(s) \notin \Sigma_i$, so another controller should be selected. In the second case, we will not know that the controller is wrong until t_i.

Again, switching is needed at t_i. However, the controller $C_i(s)$ has managed to decrease the test function for the period of time from $t_{i-1}+2\tau_i$ to t_i. In the third case, we will never find out that $G_P(s)\notin\Sigma_i$, so $C_i(s)$ will be applied to $G_P(s)$ forever. It follows from (21) that the test function will decay to zero exponentially, and so will the error signal $e(t)$ (see (20)). That is, the tracking requirement is satisfied. Because $(N_P(s), D_P(s))$ is a row Hermite factorization, $N_P(s)$ and $D_P(s)$ are co-prime in the closed right-half plane (see Definition 1). Hence, the exponential decay of $e(t)$ also implies the exponential stability of the closed-loop system.

Based on the analysis above, we are ready to build a switching function. Initially, we apply $C_1(s)$ and set the switching time $t_0=0$. Then, for $i=1,2,\ldots,2^{m-1}$, define the new switching instant

$$t_i=\sup\{t: t\geqslant t_{i-1}+2\tau_i; V(t,\tau_i)\leqslant\rho V(t-\tau_i,\tau_i)\}\tag{22}$$

and the switching index function

$$h(t)=i,\quad \text{for } t\in[t_{i-1},t_i)\tag{23}$$

Then, choose the switching control law given by

$$C=C_{h(t)}\tag{24}$$

In case $t_i=\infty$ for some $i<2^m-1$, the generation of t_i is terminated and the controller remains to be $C_i(t)$ indefinitely.

We make a few observations about the switching algorithm above. Firstly, there are only a finite number of switchings and the switching index $h(t)$ converges to a constant. In fact, suppose $G_P(s)\in\Sigma_j, 1\leqslant j\leqslant 2^m$, then switching stops when or before the switching index reaches j. Secondly, for each switching index $h(t)=i$, the testing function diverges for at most $2\tau_i$ time long. So the overall behaviour of the testing function is that it decays exponentially everywhere (except for a bounded finite period of time which is negligible). Consequently, the error function $e(t)$ exponentially converges to zero. This, in turn, guarantees the exponential stability of the closed-loop system. The detailed analysis can be found in Reference [7]. Finally, one may perceive that the dwell time τ_i is necessarily large when each controller $C_i(s)$ needs to cover a potentially large subfamily of plants Σ_i. We stress that this is not necessarily the case for two reasons: (i) The dwell time can be reduced by increasing feedback gain, although this has a possible negative effect of increasing the overshoot; (ii) The dwell time is often dominated by a 'worst-case' plant rather than the 'size' of Σ_i.

The switching law above is the so-called *pre-routed* in the sense that the switching sequence is predefined. One can easily replace such a pre-routed switching law by an on-line switching law where at each switching instance, a new switching index is selected by optimizing some sort of on-line measure. This is the basic idea behind the supervisory control [4]. It is easy to see that, if old switching indices are not recycled, the switching index will also converge just like in the pre-routed case because the on-line switching law can be viewed as a special pre-routed law. Hence, we can interpret (23) as permitting such an on-line switching law.

In summary, we have the following result

Theorem 2

Given a family of uncertain plants $\Sigma\subset\mathscr{R}^{m\times m}[s]$ satisfying Assumptions A1–A4 and a reference signal $r(\cdot):\mathscr{R}\to\mathscr{R}^m$ with maximum degree n_r. Let the candidate controllers $C_i(s), i=1,\ldots,2^m$ be designed according to the procedure in Section 3 and the switching control law be given by

(18)–(24). Then, for any (unknown) member plant $G_P(s) \in \Sigma$, the closed-loop system is exponentially stable and the tracking error converges to zero exponentially.

5. CONCLUSIONS

In this paper, we have considered an adaptive tracking problem for a family of uncertain m-input–m-output plants which satisfy Assumptions A1–A4. We have shown that at most 2^m candidate controllers are required such that any plant in the given plant family can be exponentially stabilized with exponential tracking performance by one of these controllers. This 2^m number is also shown to be the minimum number if no further knowledge on the plant family is given. Once these 2^m controllers are found, a simple switching algorithm is established using the idea in Reference [7]. This switching algorithm guarantees that a correct controller will be found adaptively. More importantly, both exponential stability and exponential tracking are guaranteed for the closed-loop system. The resulting switching controller is a piecewise linear time-invariant one with at most $2^m - 1$ switchings.

The key idea involved in the design of the candidate controllers comes from a result on robust stabilization of Wei and Barmish [29]. We, however, have relaxed and simplified their conditions slightly. In particular, we have used two spectral boundedness conditions (Assumptions A3–A4) on the plant family rather than the so-called standardness condition which is somewhat more complex (see Reference [29] for details).

Although we have only studied the case where the output of the system is measured perfectly, if the output measurement is noisy, the switching law can be modified by allowing recycling old switching indices. If the measurement noise is additive and uniformly bounded, the tracking error becomes bounded by cycling through the switching indices. If the intention is to minimize the tracking error, some sort of on-line switching algorithm (such as supervisory control) would be preferred.

Finally, we comment that the result in this paper relies on the key assumption of minimum phase invariance. It remains an interesting open problem to understand how to partition a family of non-minimum phase uncertain plants.

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