Linear Quadratic Regulation and Stabilization of Discrete-Time Systems With Delay and Multiplicative Noise

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***Abstract*—This paper is concerned with the long-standing problems of linear quadratic regulation (LQR) control and stabilization for a class of discrete-time stochastic systems involving multiplicative noises and input delay. These fundamental problems have attracted resurgent interests due to development of networked control systems. An explicit analytical expression is given for the optimal LQR controller. More specifically, the optimal LQR controller is shown to be a linear function of the conditional expectation of the state, with the feedback gain based on a Riccati-ZXL difference equation. It is also shown that the system is stabilizable in the mean-square sense if and only if an algebraic Riccati-ZXL equation has a particular solution. These results are based on a new technical tool, which establishes a non-homogeneous relationship between the state and the costate of this class of systems, and the introduction of a new Lyapunov function for the finite-horizon optimal control design.**

***Index Terms*—LQR control, multiplicative noise, networked control, stabilization, stochastic system.**

I. Introduction

THE stochastic optimal linear quadratic regulation (LQR) problem, pioneered by Wonham [1], has received paramount attention since 1960's; see [2]–[9] and references therein. For stochastic linear systems without time delay, the LQR theory is well established [7]. However, when time delays in the control input and/or the state are present, the stochastic LQR problem becomes very complicated and remains challenging, despite the fact that a huge amount of research has been devoted to it since 1970's; see [10] and [11].

Different from the stochastic LQR problem, the deterministic LQR problem with input delay has been extensively studied since 1970's for both the single-delay case [12], [13] and the multiple-delay case [14]–[16]. A deterministic LQR problem is a special case of the stochastic LQR problem without noises, and its optimal controller is a linear state feedback with its feedback gain given by solving a Riccati equation. When an input delay is present, the optimal controller is known to take the same form and a predictor approach can be used to handle the time delay.

More specifically, the deterministic LQR problem with single input delay is concerned with

$$\min \sum_{k=0}^{\infty} x_k' Q x_k + \sum_{k=d}^{\infty} u_{k-d}' R u_{k-d} \tag{1}$$
$$\text{subject to } x_{k+1} = A x_k + B u_{k-d} \tag{2}$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ is the input control with delay $d > 0$, A and B are constant matrices with compatible dimensions, and Q and R are positive semi-definite matrices. The initial values $x_0, u_i, i = -d, \ldots, -1$, are known. Owing to the input delay, the controller is required to obey the *causality constraint*, i.e., u_k must be in the form of

$$u_k = f_k(x_k, x_{k-1}, \ldots, x_0, u_{k-1}, u_{k-2}, \ldots, u_{-d}) \tag{3}$$

for some function $f_k(\cdot)$.

Under the assumption that (A, B) is stabilizable, the optimal controller for (1), (2) can be obtained by invoking the well-known Smith predictor theory [17], and the result is given by

$$u_k = K\left(A^d x_k + \sum_{i=1}^{d} A^{i-1} B u_{k-i}\right) \tag{4}$$

where the feedback gain matrix K is the same as in the delay-free case [18]. Indeed, it is easy to see that the term $(A^d x_k + \sum_{i=1}^{d} A^{i-1} B u_{k-i})$ is the d-step prediction of the future state x_{k+d}.

Unfortunately, it is recognized that the theory for deterministic LQR with input delay can not be dirtily generalized to stochastic LQR involving multiplicative noises. In fact, let system (2) involve a multiplicative noise as

$$x_{k+1} = (A + \omega_k \bar{A}) x_k + (B + \omega_k \bar{B}) u_{k-d} \tag{5}$$

where ω_k is a scalar random white noise with zero mean and variance σ^2 and $\bar{A}$ and $\bar{B}$, like A and B, are constant matrices with compatible dimensions. Accordingly, consider the following cost function:

$$J = E\left\{\sum_{k=0}^{\infty} x_k' Q x_k + \sum_{k=d}^{\infty} u_{k-d}' R u_{k-d}\right\} \tag{6}$$

Manuscript received April 12, 2014; revised November 17, 2014; accepted February 5, 2015. Date of publication March 11, 2015; date of current version September 23, 2015. This work was supported by the Taishan Scholar Construction Engineering by Shandong Government, the National Natural Science Foundation of China under Grants 61120106011, 61203029, and 61034007. Recommended by Associate Editor H. L. Trentelman.

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Digital Object Identifier 10.1109/TAC.2015.2411911

where E is the mathematical expectation over the noise $\{\omega_0, \omega_1, \ldots\}$ and the weighting matrices Q and R are as in (1). The same causality constraint (3) must be obeyed in minimizing the cost function J. Although it is easy to verify that the term $(A^d x_k + \sum_{i=1}^{d} A^{i-1} B u_{k-i})$ remains as the (optimal) d-step prediction of the future state x_{k+d}, it is unfortunate that (4) is no longer an optimal solution if the gain matrix K is the same as the delay-free case. One such example is given below to show that (4) is not the controller to minimize (6). Let us consider the system (5) and the cost function (6) with

$$A = 1.1, \bar{A} = 0.1, B = 0.2, \bar{B} = 0.22, d = 2, \sigma^2 = 4$$
$$Q = 1, R = 1$$

and the initial values

$$x_0 = 1, u_{-1} = 0, u_{-2} = 0.$$

By solving the algebraic Riccati equation for K [9], the gain matrix is given by $K = -1.0270$ and the corresponding cost of (6) is $J^* = 38.6937$. However, if we chose $u_k = -0.8807\hat{x}_{k+2|k}$, the cost of (6) is calculated to be $J^\star = 35.6268$. It is obvious that $J^* > J^\star$ and controller with gain $K = -1.0270$ is not optimal.

The discussion above leads to a fundamental difficulty for stochastic systems with multiplicative noises: The well-celebrated separation principle for stochastic systems with additive noises fails to have a similar counterpart for stochastic systems with multiplicative noises. That is, it is not possible to simply "plug in" an optimal prediction of the state into a delay-free design. We also note that the separation principle does not hold in the case of control-dependent noise and/or state-dependent noise, as pointed out in [19], [20] and references therein. Only a suboptimal controller can be obtained there by applying "enforced separation principle."

This paper focuses on the LQR control and stabilization problems for stochastic discrete-time systems as described by (5). Apart from the general interest of solving these long-standing problems, we are motivated by recent development in networked control systems where multiplicative noises and feedback time delays arise naturally [21]–[25]. Indeed, multiplicative noises have been used to model packet loss [21], [24] and time delay that occur for packet transmission in a communication network [23], [25]. For example, packet loss and communication delay of the control input in a wireless networked system can be described as

$$x_{k+1} = E x_k + \gamma_k F u_{k-d} \tag{7}$$

where d is the transmission delay and γ_k is a random variable representing the packet loss, taking value of either 1 (no loss) or 0 (loss), with packet loss probability of $p \in (0,1)$, i.e., $P(\gamma_k = 0) = p$ and $P(\gamma_k = 1) = 1 - p$. It is easy to verify that (7) is a special case of (5) with $A = E, \bar{A} = 0, B = (1-p)F, \bar{B} = F$, and $\omega_k = \gamma_k - 1 + p$. The stabilization problem for the system (7) with packet loss only (i.e., $d = 0$) has been extensively studied in the recent literature; see, e.g., [21]–[23]. However, when both packet loss and input delay occur simultaneously, the LQR control and stabilization problems for the system (7) are much more complicated and largely unsolved.

It is true that any control problem for discrete-time systems with time delays can be converted into one for a delay-free system using the well-known lifting technique, but this will lead to computational burden, as pointed out by Tadmor and Mirkin [26]. This approach is not elegant conceptually. Besides, the state feedback control problem will become an output feedback problem which tends to alter the nature of the original problem.

Instead, this paper shall develop a direct approach based on the solution to a delayed forward backward stochastic difference equation (D-FBSDE), which will lead to a non-homogeneous relationship between the optimal state and the costate. The main contributions of this paper are as follows: An explicit solution to the D-FBSDE is presented. Using this solution, a necessary and sufficient condition for the finite-horizon optimal control problem is given in terms of the solution to a Riccati-ZXL difference equation. We then generalize the solution to the infinite-horizon case. Subsequently, a necessary and sufficient condition for the stabilization of the stochastic delayed systems is developed.

The rest of the paper is organized as follows. The finite-horizon LQR problem is studied in Section II. The solutions for the infinite-horizon case are given in Section III. Section IV generalizes the above results to systems with multiple multiplicative noises. Numerical examples are given in Section V. Conclusions are provided in Section VI. Relevant proofs are detailed in Appendices. This paper is a companion of our earlier work [27] where results for continuous-time systems are presented.

Notation: R^n stands for the $n-$dimensional Euclidean space; I denotes the unit matrix; The superscript $'$ represents the matrix transpose; A symmetric matrix $M > 0$ (reps. ≥ 0) means that it is positive definite (reps. positive semi-definite); $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k\geq 0}\}$ denotes a complete probability space on which a scalar white noise ω_k is defined such that $\{\mathcal{F}_k\}_{k\geq 0}$ is the natural filtration generated by ω_k, i.e., $\mathcal{F}_k = \sigma\{\omega_0, \ldots, \omega_k\}$, augmented by all the $\mathcal{P}-$null sets in $\mathcal{F}$ [28]; $\hat{x}_{k|m} \doteq E[x_k|\mathcal{F}_{m-1}]$ denotes the conditional expectation of x_k with respect to $\mathcal{F}_{m-1}$; a.s. means almost surely as in the probability theory.

II. Finite-horizon Stochastic LQR

A. Problem Statement

Consider the discrete-time stochastic system (5) and the following finite-horizon cost function:

$$J_N = E\left(\sum_{k=0}^{N} x_k' Q x_k + \sum_{k=d}^{N} u_{k-d}' R u_{k-d} + x_{N+1}' P_{N+1} x_{N+1}\right) \tag{8}$$

where Q, R, and P_{N+1} are positive semi-definite matrices, and N is the horizon length. In view of the fact x_k depends on $\omega_{k-1}, \omega_{k-2}, \ldots$ (from (5)), the causality constraint (3) means that u_k must be $\mathcal{F}_{k-1}$-measurable, where $\mathcal{F}_{k-1}$ has been defined in Introduction. Thus, the optimal stochastic LQR problem to be addressed is stated as follows:

Problem 1: Find a $\mathcal{F}_{k-1}$-measurable u_k such that (8) is minimized, subject to (5).

B. Solution to Problem 1

Following the results in [29], we apply Pontryagin's maximum principle to the system (5) with the cost function (8) to yield the following costate equations:

$$\lambda_N = P_{N+1}x_{N+1} \tag{9}$$
$$\lambda_{k-1} = E\left[A_k'\lambda_k|\mathcal{F}_{k-1}\right] + Qx_k,\ k=0,\ldots,N \tag{10}$$
$$0 = E\left[B_k'\lambda_k|\mathcal{F}_{k-d-1}\right] + Ru_{k-d},\ k=d,\ldots,N \tag{11}$$

where λ_k is the costate and

$$A_k \doteq A + \omega_k\bar{A}, \quad B_k \doteq B + \omega_k\bar{B}.$$

Next, we define a set of matrix sequences Υ_k, M_k, and P_k^i, $i=1,2,\ldots,d+1$, by initializing the terminal values $P_{N+1}^1 = P_{N+1}, P_{N+1}^i = 0, i\geq 2$ and making the following backwards recursion for $k=N,N-1,\ldots,d$:

$$\Upsilon_k = \sum_{j=1}^{d+1} B'P_{k+1}^jB + \sigma^2\bar{B}'P_{k+1}^1\bar{B} + R \tag{12}$$
$$M_k = \sum_{j=1}^{d+1} B'P_{k+1}^jA + \sigma^2\bar{B}'P_{k+1}^1\bar{A} \tag{13}$$
$$P_k^1 = A'P_{k+1}^1A + \sigma^2\bar{A}'P_{k+1}^1\bar{A} + A'P_{k+1}^{d+1}A + Q \tag{14}$$
$$P_k^2 = -M_k'\Upsilon_k^{-1}M_k \tag{15}$$
$$P_k^i = A'P_{k+1}^{i-1}A, i=3,\ldots,d+1. \tag{16}$$

In (15), it is assumed that Υ_k is invertible. If this is not the case, the recursion stops.

Remark 1: Let $Z_k = \sum_{i=1}^{d+1} P_k^i$ and $X_k = P_k^1$. By taking the sum on both sides of (14), (15), and (16) from $i=3$ to $d+1$, we obtain the following coupled equations:

$$Z_k = A'Z_{k+1}A + \sigma^2\bar{A}'X_{k+1}\bar{A} + Q - L_k \tag{17}$$
$$X_k = Z_k + \sum_{i=0}^{d-1}(A')^iL_{k+i}A^i \tag{18}$$

where

$$L_k = M_k'\Upsilon_k^{-1}M_k \tag{19}$$
$$\Upsilon_k = B'Z_{k+1}B + \sigma^2\bar{B}'X_{k+1}\bar{B} + R \tag{20}$$
$$M_k = B'Z_{k+1}A + \sigma^2\bar{B}'X_{k+1}\bar{A} \tag{21}$$

with the terminal values $Z_{N+1} = P_{N+1}$ and $X_{N+1} = P_{N+1}$. Conversely, suppose there exist matrices Z_k and X_k obeying (17)–(21), it is easy to construct $P_k^i, i=1,\ldots,d+1$ to satisfy (12)–(16).

For the convenience of the following discussions, equation (17), (18) will be termed Riccati-ZXL difference equation.

The main result of this section is given below.

Theorem 1: Problem 1 has a unique solution if and only if the recursion (12)–(16) is well defined, i.e., $\Upsilon_k, k=N, N-1,\ldots,d$, are all invertible. If this condition holds, then the optimal controller u_k is given by

$$u_k = -\Upsilon_{k+d}^{-1}M_{k+d}\hat{x}_{k+d|k} \tag{22}$$

for $k=0,1,\ldots,N-d$, where

$$\hat{x}_{k+d|k} \doteq E[x_{k+d}|\mathcal{F}_{k-1}] = A^dx_k + \sum_{i=1}^{d}A^{i-1}Bu_{k-i}. \tag{23}$$

The associated optimal cost is given by

$$J_N^\star = E\left[\sum_{i=0}^{d-1}x_i'Qx_i + x_d'P_d^1x_d + x_d'\sum_{i=0}^{d-1}P_d^{i+2}\hat{x}_{d|i}\right] \tag{24}$$

which depends solely on the initial values $x_0, u_{-1},\ldots,u_{-d}$, where

$$\hat{x}_{d|i} = E[x_d|\mathcal{F}_{i-1}] = A^{d-i}x_i + \sum_{j=1}^{d-i}A^{j-1}Bu_{-j} \tag{25}$$

for $i=0,\ldots,d-1$. Moreover, the optimal costate λ_{k-1} and state x_k satisfy the following non-homogeneous relationship:

$$\lambda_{k-1} = P_k^1x_k + \sum_{i=2}^{d+1}P_k^i\hat{x}_{k|k-d+i-2},\ k=d,\ldots,N+1. \tag{26}$$

Proof: See Appendix A. □

Remark 2: For a delay-free stochastic system, i.e., $d=0$, (18) implies that $Z_k = X_k$, then Riccati-ZXL difference equation (17), (18) is reduced to the following standard generalized Riccati equation:

$$Z_k = A'Z_{k+1}A + \sigma^2\bar{A}'Z_{k+1}\bar{A} + Q - L_k \tag{27}$$

where

$$L_k = M_k'\Upsilon_k^{-1}M_k \tag{28}$$

$$\Upsilon_k = B'Z_{k+1}B + \sigma^2\bar{B}'Z_{k+1}\bar{B} + R \tag{29}$$

$$M_k = B'Z_{k+1}A + \sigma^2\bar{B}'Z_{k+1}\bar{A} \tag{30}$$

with the terminal value $Z_{N+1} = P_{N+1}$. The optimal controller reduces to

$$u_k = -\Upsilon_k^{-1}M_kx_k$$

which is exactly the result of the delay-free stochastic LQR [8].

Remark 3: For a deterministic system, i.e., $\bar{A}$ and $\bar{B}$ are zero, Riccati-ZXL difference equation (17), (18) is then reduced to the following standard Riccati equation:

$$Z_k = A'Z_{k+1}A + Q - L_k$$

where

$$L_k = M_k'\Upsilon_k^{-1}M_k$$
$$\Upsilon_k = B'Z_{k+1}B + R$$
$$M_k = B'Z_{k+1}A.$$

In this case, (22) and (26) are reduced to

$$u_k = -\Upsilon_{k+d}^{-1} M_{k+d} x_{k+d}$$
$$= -\Upsilon_{k+d}^{-1} M_{k+d}\left(A^d x_k + \sum_{i=1}^{d} A^{i-1} B u_{k-i}\right),\ k \geq 0$$
$$\lambda_{k-1} = \sum_{i=1}^{d+1} P_k^i x_k = Z_k x_k, \quad k \geq d.$$

Hence, Theorem 1 contains the results for deterministic LQR control with input delay as a special case.

III. Infinite-Horizon Stochastic LQR

A. Problem Formulation

In this section, we will solve the infinite-horizon stochastic LQR problem for the system (5) with the cost function (6). In conjunction with this, the stabilization problem will be studied.

We start with some definitions.

Definition 1: System (5) with $u = 0$ is called asymptotically mean-square stable if for any initial values $x_0, u_{-1}, \ldots, u_{-d}$, there holds

$$\lim_{k\to\infty} E\left(x_k' x_k\right) = 0.$$

Definition 2: System (5) is said to be stabilizable in the mean-square sense if there exists a $\mathcal{F}_{k-1}$-measurable controller $u_k = Lx_k + \sum_{i=1}^{d} L_i u_{k-i}, k \geq 0$ with constant matrices L and $L_i, i = 1, \ldots, d$, satisfying $\lim_{k\to\infty} E[u_k' u_k] = 0$, such that the closed-loop system of (5) is asymptotically mean-square stable.

Definition 3: [9] The following stochastic system:

$$x_{k+1} = (A + \omega_k \bar{A})x_k, \quad y_k = Cx_k \tag{31}$$

is said to be exactly observable (or $(A, \bar{A}, C)$ is said to be exactly observable, for short), if for any $N \geq n$

$$y_k \equiv 0,\ a.s.\ \forall 0 \leq k \leq N \Rightarrow x_0 = 0.$$

The problem to be dealt with in this section is described as follows.

Problem 2: Find the $\mathcal{F}_{k-d-1}$-measurable controller $u_{k-d} = K\hat{x}_{k|k-d}, k \geq d$, such that the system $x_{k+1} = A_k x_k + B_k K \hat{x}_{k|k-d}$ is asymptotically mean-square stable and that the cost function (6) is minimized.

To guarantee the solvability of *Problem 2*, we make two assumptions. The first one guarantees the uniqueness of the optimal controller and the second one is standard for mean-square stabilization [4].

Assumption 1: R is positive definite and Q is positive semi-definite, i.e., $Q = C'C$ for some matrix C.

Assumption 2: $(A, \bar{A}, C)$ is exactly observable.

B. Solution to Problem 2

To make the time horizon N explicit in the finite-horizon stochastic LQR problem, we rewrite Υ_k, P_k^i, and M_k in (12)–(16) as $\Upsilon_k(N), P_k^i(N)$, and $M_k(N)$. To facilitate our discussion in the sequel, the terminal weight matrix P_{N+1} in the cost function (8) will be set to be zero.

Lemma 1: Under the condition $R > 0$, *Problem 1* has a unique solution for any terminal time $N \geq d$.

Proof: See Appendix B. □

Remark 4: Under the condition $R > 0$, it follows from Lemma 1 that:

$$\Upsilon_k(N+1) > 0, \quad k = N+1, \ldots, d.$$

It is also easy to see from (12)–(16) that $\Upsilon_k(N+1)$ can be calculated for $k = d-1, d-2, \ldots, 0$ too. Moreover, noting $P_{N+1} = 0$, it follows that:

$$\Upsilon_{d-1}(N) = \Upsilon_d(N+1) > 0.$$

Inductively, it can be derived that for any $k = 0, \ldots, d-1$ and N

$$\Upsilon_k(N) > 0.$$

Lemma 2: Take any $N \geq d$. If $R > 0$, then for $k = N, \ldots, 0, P_k^i(N)$ obtained from (12)–(16) satisfy the following:

$$P_k^1(N) \geq 0 \tag{32}$$
$$P_k^i(N) \leq 0, \quad i = 2, \ldots, d+1 \tag{33}$$
$$\sum_{i=1}^{d+1} P_k^i(N) \geq 0. \tag{34}$$

Proof: See Appendix C. □

Lemma 3: Under Assumptions 1 and 2, there exists a positive integer $N_0 \geq d$, such that $\sum_{i=1}^{d+1} P_d^i(N_0)$ is positive definite.

Proof: See Appendix D. □

Theorem 2: Under Assumptions 1 and 2, if system (5) is stabilizable in the mean-square sense, we have the following properties:

1) For any $k \geq 0$ and $i = 1, \ldots, d+1$, $P_k^i(N)$ is convergent when $N \to \infty$, i.e., $P^i \doteq \lim_{N\to\infty} P_k^i(N)$ exists and it is independent of k. Moreover, P^i satisfies the following coupled algebraic equation:

$$P^1 = A'P^1A + \sigma^2 \bar{A}'P^1\bar{A} + A'P^{d+1}A + Q \tag{35}$$
$$P^2 = -M'\Upsilon^{-1}M \tag{36}$$
$$P^i = A'P^{i-1}A, \quad i = 3, \ldots, d+1 \tag{37}$$

where

$$\Upsilon = \sum_{i=1}^{d+1} B'P^iB + \sigma^2 \bar{B}'P^1\bar{B} + R > 0 \tag{38}$$
$$M = \sum_{i=1}^{d+1} B'P^iA + \sigma^2 \bar{B}'P^1\bar{A}. \tag{39}$$

2) The matrix $\sum_{i=1}^{d+1} P^i$ is positive definite.

Proof: See Appendix E. □

Now we are in the position to give the main result of this section.

Theorem 3: Under Assumptions 1 and 2, the system (5) is stabilizable in the mean-square sense if and only if there exists a unique solution to (35)–(39) such that $\sum_{i=1}^{d+1} P^i > 0$. In this case, the controller

$$u_k = -\Upsilon^{-1} M \hat{x}_{k+d|k}, k \geq 0 \tag{40}$$

stabilizes (5) in the mean-square sense and minimizes the cost function (6). The optimal cost is given by

$$J_0 = x_0' \sum_{i=1}^{d+1} P^i x_0 - \sum_{k=0}^{d-1} u_{k-d}' R u_{k-d} + \sum_{k=0}^{d-1} E\left[(u_{k-d} + \Upsilon^{-1} M \hat{x}_{k|k-d})' \Upsilon (u_{k-d} + \Upsilon^{-1} M \hat{x}_{k|k-d})\right] \tag{41}$$

where

$$\hat{x}_{k|k-d} = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_{j-d},\ k = 0, \ldots, d-1.$$

Proof: See Appendix F. □

Similar to the discussions in Remark 1, let $Z = \sum_{i=1}^{d+1} P^i$ and $X = P^1$. By taking the sum on both sides of (35), (36), and (37) from $i = 3$ to $d+1$, we obtain the following coupled algebraic equations:

$$Z = A'ZA + \sigma^2 \bar{A}' X \bar{A} + Q - L \tag{42}$$

$$X = Z + \sum_{i=0}^{d-1} (A')^i L A^i \tag{43}$$

with

$$L = M' \Upsilon^{-1} M \tag{44}$$

$$\Upsilon = B'ZB + \sigma^2 \bar{B}' X \bar{B} + R \tag{45}$$

$$M = B'ZA + \sigma^2 \bar{B}' X \bar{A}. \tag{46}$$

For the convenience of the following discussions, equation (42), (43) will be termed algebraic Riccati-ZXL equation.

Thus Theorem 3 can be restated as follows.

Corollary 1: Under Assumptions 1 and 2, the system (5) is stabilizable in the mean-square sense if and only if algebraic Riccati-ZXL equation (42), (43) admits a unique solution with $Z > 0$. In this case, the controller (40) stabilizes the system (5) and minimizes the cost function (6). The corresponding optimal cost is given by (41).

IV. Stochastic LQR With Multiple Multiplicative Noises

In this section, we generalize the results in the previous sections to stochastic systems with multiple multiplicative noises. Consider the following system:

$$x_{k+1} = \left(A + \sum_{i=1}^{r} \omega_k(i) \bar{A}_i\right) x_k + \left(B + \sum_{i=1}^{r} \omega_k(i) \bar{B}_i\right) u_{k-d} \tag{47}$$

where the variance of the noise is given by

$$E\left(\omega_k(i)\omega_k(j)\right) = \sigma_{ij}^2, \quad i,j = 1, \ldots, r.$$

The finite-horizon and infinite-horizon cost functions are still as in (8) and (6) respectively. Accordingly, E represents the mathematical expectation over the noises $\{\omega_k(i), i = 1, \ldots, r, k \geq 0\}$. It turns out that the change of the system from (5) to (47) does not cause any fundamental difficulties. Our approach developed in the previous sections is still effective to deal with (47). Thus, it is easy to develop a counterpart of Theorems 1–3.

We first generalize the backwards recursion in (12)–(16) as follows: For $k = N, N-1, \ldots, d$, compute

$$\Upsilon_k = \sum_{j=1}^{d+1} B' P_{k+1}^j B + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{B}_m' P_{k+1}^1 \bar{B}_n + R \tag{48}$$

$$M_k = \sum_{j=1}^{d+1} B' P_{k+1}^j A + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{B}_m' P_{k+1}^1 \bar{A}_n \tag{49}$$

$$P_k^1 = A' P_{k+1}^1 A + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{A}_m' P_{k+1}^1 \bar{A}_n + A' P_{k+1}^{d+1} A + Q \tag{50}$$

$$P_k^2 = -M_k' \Upsilon_k^{-1} M_k \tag{51}$$

$$P_k^i = A' P_{k+1}^{i-1} A, \quad i = 3, \ldots, d+1 \tag{52}$$

with the same terminal values $P_{N+1}^1 = P_{N+1}$, $P_{N+1}^i = 0, i = 2, \ldots, d+1$.

Theorem 4: In the finite-horizon case, the stochastic LQR problem has a unique solution if and only if Υ_k defined by the backwards recursion (48)–(52) is positive definite for all $k = N, N-1, \ldots, d$. In this case, the optimal controller is given by

$$u_k = -\Upsilon_{k+d}^{-1} M_{k+d} \hat{x}_{k+d|k}, \quad k = 0, 1, \ldots, N-d \tag{53}$$

with $\hat{x}_{k+d|k}$ given by (23).

Theorem 5: Suppose R is positive definite, $Q = C'C$ is positive semi-definite and $(A, \bar{A}_1, \ldots, \bar{A}_r | C)$ is exactly observable. Then, the system (47) is stabilizable in the mean-square sense if and only if there exists a unique solution to the following coupled algebraic equations:

$$P^1 = A' P^1 A + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{A}_m' P^1 \bar{A}_n + A' P^{d+1} A + Q$$

$$P^2 = -M' \Upsilon^{-1} M$$

$$P^i = A' P^{i-1} A, \quad i = 3, \ldots, d+1$$

with

$$\Upsilon = \sum_{i=1}^{d+1} B' P^i B + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{B}_m' P^1 \bar{B}_n + R > 0$$

$$M = \sum_{i=1}^{d+1} B' P^i A + \sum_{m=1}^{r} \sum_{n=1}^{r} \sigma_{mn}^2 \bar{B}_m' P^1 \bar{A}_n$$

such that $\sum_{i=1}^{d+1} P^i > 0$. In this case, the controller that stabilizes (47) in the mean-square sense and minimizes the cost function (6) is given by

$$u_k = -\Upsilon^{-1} M \hat{x}_{k+d|k}, \quad k \geq 0 \tag{54}$$

with $\hat{x}_{k+d|k}$ given by (23).

V. Numerical Examples

A. The Finite-Horizon Case

Consider the system (5) with

$$A = 1.1,\ \bar{A} = 0.1,\ B = 0.2, \bar{B} = 0.1,\ d = 2,\ \sigma^2 = 1$$
$$x_0 = 1\ , u_{-1} = -1,\ u_{-2} = 2$$

and the cost function (8) with

$$Q = 1, R = 1, N = 4, P_{N+1} = 0.$$

By applying Theorem 1, direct calculation yields

$$P_2^1 = 3.7084, P_3^1 = 2.2200, P_4^1 = 1$$
$$P_2^2 = -0.2250, P_3^2 = -0.0504, P_4^2 = 0$$
$$P_2^3 = -0.0610, P_3^3 = 0, P_4^3 = 0$$
$$\Upsilon_2 = 1.1090, \Upsilon_3 = 1.0500, \Upsilon_4 = 1$$
$$M_2 = 0.4995, M_3 = 0.2300, M_4 = 0.$$

Note that $\Upsilon_i > 0$ for $i = 2, 3, 4$, thus there is a unique solution to the stochastic LQR problem according to Theorem 1. The optimal controller is calculated from (22) as

$$u_0 = -0.4504\hat{x}_{2|0}$$
$$u_1 = -0.2190\hat{x}_{3|1}$$
$$u_2 = 0$$

and the optimal value of (8) is $J_N^\star = 10.9455$.

B. The Infinite-Horizon Case

Consider the system (5) with

$$A = 1.3, \bar{A} = 0.1, B = 0.2, \bar{B} = 0.1, d = 5, \sigma^2 = 1$$
$$x_0 = 0.1, u_{-1} = 0.1, u_{-2} = -0.2, u_{-3} = -0.1$$
$$u_{-4} = 0.3, u_{-5} = -0.2$$

and the cost function (6) with $R = Q = 1 > 0$. Note that the Assumptions 1 and 2 are satisfied. It can be verified by direct calculation that

$$P^1 = 4274.8218, P^2 = -217.1339, P^3 = -366.9563$$
$$P^4 = -620.1561, P^5 = -1048.0639, P^6 = -1771.2279$$
$$\Upsilon = 53.7996, M = 108.0820$$

is the unique solution to (35)–(39) and $\sum_{i=1}^{d+1} P^i = 251.2836 > 0$. According to Theorem 3, there exists a unique optimal controller to stabilize system (5) in the mean-square sense, and the controller is given by

$$u_k = -\Upsilon^{-1} M\hat{x}_{k+5|k} = -2.0090\hat{x}_{k+5|k}, \quad k \geq 0.$$

From (41), it can be derived that the optimal cost is $J_0 = 38.3066$. A simulation result of the designed controller is shown in Fig. 1. It can be seen that the regulated state is asymptotically mean-square stable. To show the effectiveness of our approach, we select the following controller:

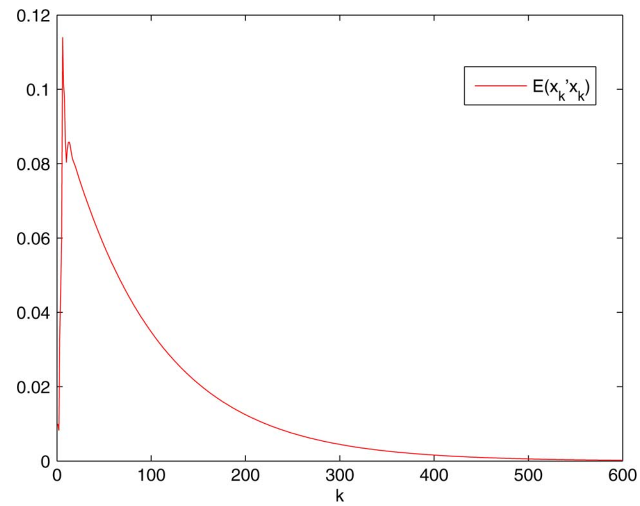

Fig. 1. Dynamic Behavior of $E(x_k' x_k)$.

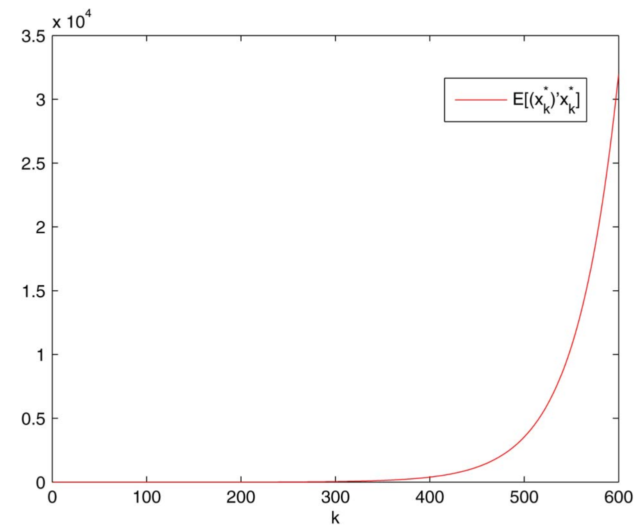

Fig. 2. Dynamic Behavior of $E[(x_k^\star)' x_k^\star]$.

$$u_k^\star = KE[x_{k+5}^\star | \mathcal{F}_{k-1}] = -2.7686E[x_{k+5}^\star | \mathcal{F}_{k-1}],\ k \geq 0$$

whose gain K and the predictor are separated enforcedly, i.e., $K = -(B'ZB + \sigma^2\bar{B}'Z\bar{B})^{-1}(B'ZA + \sigma^2\bar{B}'Z\bar{A})$ with Z being the solution to the standard generalized algebraic Riccati equation (the algebraic version of (27)–(30)). In this case, the corresponding state is given in Fig. 2. It can be observed that the chosen controller fails to stabilize system (5).

VI. Conclusion

In this paper, the optimal control and stabilization problems for stochastic discrete-time systems with multiplicative noises and input delay have been studied. The necessary and sufficient condition for the existence of a unique solution to the finite-horizon stochastic control has been obtained, and optimal controller for stochastic LQR has been presented. Under the standard assumption of exactly observability, it has been proved that the stochastic system with a single input delay is stabilizable in the mean-square sense if and only if one algebraic Riccati-ZXL equation has a unique solution such that the specific matrix $(\sum_{i=1}^{d+1} P^i)$ is positive definite. The presented results show that optimal control for *stochastic* systems with input delay is fundamentally different from optimal control for *deterministic* systems with input delay and much more complicated than the optimal control for stochastic systems *without* input delay. Although a single input delay is considered, we expect that the results in this paper pave new ways for optimal control of stochastic systems with multiple input delays and/or state delays.

Appendix A
Proof of Theorem 1

Proof—"Necessity": Suppose *Problem 1* has a unique solution. We will show by induction that Υ_k in (12) is invertible for all $k = N, N-1, \ldots, d$ and the optimal controller is given by (22). Denote

$$J(k) \doteq E\left[\sum_{i=k}^{N}\left(x_i' Q x_i + u_{i-d}' R u_{i-d}\right) + x_{N+1}' P_{N+1} x_{N+1}\right]$$

for $k = d, \ldots, N$. Firstly, we note the terminal conditions (9) and $P_{N+1}^1 = P_{N+1}, P_{N+1}^i = 0$ for $i \geq 2$. It is obvious that (26) holds for $k = N+1$. For $k = N$, note that

$$J(N) = E\left[x_N' Q x_N + u_{N-d}' R u_{N-d} + x_{N+1}' P_{N+1} x_{N+1}\right].$$

Using (5), it is clear that $J(N)$ can be expressed as a quadratic function of x_N and u_{N-d}. The uniqueness of the optimal u_{N-d} implies that the quadratic term of u_{N-d} is positive for any nonzero u_{N-d}. Setting $x_N = 0$, we obtain

$$\begin{aligned} J(N) &= E\left[u_{N-d}' R u_{N-d} + (B_N u_{N-d})' P_{N+1} B_N u_{N-d}\right] \\ &= u_{N-d}' \Upsilon_N u_{N-d} > 0. \end{aligned}$$

It follows that $\Upsilon_N > 0$.

Next the optimal u_{N-d} is to be calculated. By making use of (5), (9), and (11), it yields that

$$0 = E\left[B_N' P_{N+1} A_N\right] \hat{x}_{N|N-d} + \Upsilon_N u_{N-d}.$$

Hence, the optimal u_{N-d} is given by

$$u_{N-d} = -\Upsilon_N^{-1} M_N \hat{x}_{N|N-d} \tag{55}$$

which is (22) with $k = N - d$.

Now let us show that λ_{N-1} has the form as (26). From (10), (5), and (55), it follows that:

$$\begin{aligned} \lambda_{N-1} &= E\left[A_N' \lambda_N | \mathcal{F}_{N-1}\right] + Q x_N \\ &= E\left[A_N' P_{N+1} x_{N+1} | \mathcal{F}_{N-1}\right] + Q x_N \\ &= E\left[A_N' P_{N+1} A_N x_N | \mathcal{F}_{N-1}\right] \\ &\quad + E\left(A_N' P_{N+1} B_N u_{N-d} | \mathcal{F}_{N-1}\right] + Q x_N \\ &= \left(E\left[A_N' P_{N+1} A_N\right] + Q\right) x_N \\ &\quad - E\left[A_N' P_{N+1} B_N\right] \Upsilon_N^{-1} M_N \hat{x}_{N|N-d}. \end{aligned}$$

In view of the definition of P_N^i in (14)–(16), we have verified (26) for $k = N$.

To complete the induction proof, we take any n with $d \leq n \leq N$, and assume that Υ_k in (12) is invertible and that the optimal u_{k-d} and λ_{k-1} are as (22) and (26) for all $k \geq n+1$. We show that these conditions will also hold for $k = n$. Set u_{k-d} to be optimal for all $k \geq n+1$. We first verify the invertibility of Υ_n. For this, following the argument for Υ_N above, we set $x_n = 0$ and then check the quadratic term of u_{n-d} in $J(n)$. By applying (5), (10), and (11), for $k \geq n+1$, we get

$$\begin{aligned} & E\left[x_k' \lambda_{k-1} - x_{k+1}' \lambda_k\right] \\ &= E\left[x_k' E\left(A_k' \lambda_k | \mathcal{F}_{k-1}\right)\right. \\ &\qquad \left. + x_k' Q x_k - x_k' A_k' \lambda_k - u_{k-d}' B_k' \lambda_k\right] \\ &= E\left(x_k' Q x_k\right) - E\left[E\left(u_{k-d}' B_k' \lambda_k | \mathcal{F}_{k-d-1}\right)\right] \\ &= E\left(x_k' Q x_k\right) - E\left[u_{k-d}' E\left(B_k' \lambda_k | \mathcal{F}_{k-d-1}\right)\right] \\ &= E\left(x_k' Q x_k\right) + E\left(u_{k-d}' R u_{k-d}\right). \end{aligned}$$

Adding from $k = n+1$ to $k = N$ on both sides of the above equation yields that

$$\begin{aligned} & E\left[x_{n+1}' \lambda_n - x_{N+1}' P_{N+1} x_{N+1}\right] \\ &= \sum_{k=n+1}^{N} E\left[x_k' \lambda_{k-1} - x_{k+1}' \lambda_k\right] \\ &= \sum_{k=n+1}^{N} E\left[x_k' Q x_k + u_{k-d}' R u_{k-d}\right]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & J(n) \\ &= E\left[x_n' Q x_n + u_{n-d}' R u_{n-d}\right] \\ &\quad + E\left[\sum_{k=n+1}^{N}\left(x_k' Q x_k + u_{k-d}' R u_{k-d}\right) + x_{N+1}' P_{N+1} x_{N+1}\right] \\ &= E\left[x_n' Q x_n + u_{n-d}' R u_{n-d}\right] + E\left[x_{n+1}' \lambda_n\right] \\ &= E\left(u_{n-d}' R u_{n-d}\right) + E\left(u_{n-d}' B_n' \lambda_n\right). \end{aligned} \tag{56}$$

By applying (26) and (5), λ_n can be written as

$$\begin{aligned}\lambda_n &= P_{n+1}^1 x_{n+1} + \sum_{i=2}^{d+1} P_{n+1}^i \hat{x}_{n+1|n-d+i-1}\\ &= P_{n+1}^1 A_n x_n + P_{n+1}^1 B_n u_{n-d}\\ &\quad + \sum_{i=2}^{d+1} P_{n+1}^i \left[A\hat{x}_{n|n-d+i-1} + Bu_{n-d}\right]\\ &= P_{n+1}^1 A_n x_n + \sum_{i=2}^{d+1} P_{n+1}^i A\hat{x}_{n|n-d+i-1}\\ &\quad + \left(P_{n+1}^1 B_n + \sum_{i=2}^{d+1} P_{n+1}^i B\right) u_{n-d}. \end{aligned} \tag{57}$$

Using $x_n = 0$ (thus $\hat{x}_{n|n-d+i-1} = 0$) and plugging the above equation into (56), we get

$$\begin{aligned}J(n) &= E\left[u'_{n-d} B'_n P_{n+1}^1 B_n u_{n-d}\right]\\ &\quad + E\left[u'_{n-d}\sum_{i=2}^{d+1} B'_n P_{n+1}^i B u_{n-d}\right]\\ &\quad + E\left[u'_{n-d} R u_{n-d}\right]\\ &= u'_{n-d}\Upsilon_n u_{n-d}.\end{aligned}$$

The uniqueness of the optimal control implies that $J(n)$ must be positive for any $u_{n-d} \neq 0$. Hence, $\Upsilon_n > 0$.

To compute the optimal u_{n-d}, substituting (57) into (11) yields

$$\begin{aligned}0 &= E\Bigg[B'_n P_{n+1}^1 A_n x_n + \sum_{i=2}^{d+1} B'_n P_{n+1}^i A\hat{x}_{n|n-d+i-1}\\ &\qquad + \left(B'_n P_{n+1}^1 B_n + \sum_{i=2}^{d+1} B'_n P_{n+1}^i B\right) u_{n-d}\Big|\mathcal{F}_{n-d-1}\Bigg]\\ &\quad + Ru_{n-d}\\ &= E\Bigg[E\left(B'_n P_{n+1}^1 A_n\right) x_n + \sum_{i=2}^{d+1} E\left(B'_n P_{n+1}^i A\right)\hat{x}_{n|n-d+i-1}\\ &\qquad \Big|\mathcal{F}_{n-d-1}\Bigg] + E\left[B'_n P_{n+1}^1 B_n + \sum_{i=2}^{d+1} B'_n P_{n+1}^i B\right] u_{n-d}\\ &\quad + Ru_{n-d}\\ &= \left(E\left[B'_n P_{n+1}^1 A_n\right] + \sum_{i=2}^{d+1} B' P_{n+1}^i A\right)\hat{x}_{n|n-d}\\ &\quad + \Upsilon_n u_{n-d}.\end{aligned}$$

The optimal u_{n-d} is derived as

$$u_{n-d} = -\Upsilon_n^{-1} M_n \hat{x}_{n|n-d} \tag{58}$$

where M_n is as (16). Hence, (22) holds for $k = n$.

Now we show that λ_{n-1} is of the form as (26). In terms of (10), (57), and (58), we get

$$\begin{aligned}&\lambda_{n-1}\\ &\quad = E\Bigg[\left(A'_n P_{n+1}^1 A_n + A'_n P_{n+1}^{d+1} A\right) x_n\\ &\qquad - \left(\sum_{i=2}^{d+1} A'_n P_{n+1}^i B + A'_n P_{n+1}^1 B_n\right)\Upsilon_n^{-1} M_n \hat{x}_{n|n-d}\\ &\qquad + \sum_{i=2}^{d} A'_n P_{n+1}^i A\hat{x}_{n|n-d+i-1}\Big|\mathcal{F}_{n-1}\Bigg] + Qx_n\\ &\quad = \left[E\left(A'_n P_{n+1}^1 A_n + A'_n P_{n+1}^{d+1} A\right) + Q\right] x_n\\ &\qquad - E\left(\sum_{i=2}^{d+1} A'_n P_{n+1}^i B + A'_n P_{n+1}^1 B_n\right)\Upsilon_n^{-1} M_n \hat{x}_{n|n-d}\\ &\qquad + \sum_{i=3}^{d+1} A' P_{n+1}^{i-1} A\hat{x}_{n|i+n-d-2}.\end{aligned}$$

By means of (14)–(16), the above equation can be written as (26). This ends the proof of necessity.

"*Sufficiency*" Suppose (12) is true, i.e., $\Upsilon_k > 0$ for $k \geq d$. The uniqueness of the solution to *Problem 1* is to be shown. Denote by

$$V_N(k, x_k) \doteq E\left[x'_k P_k^1 x_k + x'_k \sum_{i=2}^{d+1} P_k^i \hat{x}_{k|i+k-d-2}\right]. \tag{59}$$

Using (5) and (12)–(16), we have

$$\begin{aligned}&V_N(k, x_k) - V_N(k+1, x_{k+1})\\ &\quad = E\Bigg[-x'_k\left(A' P_{k+1}^1 A + \sigma^2 \bar{A}' P_{k+1}^1 \bar{A} + A' P_{k+1}^{d+1} A - P_k^1\right) x_k\\ &\qquad - 2u'_{k-d}\left(\sigma^2 \bar{B}' P_{k+1}^1 \bar{A} + B' \sum_{i=1}^{d+1} P_{k+1}^i A\right)\hat{x}_{k|k-d}\\ &\qquad - x'_k A' \sum_{i=3}^{d+1} P_{k+1}^{i-1} A\hat{x}_{k|i+k-d-2}\\ &\qquad + x'_k \sum_{i=3}^{d+1} P_k^i \hat{x}_{k|i+k-d-2} + x'_k P_k^2 \hat{x}_{k|k-d}\\ &\qquad - u'_{k-d}\left(\sigma^2 \bar{B}' P_{k+1}^1 \bar{B} + B' \sum_{i=1}^{d+1} P_{k+1}^i B\right) u_{k-d}\Bigg]\\ &\quad = E\Big[x'_k Q x_k - 2u'_{k-d} M_k \hat{x}_{k|k-d}\\ &\qquad + \hat{x}'_{k|k-d} P_k^2 \hat{x}_{k|k-d} - u'_{k-d}(\Upsilon_k - R) u_{k-d}\Big]\\ &\quad = E\Big[x'_k Q x_k + u'_{k-d} R u_{k-d} - \left(u_{k-d} + \Upsilon_k^{-1} M_k \hat{x}_{k|k-d}\right)'\\ &\qquad \times \Upsilon_k\left(u_{k-d} + \Upsilon_k^{-1} M_k \hat{x}_{k|k-d}\right)\Big].\end{aligned} \tag{60}$$

Adding from $k = d$ to $k = N$ on both sides of (60), the cost function (8) is rewritten as

$$J_N = E\left[\sum_{k=0}^{d-1} x_k' Q x_k + x_d' P_d^1 x_d + x_d' \sum_{i=2}^{d+1} P_d^i \hat{x}_{d|i-2}\right] + E\left[\sum_{k=d}^{N} \left(u_{k-d} + \Upsilon_k^{-1} M_k \hat{x}_{k|k-d}\right)' \Upsilon_k \times \left(u_{k-d} + \Upsilon_k^{-1} M_k \hat{x}_{k|k-d}\right)\right]. \quad (61)$$

Note that for $k \le d$, x_k is determined by the initial value $x_0, u_{-1}, \dots, u_{-d}$, and that Υ_k is positive definite. Thus, the unique optimal control exists and is given by (22) and the optimal cost is given by (24). This completes the sufficiency proof. □

Appendix B
Proof of Lemma 1

Proof: We show by induction that *Problem 1* has a unique solution for any terminal time $N \ge d$. When $N = d$, $\Upsilon_N(N)$ is given by

$$\Upsilon_N(N) = B' \sum_{i=1}^{d+1} P_{N+1}^i(N) B + \sigma^2 \bar{B}' P_{N+1}^1(N) \bar{B} + R = R > 0.$$

In view of Theorem 1, *Problem 1* with $N = d$ has a unique solution.

Now suppose the solution to *Problem 1* with $N = m$ is unique for some $m \ge d$, i.e.,

$$\Upsilon_k(m) > 0, \quad k = m, \dots, d. \quad (62)$$

It follows that:

$$P_k^2(m) = -M_k'(m) \Upsilon_k^{-1}(m) M_k(m) \le 0 \quad (63)$$

$$P_k^i(m) = A' P_{k+1}^{i-1}(m) A \le 0, \quad i = 3, \dots, d+1. \quad (64)$$

Let the system (5) start at d with an arbitrary initial value x_d and denote

$$F_m \doteq \sum_{k=d}^{m} E\left(x_k' Q x_k + u_{k-d}' R u_{k-d}\right). \quad (65)$$

By applying (24) of Theorem 1, it is apparent that the optimal value of (65) is given by

$$F_m = E\left[x_d' P_d^1(m) x_d + x_d' \sum_{i=2}^{d+1} P_d^i(m) \hat{x}_{d|i-2}\right] = x_d' P_d^1(m) x_d + \sum_{i=2}^{d+1} x_d' P_d^i(m) x_d = x_d' \sum_{i=1}^{d+1} P_d^i(m) x_d \ge 0 \quad (66)$$

where the second equality is based on the fact that $\hat{x}_{d|i-2} = x_d$ with $i = 2, \dots, d+1$. The arbitrariness of x_d yields

$$\sum_{i=1}^{d+1} P_d^i(m) \ge 0 \quad (67)$$

and further

$$P_d^1(m) \ge -\sum_{i=2}^{d+1} P_d^i(m) \ge 0.$$

Note that the variables given in (12)–(16) are time invariant for N due to the choice that $P_{N+1} = 0$, i.e.,

$$P_k^i(N) = P_{k-s}^i(N-s), \quad i = 1, \dots, d+1 \quad (68)$$

$$\Upsilon_k(N) = \Upsilon_{k-s}(N-s), M_k(N) = M_{k-s}(N-s), \quad d+s \le k \le N,\ N \ge d,\ 0 \le s \le N-d. \quad (69)$$

For $N = m+1$, it follows from (62) that:

$$\Upsilon_k(m+1) = \Upsilon_{k-1}(m) > 0, \quad k = m+1, \dots, d+1. \quad (70)$$

For $k = d$, we have

$$\Upsilon_d(m+1) = B' \sum_{i=1}^{d+1} P_{d+1}^i(m+1) B + \sigma^2 \bar{B}' P_{d+1}^1(m+1) \bar{B} + R = B' \sum_{i=1}^{d+1} P_d^i(m) B + \sigma^2 \bar{B}' P_d^1(m) \bar{B} + R \ge R > 0.$$

Thus *Problem 1* has a unique solution with $N = m+1$ from Theorem 1. Now the uniqueness of the solution to *Problem 1* for any terminal time $N \ge d$ is shown. □

Appendix C
Proof of Lemma 2

Proof: Recall the results of Theorem 1 that *Problem* 1 has unique solution implies that $\Upsilon_k(N) > 0$ for any $N \ge d$ and $d \le k \le N$. If $0 \le k < d$, there holds $\Upsilon_k(N) = \Upsilon_{k+d}(N+d)$ due to the time-invariance of (12)–(16). $\Upsilon_{k+d}(N+d) > 0$ since the solution to *Problem* 1 with terminal time $N+d$ is unique. Thus $\Upsilon_k(N) > 0$ for any $0 \le k \le N$. Therefore, similar to (63) and (64) in Appendix B, it is easily known that $P_k^i(N) \le 0\,(i = 2, \dots, d)$ for any $N \ge d$. Noting the time-invariance of $P_k^i(N) = P_d^i(N+d-k)$ and (67), we have $\sum_{i=1}^{d} P_k^i(N) \ge 0$. Then $P_k^1(N) \ge 0$ follows immediately. This ends the proof of Lemma 2. □

Appendix D
Proof of Lemma 3

Proof: Under Assumption 1, it follows from Lemma 2 that $\sum_{i=1}^{d+1} P_d^i(N) \ge 0$ for all $N \ge d$, we only need to show that there exists $N_0 \ge d$ such that $\sum_{i=1}^{d+1} P_d^i(N_0)$ is invertible. Suppose this is not the case. Then we get an non-empty set

$$X_N \doteq \left\{x \in R^n : x \ne 0, \sum_{i=1}^{d+1} P_d^i(N) x = 0\right\}.$$

The positive semi-definiteness of $\sum_{i=1}^{d+1} P_d^i(N)$ implies that

$$X_N = \left\{ x \in R^n : x \neq 0, x' \sum_{i=1}^{d+1} P_d^i(N) x = 0 \right\}.$$

Now the monotonicity of $\sum_{i=1}^{d+1} P_d^i(N)$ with respect to N is to be proven. Similar to the proof of Lemma 1, the optimal cost of (65) satisfies

$$x_d' \sum_{i=1}^{d+1} P_d^i(N) x_d = F_N^0 \leq F_{N+1}^0 = x_d' \sum_{i=1}^{d+1} P_d^i(N+1) x_d.$$

The arbitrariness of x_d implies

$$\sum_{i=1}^{d+1} P_d^i(N) \leq \sum_{i=1}^{d+1} P_d^i(N+1) \tag{71}$$

i.e., $\sum_{i=1}^{d+1} P_d^i(N)$ increases with respect to N. Furthermore, it follows that:

$$x' \sum_{i=1}^{d+1} P_d^i(N+1) x = 0 \Rightarrow x' \sum_{i=1}^{d+1} P_d^i(N) x = 0$$

i.e., $X_{N+1} \subset X_N$. Each X_N is a non-empty finite-dimensional set. So

$$1 \leq \cdots \leq dim(X_{d+2}) \leq dim(X_{d+1}) \leq dim(X_d) \leq n$$

where dim represents the dimension of the set. Thus there must exist N_1, such that for any $N \geq N_1$

$$dim(X_N) = dim(X_{N_1})$$

which yields that $X_N = X_{N_1}$, and thus

$$\bigcap_{N \geq d} X_N = X_{N_1} \neq 0.$$

So there exists a nonzero vector $x \in X_{N_1}$ such that

$$x' \sum_{i=1}^{d+1} P_d^i(N) x = 0, \quad \forall N \geq d.$$

Let the value x_d be equal to x. Then the optimal value of (65) is as

$$\begin{aligned} F_N &= \min \left\{ \sum_{k=d}^{N} E\left(x_k' Q x_k + u_{k-d}' R u_{k-d} \right) \right\} \\ &= \sum_{k=d}^{N} E\left(x_k^{*\prime} Q x_k^* + u_{k-d}^{*\prime} R u_{k-d}^* \right) \\ &= x' \sum_{i=1}^{d+1} P_d^i(N) x \\ &= 0 \end{aligned}$$

where u_{k-d}^* and x_k^* represent the optimal controller and the optimal state trajectory, respectively. Note that $R > 0$ and $Q = C'C \geq 0$. It follows that:

$$u_{k-d}^* = 0,\ C x_k^* = 0,\ d \leq k \leq N,\ N \geq d.$$

Then system (5) is reduced to

$$\begin{aligned} x_{k+1}^* &= A_k x_k^* \\ C x_k^* &= 0, \quad \forall k \geq d. \end{aligned}$$

Recalling the definition of exactly observability of $(A, \bar{A}, C)$, it yields that $x_d = x = 0$, which is a contradiction to $x \neq 0$. Therefore, there exists some $N_0 \geq d$ such that $\sum_{i=1}^{d+1} P_d^i(N_0) > 0$. □

Appendix E
Proof of Theorem 2

Proof: 1). First, we show that $P_0^1(N)$ is increasing with respect to N. To this end, we will calculate the optimal J_N for the case $u_i = 0$ for all $i = -d, \ldots, -1$ but x_0 is arbitrary. Recall that (60) with $k \geq d$ was derived by using (5) and (12)–(16). As pointed out in Remark 4, the recursion (12)–(16) is also meaningful for $k = d-1, \ldots, 0$. So (60) holds for $k = d-1, \ldots, 0$ too. Then adding from $k = 0$ to $k = N$ on both sides of (60) yields

$$\begin{aligned} &V_N(0, x_0) \\ &\quad = \sum_{k=0}^{N} \left[V_N(k, x_k) - V_N(k+1, x_{k+1}) \right] \\ &\quad = \sum_{k=0}^{N} E\left[x_k' Q x_k + u_{k-d}' R u_{k-d} \right. \\ &\qquad - \left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right)' \Upsilon_k(N) \\ &\qquad \left. \times \left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right) \right]. \end{aligned}$$

So

$$\begin{aligned} J_N &= \sum_{k=0}^{N} E\left(x_k' Q x_k \right) + \sum_{k=d}^{N} E\left(u_{k-d}' R u_{k-d} \right) \\ &= V_N(0, x_0) - \sum_{k=0}^{d-1} E\left(u_{k-d}' R u_{k-d} \right) \\ &\quad + \sum_{k=0}^{N} E\Bigg[\left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right)' \Upsilon_k(N) \\ &\qquad \times \left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right) \Bigg]. \end{aligned}$$

In the above equation, let u_{k-d}, $k = d, \ldots, N$, be optimal, i.e., $u_{k-d} = -\Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d}$. Then

$$\begin{aligned} J_N^\star &= V_N(0, x_0) - \sum_{k=0}^{d-1} E\left(u_{k-d}' R u_{k-d} \right) \\ &\quad + \sum_{k=0}^{d-1} E\Bigg[\left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right)' \Upsilon_k(N) \\ &\qquad \times \left(u_{k-d} + \Upsilon_k^{-1}(N) M_k(N) \hat{x}_{k|k-d} \right) \Bigg]. \end{aligned}$$

By (59), $V_N(0, x_0)$ is as

$$V_N(0, x_0) = E\left[x_0' P_0^1 x_0 + x_0' \sum_{i=2}^{d+1} P_0^i \hat{x}_{0|i-d-2} \right] = x_0' \sum_{i=1}^{d+1} P_0^i x_0.$$

where $\hat{x}_{0|i-d-2} = x_0$ has been applied. Hence

$$J_N^\star = E\Bigg[x_0'\sum_{i=1}^{d+1} P_0^i(N)x_0 - \sum_{k=0}^{d-1} u_{k-d}'Ru_{k-d} + \sum_{k=0}^{d-1}\left(u_{k-d}+\Upsilon_k^{-1}(N)M_k(N)\hat{x}_{k|k-d}\right)'\Upsilon_k(N) \times\left(u_{k-d}+\Upsilon_k^{-1}(N)M_k(N)\hat{x}_{k|k-d}\right)\Bigg].$$

Using $u_i = 0, i = -d, \ldots, -1$, the optimal cost becomes

$$\begin{aligned}J_N^\star &= E\left[x_0'\sum_{i=1}^{d+1} P_0^i(N)x_0 - \sum_{k=0}^{d-1}\hat{x}_{k|k-d}'P_k^2(N)\hat{x}_{k|k-d}\right]\\ &= x_0'\sum_{i=1}^{d+1} P_0^i(N)x_0 - \sum_{k=0}^{d-1} x_0'A^{k'}P_k^2(N)A^k x_0\\ &= x_0'\sum_{i=1}^{d+1} P_0^i(N)x_0 - \sum_{k=0}^{d-1} x_0'P_0^{k+2}(N)x_0\\ &= x_0'P_0^1(N)x_0. \end{aligned}\tag{72}$$

Hence, we have

$$x_0'P_0^1(N)x_0 = J_N^\star \le J_{N+1}^\star = x_0'P_0^1(N+1)x_0.$$

The arbitrariness of x_0 implies that $P_0^1(N)$ increases with respect to N, i.e., $P_0^1(N+1) \ge P_0^1(N)$.

Now the boundedness of $P_0^1(N)$ is to be clarified. Since system (5) is stabilizable in the mean-square sense, there exists

$$u_k = Lx_k + \sum_{i=1}^{d} L_i u_{k-i}, k \ge 0 \tag{73}$$

with L and L_i, $i = 1, \ldots, d$, being constant matrices, such that the closed-loop system of (5) satisfies

$$\lim_{k\to\infty} E(x_k'x_k) = 0,\ \lim_{k\to\infty} E(u_k'u_k) = 0. \tag{74}$$

By defining

$$\bar{x}_k \doteq \begin{pmatrix} x_k\\ u_{k-1}\\ \vdots\\ u_{k-d}\end{pmatrix},\ \bar{A}_k \doteq \begin{pmatrix} A_k & 0 & \cdots & 0 & B_k\\ 0 & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0\end{pmatrix}$$

$$\hat{B} \doteq \begin{pmatrix} 0\\ I\\ 0\\ \vdots\\ 0\end{pmatrix}$$

the system (5) is converted into the following "delay-free" system:

$$\bar{x}_{k+1} = \bar{A}_k\bar{x}_k + \hat{B}u_k. \tag{75}$$

Note that (73) can be rewritten as

$$u_k = Lx_k + \sum_{i=1}^{d} L_i u_{k-i} = \begin{pmatrix} L & L_1 & \cdots & L_d\end{pmatrix}\begin{pmatrix} x_k\\ u_{k-1}\\ \vdots\\ u_{k-d}\end{pmatrix}.$$

Denote $\bar{L} \doteq (L\ L_1\ \cdots\ L_d)$. Thus $u_k = \bar{L}\bar{x}_k, k \ge 0$. Then the closed-loop system of (75) is

$$\bar{x}_{k+1} = (\bar{A}_k + \hat{B}\bar{L})\bar{x}_k. \tag{76}$$

From (74), it can be derived that

$$\lim_{k\to\infty} E\left(\bar{x}_k'\bar{x}_k\right) = \lim_{k\to\infty} E\left(x_k'x_k + \sum_{i=1}^{d} u_{k-i}'u_{k-i}\right) = 0$$

i.e., (76) is asymptotically mean-square stable. By [30], there exists a constant $c > 0$, such that for any deterministic initial value $\bar{x}_0$, we have

$$\sum_{k=0}^{\infty} E(\bar{x}_k'\bar{x}_k) \le c\bar{x}_0'\bar{x}_0. \tag{77}$$

Select a constant λ, such that $Q \le \lambda I$ and $\bar{L}'R\bar{L} \le \lambda I$. Then

$$\begin{aligned}J &= \sum_{k=0}^{\infty} E\left(x_k'Qx_k\right) + \sum_{k=d}^{\infty} E\left(u_{k-d}'Ru_{k-d}\right)\\ &= \sum_{k=0}^{\infty} E\left(x_k'Qx_k\right) + \sum_{k=0}^{\infty} E\left(\bar{x}_k'\bar{L}'R\bar{L}\bar{x}_k\right)\\ &\le \lambda\left[\sum_{k=0}^{\infty} E\left(x_k'x_k\right) + \sum_{k=0}^{\infty} E\left(\bar{x}_k'\bar{x}_k\right)\right]\\ &\le 2\lambda\sum_{k=0}^{\infty} E\left(\bar{x}_k'\bar{x}_k\right) \le 2\lambda c\bar{x}_0'\bar{x}_0.\end{aligned}$$

Let $u_{-i} = 0, i = 1, \ldots, d$. We have

$$0 \le x_0'P_0^1(N)x_0 = J_N^\star \le J \le 2\lambda c x_0'x_0$$

which indicates that $0 \le P_0^1(N) \le 2\lambda c I$, i.e., $P_0^1(N)$ is bounded. Recall that $P_0^1(N)$ is monotonically increasing. Hence, it is convergent, i.e.,

$$\lim_{N\to\infty} P_0^1(N) = P^1.$$

From (68), we obtain

$$\lim_{N\to\infty} P_k^1(N) = \lim_{N\to\infty} P_0^1(N-k) = P^1.$$

Therefore, $P_k^1(N)$ is convergent for any $k \ge 0$.

In order to show the convergence of P_k^i for $i = 2, \ldots, d+1$, we shall prove the convergence of $\sum_{i=1}^{d+1} P_k^i(N)$ first. In fact, via (32)–(34), we get

$$0 \le \sum_{i=1}^{d+1} P_k^i(N) \le P_k^1(N) = P_0^1(N-k) \le 2\lambda c I$$

where the last inequality holds, as shown in the above. Thus $\sum_{i=1}^{d+1} P_k^i(N)$ is bounded. With a similar line to the proof of Lemma 3, it is easily known that $\sum_{i=1}^{d+1} P_d^i(N)$ is increasing on N. So $\sum_{i=1}^{d+1} P_k^i(N) = \sum_{i=1}^{d+1} P_d^i(N-k+d)$ is monotonically increasing with regards to N and thus convergent, in view of its boundedness.

Further, from (12), (16), (33), and (34)

$$\Upsilon_k(N) = B' \sum_{i=1}^{d+1} P_{k+1}^i(N)B + \sigma^2 \bar{B}' P_{k+1}^1(N)\bar{B} + R \geq R$$
$$M_k(N) = B' \sum_{i=1}^{d+1} P_{k+1}^i(N)A + \sigma^2 \bar{B}' P_{k+1}^1(N)\bar{A}.$$

Since $\sum_{i=1}^{d+1} P_{k+1}^i(N)$ and $P_{k+1}^1(N)$ are convergent as shown in the above, $\Upsilon_k(N)$ and $M_k(N)$ converge, i.e.,

$$\lim_{N\to\infty} \Upsilon_k(N) = \Upsilon \geq R > 0, \ \lim_{N\to\infty} M_k(N) = M.$$

Notice that

$$P_k^2(N) = -M_k'(N)\Upsilon_k^{-1}(N)M_k(N)$$
$$P_k^i(N) = A'P_{k+1}^{i-1}(N)A, \quad i = 3, \dots, d+1.$$

Thus $P_k^i(N)$ is convergent for $i = 2, \dots, d+1$ and any $k \geq 0$. By letting $N \to \infty$ on both sides of (12)–(16), it can be derived that P^i, $i = 1, \dots, d+1$ obeys (35)–(39).

2). From Lemma 3, there exists a positive integer N_0, such that $\sum_{i=1}^{d+1} P_d^i(N_0) > 0$. Since $\sum_{i=1}^{d+1} P_d^i(N)$ is monotonically increasing with respect to N, there holds

$$\sum_{i=1}^{d+1} P^i = \lim_{N\to\infty} \sum_{i=1}^{d+1} P_d^i(N) \geq \sum_{i=1}^{d+1} P_d^i(N_0) > 0.$$

Thus the positive definiteness of $\sum_{i=1}^{d+1} P^i$ is shown. The proof of Theorem 2 is now completed. □

Appendix F
Proof of Theorem 3

Proof—"Sufficiency": Assume P^i, $i = 1, \dots, d+1$ is a solution to (35)–(39) such that $\sum_{i=1}^{d+1} P^i > 0$. We shall show that (40) stabilizes system (5) in the mean-square sense. To this end, define the Lyapunov function candidate $V(k, x_k)$ as

$$V(k, x_k) \doteq E\left[x_k' P^1 x_k + x_k' \sum_{i=2}^{d+1} P^i \hat{x}_{k|i+k-d-2}\right]. \tag{78}$$

The claim that $V(k, x_k)$ is monotonically decreasing and bounded with regard to k for $k \geq d$ is to be proven. Using (5) and (35)–(39) yields

$$\begin{aligned}
&V(k, x_k) - V(k+1, x_{k+1}) \\
&= E\Bigg\{x_k' \left(P^1 - A'P^1A - \sigma^2 \bar{A}'P^1\bar{A} - A'P^{d+1}A\right) x_k \\
&\quad - 2\hat{x}_{k|k-d}' \left(A'P^1B + \sigma^2 \bar{A}'P^1\bar{B} + A' \sum_{i=2}^{d+1} P^i B\right) u_{k-d} \\
&\quad - u_{k-d}' \left(B'P^1B + \sigma^2 \bar{B}'P^1\bar{B} + B' \sum_{i=2}^{d+1} P^i B\right) u_{k-d} \\
&\quad - x_k'A' \sum_{i=2}^{d} P^i A \hat{x}_{k|i+k-d-1} + x_k' \sum_{i=2}^{d+1} P^i \hat{x}_{k|i+k-d-2}\Bigg\} \\
&= E\Big\{x_k'Qx_k - \hat{x}_{k|k-d-1}' M' u_{k-d} - u_{k-d}' M \hat{x}_{k|k-d} \\
&\quad - u_{k-d}'(\Upsilon - R)u_{k-d} + x_k' P^2 \hat{x}_{k|k-d}\Big\} \\
&= E\Big[x_k'Qx_k + u_{k-d}'Ru_{k-d} - \left(u_{k-d} + \Upsilon^{-1}M\hat{x}_{k|k-d}\right)' \Upsilon \\
&\quad \times \left(u_{k-d} + \Upsilon^{-1}M\hat{x}_{k|k-d}\right)\Big] \qquad (79) \\
&= E[x_k'Qx_k + u_{k-d}'Ru_{k-d}] \geq 0, \quad k \geq d \qquad (80)
\end{aligned}$$

where $u_{k-d} = -\Upsilon^{-1}M\hat{x}_{k|k-d}$ for $k \geq d$ has been imposed in the last identity. The above inequality indicates that $V(k, x_k)$ decreases with respect to k. In virtue of the orthogonality of x_k and $x_k - \hat{x}_{k|i+k-d-2}$, and $P^i \leq 0, i = 2, \dots, d+1$, the Lyapunov function $V(k, x_k)$ becomes

$$\begin{aligned}
V(k, x_k) &= E\Bigg\{x_k' \sum_{i=1}^{d+1} P^i x_k - \sum_{i=2}^{d+1} [x_k - \hat{x}_{k|i+k-d-2}]' P^i \\
&\qquad \times [x_k - \hat{x}_{k|i+k-d-2}]\Bigg\} \\
&\geq E\left(x_k' \sum_{i=1}^{d+1} P^i x_k\right) \geq 0 \qquad (81)
\end{aligned}$$

i.e., $V(k, x_k)$ is bounded below and thus is convergent.

Now let m be any nonnegative integer. By adding from $k = m+d$ to $k = m+N$ on both sides of (80) and letting $m \to +\infty$, it yields that

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{k=m+d}^{m+N} E\left[x_k'Qx_k + u_{k-d}'Ru_{k-d}\right] \\
&\quad = \lim_{m\to\infty} V(m+d, x_{m+d}) - V(m+N+1, x_{m+N+1}) = 0
\end{aligned} \tag{82}$$

where the last equality holds because of the convergence of $V(k, x_k)$. Recall that

$$\sum_{k=d}^{N} E\left(x_k'Qx_k + u_{k-d}'Ru_{k-d}\right) \geq x_d' \sum_{i=1}^{d+1} P_d^i(N) x_d.$$

Via a time-shift of length of m, it leads to

$$\begin{aligned}
&\sum_{k=m+d}^{m+N} E\left(x_k'Qx_k + u_{k-d}'Ru_{k-d}\right) \\
&\qquad \geq E\left(x_{m+d}' \sum_{i=1}^{d+1} P_{m+d}^i(m+N) x_{m+d}\right) \\
&\qquad = E\left(x_{m+d}' \sum_{i=1}^{d+1} P_d^i(N) x_{m+d}\right) \geq 0.
\end{aligned}$$

From (82), it follows that:

$$\lim_{m\to\infty} E\left[x'_{m+d}\sum_{i=1}^{d+1} P_d^i(N)x_{m+d}\right] = 0, \quad \forall N \geq d. \tag{83}$$

According to Lemma 3, there exists N_0, such that $\sum_{i=1}^{d+1} P_d^i(N_0)$ is positive definite. Thus (83) implies that $\lim_{m\to\infty} E[x'_{m+d}x_{m+d}] = 0$. Therefore, the controller (40) stabilizes (5) in the mean-square sense.

Next we will show that (40) minimizes the cost function (6). Apply (79) again. Adding from $k=0$ to $k=N$ to (79) leads to

$$\begin{aligned} E&\left[\sum_{k=0}^{N} x'_k Q x_k + \sum_{k=d}^{N} u'_{k-d}Ru_{k-d}\right] \\ &= V(0,x_0) - V(N+1,x_{N+1}) \\ &\quad + \sum_{k=d}^{N} E\left[(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})'\Upsilon \right.\\ &\qquad\qquad \left.\times(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})\right] \\ &\quad + \sum_{k=0}^{d-1} E\left[(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})'\Upsilon \right.\\ &\qquad\qquad \left.\times(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})\right] \\ &\quad - \sum_{k=0}^{d-1} E(u'_{k-d}Ru_{k-d}) \end{aligned} \tag{84}$$

where $V(0,x_0)$ and $V(N+1,x_{N+1})$ are defined in (78). Then $\lim_{k\to\infty} V(k,x_k) = 0$ is to be shown. In fact, in view of (81) and $P^i \leq 0, i=2,\ldots,d+1$, it follows that:

$$\begin{aligned} 0 \leq V(k,x_k) &= E\left[x'_kP^1x_k + \hat{x}'_{k|i+k-d-2}\sum_{i=2}^{d+1} P^i\hat{x}_{k|i+k-d-2}\right] \\ &\leq E\left(x'_kP^1x_k\right). \end{aligned}$$

Now we only consider the controller which stabilizes system (5). Thus $\lim_{k\to\infty} E(x'_kP^1x_k) = 0$. So $\lim_{k\to\infty} V(k,x_k) = 0$. By letting $N\to\infty$ on both sides of (84), the cost function (6) is rewritten as

$$\begin{aligned} J &= x'_0\sum_{i=1}^{d+1} P^i x_0 \\ &\quad + \sum_{k=0}^{d-1} E\left[(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})'\Upsilon \right.\\ &\qquad\qquad \left.\times(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})\right] - \sum_{k=0}^{d-1} u'_{k-d}Ru_{k-d} \\ &\quad + \sum_{k=d}^{\infty} E\left[(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})'\Upsilon \right.\\ &\qquad\qquad \left.\times(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d})\right]. \end{aligned} \tag{85}$$

In view of the positive definiteness of Υ, the optimal controller to minimize (85) must be (40) and the corresponding optimal cost is as (41). Therefore the proof of sufficiency is finished.

"*Necessity*": Suppose the system (5) is stabilizable in the mean-square sense. In Theorem 2, the existence of the solution to (35)–(39) satisfying $\sum_{i=1}^{d+1} P^i > 0$ has been verified. We just need to show the uniqueness. Let $S^i, i=1,\ldots,d+1$, be another solution to (35)–(39) satisfying $\sum_{i=1}^{d+1} S^i > 0$, i.e.,

$$S^1 = A'S^1A + \sigma^2\bar{A}'S^1\bar{A} + A'S^{d+1}A + Q \tag{86}$$

$$S^2 = -\Pi'\Delta^{-1}\Pi \tag{87}$$

$$S^i = A'S^{i-1}A, \quad i=3,\ldots,d+1 \tag{88}$$

where

$$\Delta = \sum_{i=1}^{d+1} B'S^iB + \sigma^2\bar{B}'S^1\bar{B} + R > 0 \tag{89}$$

$$\Pi = \sum_{i=1}^{d+1} B'S^iA + \sigma^2\bar{B}'S^1\bar{A}. \tag{90}$$

In view of the proof of sufficiency as in the above, the optimal value of the cost function (6) is as

$$\begin{aligned} J_0 &= x'_0\sum_{i=1}^{d+1} P^ix_0 + \sum_{k=0}^{d-1} E\Big[\left(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d}\right)'\Upsilon \\ &\quad\times\left(u_{k-d}+\Upsilon^{-1}M\hat{x}_{k|k-d}\right)\Big] - \sum_{k=0}^{d-1} u'_{k-d}Ru_{k-d} \\ &= x'_0\sum_{i=1}^{d+1} S^ix_0 + \sum_{k=0}^{d-1} E\Big[\left(u_{k-d}+\Delta^{-1}\Pi\hat{x}_{k|k-d}\right)'\Delta \\ &\quad\times\left(u_{k-d}+\Delta^{-1}\Pi\hat{x}_{k|k-d}\right)\Big] - \sum_{k=0}^{d-1} u'_{k-d}Ru_{k-d}. \end{aligned} \tag{91}$$

If $u_{-1},\ldots,u_{-d}$ are zero, the identity (91) becomes

$$J_0 = x'_0P^1x_0 = x'_0S^1x_0.$$

As x_0 is arbitrary, the above equation implies that

$$P^1 = S^1. \tag{92}$$

If we let $u_{k-d} = -\Upsilon^{-1}M\hat{x}_{k|k-d}, k=0,\ldots,d-1$, it follows from (91) and $\Delta > 0$ that:

$$\begin{aligned} &x'_0\sum_{i=1}^{d+1} P^ix_0 \\ &= x'_0\sum_{i=1}^{d+1} S^ix_0 + \sum_{k=0}^{d-1} E\Big[\left(u_{k-d}+\Delta^{-1}\Pi\hat{x}_{k|k-d}\right)'\Delta\,(u_{k-d} \\ &\quad + \Delta^{-1}\Pi\hat{x}_{k|k-d})\Big] \geq x'_0\sum_{i=1}^{d+1} S^ix_0. \end{aligned}$$

Since x_0 is arbitrary, $\sum_{i=1}^{d+1} P^i \geq \sum_{i=1}^{d+1} S^i$. In a similar way, we can show that $\sum_{i=1}^{d+1} P^i \leq \sum_{i=1}^{d+1} S^i$. Therefore

$$\sum_{i=1}^{d+1} P^i = \sum_{i=1}^{d+1} S^i. \tag{93}$$

Furthermore, via (92), (93), (38)–(39), and (89)–(90), it follows that $\Upsilon = \Delta, M = \Pi$ and (36), (37), (87), and (88) result in $P^i = S^i, i = 2, \ldots, d+1$. Thus the uniqueness has been proven. The proof of necessity is now complete. □

ACKNOWLEDGMENT

The authors would like to thank Dr. C. Tan, Shandong University, for his comments in Remark 1 and Corollary 1.

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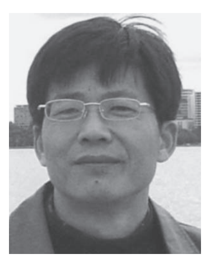

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