# A Graph Laplacian Approach to Coordinate-Free Formation Stabilization for Directed Networks

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***Abstract*—This paper concentrates on coordinate-free formation control for directed networks, for which the dynamic motion of each agent is assumed to be governed only by a local control. We develop a graph Laplacian approach to solve the global and exponential formation stablization problem using merely relative position measurements between neighbors. First, to capture the sensing and control architectures that are needed to maintain the shape of a formation, a necessary and sufficient topological condition is proposed. Second, a Laplacian-based control law is developed for the stablization problem of a group of mobile agents to a desired formation shape under both fixed and switching topologies due to temporal node failures. Simulation results are provided to demonstrate that our Laplacian-based formation control strategy is inherently fault-tolerant and robust to node failures.**

***Index Terms*—Directed graph, distributed control, formation, multi-agent systems.**

## I. Introduction

MULTI-AGENT systems have received a lot of attention by researchers from many different disciplines. On one hand, a huge number of amazing examples of multi-agent systems are observed in nature, and researchers aim to understand their underlying mechanisms of how local interactions lead to collective patterns such as weaver ants forming a line to collaboratively pull nest leaves together, geese flying in V-formation for energy saving, and multi-cellular organisms establishing spatial patterns of gene expression during development [1], [2]. On the other hand, technological advances in embedded computing and control continuously enable the development of new artificial multi-agent systems of higher complexity, including autonomous underwater, ground or aerial vehicles, which are able to perform coordinated tasks effectively in many applications such as search and rescue in hazardous environments, ocean data retrieval and sampling, surveillance/combat tasks, etc. [3], [4]. For these engineering systems, distributed implementations are preferred as they provide higher flexibilities, scalability and robustness compared with centralized implementations.

We address in this paper a fundamental coordinated task for multi-agent systems, namely, the formation control problem. There has been extensive literature on formation control. Depending on whether a global coordinate system is needed, formation control can be classified into two categories: coordinate-dependent and coordinate-free. Coordinate-dependent formation control either assumes that the agents know their absolute coordinates in a global coordinate system such as [5]–[7] or assumes that the agents share a common north such as [8]–[13]. When the agents are able to access their global coordinates, then the formation control problem can be addressed in a trajectory tracking framework given a desired formation and a desired trajectory for the center of mass of the formation [6]. On the other hand, when the agents share a common north, then the formation control problem can be dealt with in the consensus framework by adding the desired relative positions with respect to the common north, for which the orientation of achieved formations is fixed by the common north [8]–[12]. In certain scenarios, the access of global coordinates or a common north may be difficult, e.g., in GPS-denied areas, indoors etc. To alleviate the requirement of such global information, it is therefore important to develop coordinate-free formation control strategies.

Coordinate-free formation control typically relies on how to define a formation shape based only on inter-agent constraints expressed without using a common coordinate system. One way is to specify distance constraints between pairs of neighboring agents and thus removes the requirement for a common coordinate system. This develops distance-based formation control [14]–[23]. The other way is to specify relative position constraints using the barycentric coordinates and thus does not require a common coordinate system. This leads to Laplacian-based formation control [24]–[30]. It is worth to point out that although the distances between pairs of neighboring agents are expected to be controlled in distance-based formation control, the control law still uses relative positions and not just the current distances between neighboring agents [17]–[23]. That is, more needs to be sensed than being controlled. The only exception is [31], in which the current distances are used as feedback information. However, the algorithm in [31] is a cyclic stop-and-go strategy, i.e., at each step, only one agent can move while its neighbors have to remain stationary. Compared with distance-based formation control, the advantages of Laplacian-based formation control are two-folds. First, the Laplacian-based approach requires less links for formation control as it uses relative positions as feedback and the same needs to be sensed as being controlled. Second, the Laplacian-based formation control law is a linear one and thus ensures globally

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exponential convergence, while distance-based formation control takes nonlinear gradient-type laws, which only assure local convergence in general. However, the drawback of Laplacian-based formation control is that the resulting formation has one more degree of freedom (namely, scaling of the formation) in addition to rotations and translations, which are the three degrees of freedom for distance-based formation. Nevertheless, as shown in [29], by additionally controlling a pair of agents to reach different distances, the scale of the formation can be altered. In this sense, it is also an advantage that the agents can scale the formation using the Laplacian-based approach for better adaptivity to possibly changing environments, e.g., shrinking the formation to pass through a narrow corridor. [32] also develops a scheme to scale the formation by modifying distance-based formation control, but the control law is much more complicated.

Regarding Laplacian-based formation control, two fundamental questions need to be addressed. That is, what are the types of sensing and control architectures that are needed to maintain the shape of a formation and what are the distributed control laws that can steer a group of mobile agents to form a desired geometric shape? Assuming a directed acyclic network for the agents, [33] developed the rudiments of complex Laplacian based formation control and showed that the leader-follower system is bounded input-bounded output (BIBO) stable under certain conditions. Later in [27], the concept of complex Laplacian was formally introduced to study formation control in the plane for leader-follower networks with cycles. Only till very recently, [29] systematically addressed the aforementioned two fundamental questions in the leader-less setup. But it assumed that the network is undirected and static. No results have been reported for Laplacian-based formation control over directed leader-less networks except the preliminary conference version [24], [26] of this paper. However, as commented in [15], using directed networks as opposed to undirected networks shows many advantages such as halved sensing information and reduced difficulties with interference, and has more practical implication due to non-identical sensing capabilities of the agents. But formation control becomes technically more challenging in the directed graph case because many good properties due to the symmetric structure of undirected graphs do not exist any more and many tools that usually take advantages of the symmetric property of undirected graphs are not applicable to directed networks.

This paper aims to solve the aforementioned two fundamental problems for directed networks. First, to capture the necessary and sufficient graphical characterization for formations of directed networks such that a formation shape can be uniquely determined by the linear constraints on the agents regarding their neighbors, this paper develops a new approach that does not rely on the symmetric structure property as done in [29]. We then show the connection of topological connectivity and the generic rank of complex-valued Laplacian, which is defined to be the maximum of the rank of any complex-valued Laplacian subject to the zero/nonzero structure constraint. Using this connection, the sensing and control architecture that is needed to maintain the shape of a formation in a directed graph setup is uncovered. Second, assuming that the dynamical motion of agents is governed only by a local control, this paper solves the stabilization problem of a group of agents to a desired formation shape under a fixed topology or a switching interaction topology due to temporal node failures. The proposed control law is valid for directed graphs with cycles. Due to the nature of the onboard sensor devices and the usually harsh environments, it becomes possible that one or more agents may temporarily fail to get the relative position measurements, which leads to a switching topology. Switching information graphs have been considered in collective motion control [34] and flocking [35], which however does not aim to form a specific formation for the agents but stay cohesively. This paper provides a feasible solution to achieve an arbitrary formation shape using the Laplacian-based formation control law. We show that a group of agents can globally exponentially reach the desired formation shape with any desired exponential convergence rate if a scalar control parameter is designed to make the averaged dominant eigenvalue of the switched system surpass the divergence rate of the system due to node failures. In this sense, our Laplacian-based formation control strategy is inherently fault-tolerant and robust to node failures.

*Notation:* $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex and real numbers, respectively. $\iota = \sqrt{-1}$ denotes the imaginary unit. For a complex number $p \in \mathbb{C}$, $|p|$ represents its modulus. For a set $\mathcal{E}$, $|\mathcal{E}|$ represents its cardinality. For a matrix $M \in \mathbb{C}^{n\times n}$, $|M|$ represents its matrix norm induced by Euclidean norm. $\mathbf{1}_n$ represents the $n$-dimensional vector of ones and $I_n$ denotes the identity matrix of order $n$.

## II. Preliminaries and Problem Formulation

In this section, we introduce several notions and a preliminary result for directed graphs, and formulate the problems.

### A. Preliminaries From Graph Theory

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ whose elements are ordered pairs of nodes. If $(j, i) \in \mathcal{E}$, node $j$ is called an *in-neighbor* of $i$. We let $\mathcal{N}_i$ denote the *in-neighbor set* of node $i$, i.e., $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. A *path* in a directed graph $\mathcal{G}$ is an alternating sequence $p : v_1 e_1 v_2 e_2 \cdots e_{k-1} v_k$ of nodes and edges such that $e_i = (v_i, v_{i+1})$ for every $i = 1, 2, \ldots, k-1$ where the nodes in the sequence are distinct. Two paths from a subset of nodes, $\mathcal{U} \subset \mathcal{V}$, to a node $v$ are said to be *disjoint* if there are no common nodes in the two paths except $v$. A node $v$ is said to be *reachable* from a subset of nodes, $\mathcal{U} \subset \mathcal{V}$, if there is a path from a node in $\mathcal{U}$ to $v$.

Next, we introduce several new concepts, which extend the same notions for undirected graphs [29] to directed graphs.

*Definition 2.1:* A node $v$ is said to be *2-reachable* from a non-singleton set $\mathcal{U}$ of nodes in $\mathcal{G}$ if there exists a path from a node in $\mathcal{U}$ to $v$ after removing any one node except node $v$ (i.e., there are two disjoint paths from $\mathcal{U}$ to $v$).

*Definition 2.2:* A directed graph $\mathcal{G}$ is said to be *2-rooted* if there exists a subset of two nodes, from which every other node is 2-reachable. These two nodes are called *roots* of $\mathcal{G}$.

*Definition 2.3:* A spanning 2-tree of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, rooted at $R = \{r_1, r_2\} \subset \mathcal{V}$, is a spanning subgraph $\mathcal{T} = (\mathcal{V}, \bar{\mathcal{E}})$ such that

1) every node $r \in R$ has no in-neighbor;
2) every node $v \notin R$ has 2 in-neighbors;
3) every node $v \notin R$ is 2-reachable from $R$.

In the following, we provide a preliminary result about the 2-rooted connectivity of a directed graph.

*Lemma 2.1:* A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is 2-rooted if and only if $\mathcal{G}$ has a spanning 2-tree.

*Proof:* (Sufficiency) If $\mathcal{G}$ has a spanning 2-tree, then by the definition of a 2-rooted graph, it follows that $\mathcal{G}$ is 2-rooted.

(Necessity) By the definition of 2-rooted connectivity, we know that there exists a subset of two nodes, called roots, such that every other node is 2-reachable from it. Denote by $R = \{r_1, r_2\}$ the set of two roots. Moreover, from the definition of 2-rooted connectivity, we know that every node not in $R$ has at least two incoming edges. We then show that a spanning 2-tree rooted at $R$ can be constructed by the following operations. First, we remove all incoming edges to nodes $r_1$ and $r_2$. Second, for every node $v \notin R$, we denote all the incoming edges to node $v$ by $e_1, e_2, \ldots, e_m$. We remove some incoming edges (see for example, $e_1, e_2, \ldots, e_{m-2}$) for node $v$ such that node $v$ has only two incoming edges left and is 2-reachable from $R$. We denote the resulting graph by $\mathcal{T}$.

We now show that $\mathcal{T}$ is a spanning 2-tree of $\mathcal{G}$. First, we know that every node $v \notin R$ is still 2-reachable from $R$ in $\mathcal{T}$ according to our operation rules. Second, we show in the following that every node $u \notin R \cup \{v\}$ is also 2-reachable from $R$ in $\mathcal{T}$. For the first case, if $u$ has two disjoint paths from $R$, which do not go through $v$, then the removal of edges on node $v$ does not affect the 2-reachability from $R$ to $u$. For the second case, if $u$ has two disjoint paths from $R$ and one of them passes through $v$, then the removal of the edges $e_1, e_2, \ldots, e_{m-2}$ on node $v$ does not make the path from $R$ to node $u$ via node $v$ disappear as node $v$ still has two incoming edges left and is still 2-reachable from $R$. So there are still two disjoint paths from $R$ to node $u$. Therefore, by Definition 2.3, $\mathcal{T}$ is a spanning 2-tree of $\mathcal{G}$. ■

Finally, we introduce the Laplacian of a directed graph $\mathcal{G}$ with $n$ nodes, which is an $n$-by-$n$ matrix with its $(i, j)$th off-diagonal entry $-w_{ij}$ if $j \in \mathcal{N}_i$ and 0 otherwise, and the $(i, i)$th diagonal entry $\sum_{k \in \mathcal{N}_i} w_{ik}$. If the weights $w_{ij}$'s equal to 1 on the edges, then $L$ is called the *binary Laplacian* of $\mathcal{G}$. If the weights $w_{ij}$'s are complex numbers, then $L$ is called the *complex Laplacian* of $\mathcal{G}$. We use $\mathcal{L}(\mathcal{G})$ to denote the set of all complex Laplacian associated to $\mathcal{G}$. It is clear that for any $L \in \mathcal{L}(\mathcal{G})$, $L\mathbf{1}_n = 0$. That is, a Laplacian matrix must have an eigenvalue at the origin with its associated vector 1. The Laplacian matrix is a discrete analog of the Laplacian operator in multivariable calculus and arises in the analysis of random walk and electrical networks on graphs [36].

### B. Problem Formulation

We consider a group of $n$ agents in the plane. The positions of $n$ agents are denoted by complex numbers $z_1, \cdots, z_n \in \mathbb{C}$. The aggregate state $z = [z_1, z_2, \cdots, z_n]^T \in \mathbb{C}^n$ is called the *position vector* of the $n$ agents. The motion of each agent is governed by the following dynamics in continuous time:

$$\dot{z}_i(t) = u_i(t), \quad i = 1, \ldots, n \tag{1}$$

where $u_i(t) \in \mathbb{C}$ is the control input. That is to say, by this model, we do not explicit account for: internal dynamics of the agent, environmental forcing terms, stochastic forcing, and external forces. Also, except for node failure, the time-dependence aspects are not considered.

Define a *target configuration* $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{C}^n$ to be an assignment of the $n$ agents to points in a global reference frame $\Sigma$, which is used to characterize the formation shape that the agents try to achieve. However, it should be emphasized that the agents do not have the knowledge about the global reference frame $\Sigma$ and do not know their own absolute positions $z_i$'s.

Throughout the paper, we assume that the target configuration $\xi$ is generic. A configuration $\xi$ is said to be *generic* if the coordinates $\xi_1, \ldots, \xi_n$ do not satisfy any nontrivial algebraic equation with integer coefficients [37]. Intuitively speaking, a generic configuration has no degeneracy, i.e., no three points staying on the same line, no three lines go through the same point, etc. The set of all formations similar to the target configuration $\xi$ is described by

$$\mathcal{S}(\xi) = \{c_1\mathbf{1}_n + c_2\xi : \; c_1, c_2 \in \mathbb{C}\}.$$

In other words, the set $\mathcal{S}(\xi)$ consists of *similar formations* obtained from $\xi$ via translations $c_1$, rotations $\theta$, and scaling $\rho$ (four degrees of freedom) where $\theta$ and $\rho$ are the phase and magnitude of the complex number $c_2$.

A directed graph $\mathcal{G}$ of $n$ nodes represents the *sensing graph* in which an edge $(j, i)$ indicates that agent $i$ can measure the relative position of agent $j$. To be more specific, suppose that each agent $i$ has a local frame $\Sigma_i$, whose orientation angle (namely, the angle between the $x$-axis of this local frame and the global frame) is $\theta_i$. Then the relative position measurements by agent $i$ are

$$y_{ij} = e^{\iota\theta_i}(z_j - z_i), \quad \forall j \in \mathcal{N}_i.$$

*Remark 2.1:* It is worth to point out that the actual control equation of each agent $i$ is

$$\dot{z}_i = e^{-\iota\theta_i}u_i^a \tag{2}$$

where $u_i^a$ is the actual control input of agent $i$ defined with respect to its own local coordinate system $\Sigma_i$. But if the control law $u_i$ of the form

$$u_i = g_i(\cdots, z_j - z_i, \cdots), \quad \text{for } j \in \mathcal{N}_i \tag{3}$$

has the *rotational invariance property*, i.e.,

$$e^{\iota\theta_i}g_i(\cdots, z_j - z_i, \cdots) = g_i\left(\cdots, e^{\iota\theta_i}(z_j - z_i), \cdots\right)$$

then the actual control input $u_i^a$ uses only the relative position measurement $y_{ij}$, that is

$$\begin{aligned} u_i^a &= e^{\iota\theta_i}u_i = e^{\iota\theta_i}g_i(\cdots, z_j - z_i, \cdots) \\ &= g_i\left(\cdots, e^{\iota\theta_i}(z_j - z_i), \cdots\right) = g_i(\cdots, y_{ij}, \cdots). \end{aligned}$$

For the neatness of the presentation, in the rest of the paper, we will concentrate on the control law of the form (3) and the agent model in (1) rather than the one given in (2). By referring to the control law (3), we mean the control law (3) having the rotational invariance property.

The following assumptions are made in this paper.

*Assumption A1:* The local frames $\Sigma_i$ may not be aligned and the agents do not know the orientation difference.

*Assumption A2:* Each agent $i$ knows $\xi_j$ for all $j \in \mathcal{N}_i$.

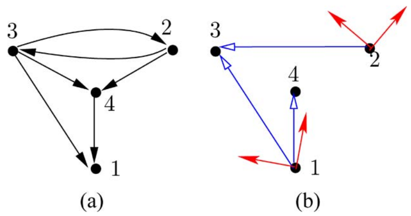


Fig. 1. A sensing graph and examples of relative measurements for agent 1 and 2 in their local frames. (a) Sensing graph. (b) Local measurements.

An illustration of a sensing graph and relative measurements in local frames is given in Fig. 1. In Fig. 1(a), the black lines with arrows indicate the edges of the graph, while in Fig. 1(b) the blue lines with arrows represent the relative state vectors (e.g., $z_3 - z_2, z_3 - z_1$, and $z_4 - z_1$) in the local frames (the red lines with arrows are the $x$-axis and $y$-axis of the local frames of different agents).

Over time, some agents may temporarily fail to sense their in-neighbors, resulting in a time-varying graph. To precisely describe the time-varying graph, we need the notion of a *switching signal*. We use the symbol $\mathcal{P}$ to denote a suitably defined set, indexing all possible sensing graphs of $n$ agents subject to some agent failures. A switching signal is a piecewise constant function $\sigma$: $[0, \infty) \to \mathcal{P}$. Such a function $\sigma$ has a finite number of discontinuities-which we call the *switching times*-on every bounded time interval and takes a constant value on every interval between two consecutive switching times. The role of $\sigma$ is to specify, at each time instant $t$, the index $\sigma(t) \in \mathcal{P}$ of the *active sensing graph*, for which we denote by $\mathcal{G}_{\sigma(t)}$.

The goal of this paper is to achieve a formation similar to the target configuration $\xi$ for arbitrarily initial distribution of $n$ agents in a general directed graph setup. We term this problem *formation shape control*. Before formally formulating our problems, we introduce a definition. Let $Q$ be an $(n-2) \times n$ matrix with orthonormal rows that are each orthogonal to $\mathbf{1}_n$ and $\xi$; that is

$$Q\mathbf{1}_n = 0,\ Q\xi = 0,\ QQ^T = I_{n-2}. \tag{4}$$

Let $z$ be the aggregate state of $z_i$'s. Then $z \to \mathcal{S}(\xi)$ is equivalent to $x := Qz \to 0$.

*Definition 2.4:* The $n$ agents are said to *globally exponentially reach a similar formation* of $\xi$ (with the exponential convergence rate $\alpha > 0$) if there exists a positive constant $c$ such that for any $x_0 = Qz_0$

$$\|Qz(t)\| = \|x(t)\| \le c\|x_0\|e^{-\alpha t}. \tag{5}$$

Thus, the formation shape control problem is summarized as follows.

(P1) Given Assumptions A1–A2 and relative position measurements $z_j - z_i$ for $j \in \mathcal{N}_i$, find the necessary and sufficient graphical condition, under which there exists a distributed local control law of the form (3) such that (5) is satisfied for some positive $\alpha$ (namely, the formation shape control problem is solvable).

(P2) Consider the scenario that the directed sensing graph $\mathcal{G}$ is fixed. Given Assumptions A1–A2 and relative position measurements $z_j - z_i$ for $j \in \mathcal{N}_i$, for a given $\alpha > 0$, find a distributed local control law of the form (3) such that (5) is satisfied.

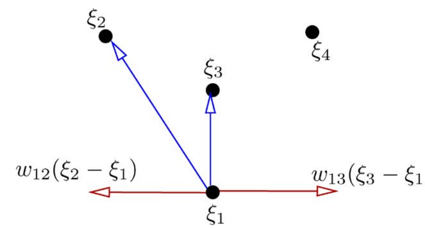


Fig. 2. An illustration of the linear constraint $\sum_{j\in\mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0$.

(P3) Consider the scenario that the directed sensing graph $\mathcal{G}$ is time-varying due to node failures or link failures which can be also treated as node failures. Given Assumptions A1–A2 and relative position measurements $z_j - z_i$ for $j \in \mathcal{N}_i$, for a given $\alpha > 0$, find a distributed local control law of the form (3) such that (5) is satisfied.

*Remark 2.2:* The solution to Problem (P1) is related to certain graph connectivity of the directed sensing graph $\mathcal{G}$. However, no appropriate known connectivity in graph theory can be found. As we will show in next section, we introduce the novel notion called "2-rooted" to characterize such graph connectivity for this problem.

## III. Graphical Condition for Formation Characterization

In this section, we will explore a necessary and sufficient graphical condition (namely, the connectivity pattern of a directed graph $\mathcal{G}$) such that any formation $z = [z_1, \ldots, z_n]^T$ satisfying

$$\sum_{j\in\mathcal{N}_i} w_{ij}(z_j - z_i) = 0, \quad i = 1, \ldots, n \tag{6}$$

for some complex weights $w_{ij}$'s on the edges of $\mathcal{G}$ is a similar formation of $\xi$ (i.e., $z \in \mathcal{S}(\xi)$). Then, we introduce a new concept, called *realizability of a similar formation on a directed graph* $\mathcal{G}$.

*Definition 3.1:* A similar formation of $\xi$ is said to be *realizable* on $\mathcal{G}$ if there are complex weights $w_{ij}$'s on the edges of $\mathcal{G}$ such that

$$\ker(L) = \mathcal{S}(\xi)$$

where $L$ is the Laplacian of $\mathcal{G}$ with complex weights $w_{ij}$'s.

*Remark 3.1:* The advantage of using (6) to define a similar formation is that no global reference frame and no absolute states are required as the condition (6) is equivalent to

$$\sum_{j\in\mathcal{N}_i} w_{ij}e^{\iota\theta_i}(z_j - z_i) = 0, \quad i = 1, \ldots, n$$

which means that the set of configurations $\{z : z \text{ satisfies (6)}\}$ is irrelevant to the orientations of each agent's local frame.

*Remark 3.2:* In order to encode the desired formation shape specified by $\xi$ in a global reference frame, the complex weights $w_{ij}$'s on the edges incident to node $i$ should be chosen to satisfy

$$\sum_{j\in\mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0. \tag{7}$$

An illustration of such a linear constraint (7) is given in Fig. 2, in which agent 1 has two in-neighbors (agent 2 and 3). The complex weights $w_{12}$ and $w_{13}$ thus rotate and scale the vectors

$\xi_2 - \xi_1$ and $\xi_3 - \xi_1$ respectively so that the sum becomes zero. We point out that such complex weights are not unique.

*Remark 3.3:* Definition 3.1 indicates that if a graph $\mathcal{G}$ makes a similar formation realizable, then the desired formation shape can be maintained by the agents that actively control themselves to satisfy (6).

A necessary and sufficient topological condition is given below for the realizability of a similar formation on $\mathcal{G}$.

*Theorem 3.1:* For a generic $\xi \in \mathbb{C}^n$, a similar formation of $\xi$ is realizable on $\mathcal{G}$ if and only if $\mathcal{G}$ is 2-rooted.

The proof of Theorem 3.1 is technically involved. We will develop two lemmas first. These two lemmas establish the relationship between the connectivity property of a directed graph and the algebraic property of the associated Laplacian matrix.

In the sequel, we use $L_R$ to represent the sub-matrix of $L$ with the rows and columns corresponding to nodes in $R \subset \mathcal{V}$ crossed out.

*Lemma 3.1:* If $\mathcal{G}$ is 2-rooted with the root set $R = \{r_1, r_2\}$, then for the Laplacian $L$ of $\mathcal{G}$ with almost all[1] complex weights $w_{ij}$'s,

a) all the principal minors of $L_R$ are distinct from zero;
b) $\det(M) \neq 0$ where $M$ is the sub-matrix of $L$ by deleting the two rows corresponding to the two roots and any two columns.

The proof of Lemma 3.1 is given in the Appendix.

*Lemma 3.2:* Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a generic $\xi \in \mathbb{C}^n$. The following statements are equivalent.

1) $\mathcal{G}$ is 2-rooted with the root set $R = \{r_1, r_2\}$.
2) For almost all[2] $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, all the principal minors of $L_R$ are distinct from zero.
3) There exists an $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ such that $\det(L_R) \neq 0$.

*Proof of Lemma 3.2:* (1) $\Longrightarrow$ (2) If (1) holds, then by Lemma 2.1, it follows that $\mathcal{G}$ has a spanning 2-tree rooted at $R = \{r_1, r_2\}$. We denote it by $\mathcal{T} = (\mathcal{V}, \bar{\mathcal{E}})$.

First, we show that for any $L \in \{L \in \mathcal{L}(\mathcal{T}) : L\xi = 0\}$ with $\xi$ being generic, all the principal minors of $L_R$ are distinct from zero. To this end, we first consider an $L \in \mathcal{L}(\mathcal{T})$, that does not need to satisfy $L\xi = 0$. Recall that for a spanning 2-tree, the two rows of $L \in \mathcal{L}(\mathcal{T})$ corresponding to the two roots ($r_1$ and $r_2$) are all zeros. On the other hand, from Lemma 3.1, we know that $\operatorname{rank}(L) = n - 2$. Thus, the kernel of $L$ is a two-dimensional subspace, for which one basis vector is 1. We denote the other linearly independent basis vector as $\eta$. We show in the following that the components, $\eta_i$'s of $\eta$, are distinct. To see this, suppose by contradiction that there exist two components, say $\eta_i$ and $\eta_j$, which are equal. Then the basis $\eta$ can be scaled to have $\eta_i = \eta_j = 1$. Let $\eta_s$ be the sub-vector composed of the remaining entries of $\eta$ after removing $\eta_i$ and $\eta_j$. Denote by $N$ the sub-matrix of $L$ by deleting the two rows corresponding to $r_1$, $r_2$, and the two columns corresponding to nodes $i$ and $j$. Moreover, denote by $M$ the sub-matrix of $L$ by deleting the two rows corresponding to $r_1$, $r_2$, and the $n-2$ columns corresponding to all nodes in $\mathcal{V} - \{i, j\}$. Thus, we have

$$N\mathbf{1}_{(n-2)} + M\mathbf{1}_2 = 0 \tag{8}$$

$$N\eta_s + M\mathbf{1}_2 = 0. \tag{9}$$

Equations (8) and (9) together imply $N(\eta_s - \mathbf{1}_{(n-2)}) = 0$. Note from Lemma 3.1, we know that $N$ is of full rank. Hence $N(\eta_s - \mathbf{1}_{(n-2)}) = 0$ implies $\eta_s = \mathbf{1}_{(n-2)}$, which contradicts the assumption that $\eta$ and 1 are linearly independent. Therefore, the components of $\eta$ are all distinct. In addition, it should be noted that each row of $L \in \{L \in \mathcal{L}(\mathcal{T}) : L\xi = 0\}$ corresponding to a node not in $R$ has exactly three non-zero entries and satisfies

$$\begin{bmatrix} 1 & 1 & 1 \\ \xi_i & \xi_{j_i^1} & \xi_{j_i^2} \end{bmatrix} \begin{bmatrix} L_{ii} \\ L_{ij_i^1} \\ L_{ij_i^2} \end{bmatrix} = 0 \tag{10}$$

where $L_{ii}$, $L_{ij_i^1}$, and $L_{ij_i^2}$ are the three nonzero entries of the $i$th row of $L$, and $\xi_i$, $\xi_{j_i^1}$, and $\xi_{j_i^2}$ are the corresponding components of $\xi$ with $j_i^1$ and $j_i^2$ being the two in-neighbors of $i$. Thus, $[L_{ii}, L_{ij_i^1}, L_{ij_i^2}]$ can be written in a general form in terms of $\xi$, i.e.,

$$\left[L_{ii}, L_{ij_i^1}, L_{ij_i^2}\right] = a\left[\xi_{j_i^1} - \xi_{j_i^2},\ \xi_i - \xi_{j_i^1},\ \xi_{j_i^2} - \xi_i\right]$$

where $a$ is any nonzero complex coefficient. Therefore, the nonzero entries of $L$ are polynomials of $\xi$, so are the principal minors of $L$. Recall from above that we have found an instance of $\xi$, namely, $\xi = \eta$, such that all the principal minors of $L_R$ are nonzero for $L \in \{L \in \mathcal{L}(\mathcal{T}) : L\xi = 0\}$. So for any generic $\xi$, the conclusion holds.

Second, we consider $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ with $\xi$ being generic. Notice that the difference between $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ and $L' \in \{L' \in \mathcal{L}(\mathcal{T}) : L'\xi = 0\}$ is that some nonzero weights in $L$ become zero in $L'$. Since the principal minors of $L_R$ are polynomials $P(\cdots, w_{ij}, \cdots)$ of the weights $w_{ij}$ in $L$, the nonzero principal minors of $L'_R$ imply that there exists a set of weights $w_{ij}$ with some of them being zero such that $P(\cdots, w_{ij}, \cdots)$ are nonzero. Thus, $P(\cdots, w_{ij}, \cdots)$ is not identically zero, from which we can infer that for all weights $w_{ij}$, except for being the roots of $P(\cdots, w_{ij}, \cdots) = 0$, which are countable, the principal minors of $L_R$ (namely, the polynomials $P(\cdots, w_{ij}, \cdots)$) are nonzero. Thus, the conclusion follows.

(2) $\Longrightarrow$ (3). It is straightforward.

(3) $\Longrightarrow$ (1) We prove it in a contrapositive form. Suppose that there exists a node $i \notin R$ such that there are no two disjoint paths in $\mathcal{G}$ from $R$ to $i$. That is, after removing a node, without loss of generality, say $k$, a subset $\mathcal{W}$ of nodes becomes not reachable from $R$. Denote the set of remaining nodes as $\bar{\mathcal{W}}$, which are reachable from $R$ after removing $k$. It is certain that $r_1, r_2 \in \bar{\mathcal{W}}$ and after removing node $k$, the nodes in $\mathcal{W}$ are not reachable from any node in $\bar{\mathcal{W}}$. Suppose the total number of nodes in $\mathcal{W}$ is $m$. If necessary, re-label the nodes in $\mathcal{W}$ as $1, \ldots, m$, change the label of node $k$ to $m+1$, and re-label the nodes in $\bar{\mathcal{W}}$ as $m+2, \ldots, n$. Then the matrix $L$ after

[1]Here "for almost all" parameter values is to be understood as "for all parameter values except for those in some proper algebraic variety in the parameter space." The proper algebraic variety for which a property is not true is the zero set of some nontrivial polynomial with real coefficients in the parameters. A proper algebraic variety has Lebesgue measure zero [38].

[2]Here "for almost all Laplacian $L$" means "for almost all complex weights used to construct the Laplacian $L$."

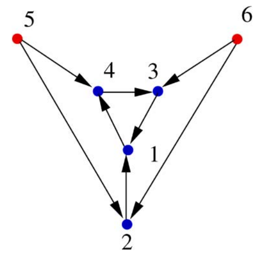


Fig. 3. An example of 2-rooted graph, for which a similar formation of some particular $\xi$ taken from a set of measure zero is not realizable.

re-labeling satisfies $L(i,j)=0$ for $i \in \mathcal{W}$ and $j \in \bar{\mathcal{W}}$. That is, $L$ is of the following form:

$$\left[\begin{array}{c|c|c} L_w & l & 0 \\ \hline * & * & * \end{array}\right]$$

where $L_w \in \mathbb{C}^{m\times m}$ and $l \in \mathbb{C}^m$. Re-order the components of $\xi$ in the same way as relabeling the nodes, and denote the resulting vector by $[\xi_a^T,\ \xi_b^T]^T$ where $\xi_a \in \mathbb{C}^{m+1}$ and $\xi_b \in \mathbb{C}^{(n-m-1)}$. According to the definition of $L$, we have

$$[L_w\ l]\mathbf{1}_{m+1} = 0 \text{ and } [L_w\ l]\xi_a = 0.$$

As $\mathbf{1}_{m+1}$ and $\xi_a$ are linearly independent for a generic $\xi$, we then know that $[L_w l]$ must be row linearly dependent, which means $\det(L_R)=0$ for any $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$. ■

*Proof of Theorem 3.1:* (Sufficiency) If $\mathcal{G}$ is 2-rooted, it follows from Lemma 3.2 that for any generic $\xi$, there exists an $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ such that $\ker(L) = \mathcal{S}(\xi)$. Thus, by Definition 3.1, a similar formation of $\xi$ is realizable.

(Necessity) Suppose that $\mathcal{G}$ is not 2-rooted. Then by Lemma 3.2, it follows that for any subset $R$ of two nodes and for any $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ we have $\det(L_R)=0$, which implies $\mathcal{S}(\xi)$ is a strict subspace of $\ker(L)$. So a similar formation of $\xi$ is not realizable on $\mathcal{G}$. ■

*Remark 3.4:* Theorem 3.1 shows that the assumption of a generic configuration $\xi$ is needed for formation shape control over a directed graph, which is different from the undirected graph case [29]. In other words, there exist special $\xi$'s taken from a set of measure zero, which may not be realizable on a directed graph $\mathcal{G}$ even though the graph is 2-rooted. An example is given in Fig. 3 to demonstrate this observation. This example has six agents, whose sensing graph is shown in Fig. 3. It can be verified that the graph is 2-rooted with nodes 5 and 6 being the two roots. Let us consider a formation vector $\xi = [1+\iota, -6\iota, -0.8-1.6\iota, -1-\iota, -3-3\iota, 0]$. It can be checked that for any Laplacian $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, $\operatorname{rank}(L) < 4$, which means there is another formation shape $\eta$ that satisfies $L\eta = 0$ but is not similar to $\xi$. However, arbitrarily perturbing the configuration $\xi$, for example, we take $\xi' = [1, -0.5-2.5\iota, -0.3-0.1\iota, -1-\iota, 3-\iota, 0.5]$. We are able to find a Laplacian $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi' = 0\}$, e.g.,

$$L = \begin{bmatrix} 2+24\iota & 13+\iota & -15-25\iota & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\iota & 1+\iota \\ 0 & 0 & 2+\iota & -1 & 0 & -1-\iota \\ -4 & 0 & 0 & 2-\iota & 2+\iota & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfying $\operatorname{rank}(L) = 4$ and $\ker(L) = \mathcal{S}(\xi')$. Indeed, in this example, for any $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi' = 0\}$ it holds that $\ker(L) = \mathcal{S}(\xi')$.

*Remark 3.5:* It should be pointed out that by Lemma 3.2, if there exists a complex Laplacian $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$ such that $\ker(L) = \mathcal{S}(\xi)$, then for almost all complex Laplacian $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, $\ker(L) = \mathcal{S}(\xi)$. In other words, if the 2-rooted condition holds, then we can randomly select an $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, which can be used for formation shape control as shown later in this section.

## IV. Formation Stabilization Under Fixed Graph

According to the preceding section, a set of complex weights $w_{ij}$'s can be designed to encode the desired formation shape. This can be done in a distributed manner. That is, the complex weights $w_{ij}$ for $j \in \mathcal{N}_i$ can be calculated by agent $i$ from the equation

$$\sum_{j\in\mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0 \tag{11}$$

as $\xi_j$'s for $j \in \mathcal{N}_i$ are available to agent $i$ by Assumption A2. It is a fact that there are infinite number of solutions $w_{ij}$ satisfying (11). For our use, we can randomly select one from the solution space. By such random selection, the properties in Lemma 3.2 will be satisfied if the directed graph $\mathcal{G}$ is 2-rooted. Then we consider the following distributed control law for formation shape control:

$$u_i = d_i \sum_{j\in\mathcal{N}_i} w_{ij}(z_j - z_i), \quad i = 1,\ldots,n \tag{12}$$

where $d_i \in \mathbb{C}$ is a control parameter, which requires global information of the group (namely, $L$) to determine. Given $w_{ij}$'s, a Newton iteration method can then be used to compute $d_i$ for the purpose of assigning the eigenvalues of the closed-loop system (see [29] for the details of the method).

*Remark 4.1:* It can be checked that the linear control law (12) does satisfy the rotational invariance property and thus is locally implementable by an onboard sensor.

With the control law (12), the closed-loop system turns out to be

$$\dot{z} = -DLz \tag{13}$$

where $D$ is the diagonal matrix with its diagonal entries $d_i$'s and $L$ is the Laplacian of $\mathcal{G}$ with the complex weights $w_{ij}$'s satisfying (11).

As whether or not a group of $n$ agents globally exponentially reaches a similar formation of $\xi$ with the exponential convergence rate $\alpha > 0$ relies on the eigenvalue distribution of $-DL$, the next result shows that it is always feasible to have exponential convergence by using the control law (12) if the directed graph $\mathcal{G}$ is 2-rooted.

*Theorem 4.1:* Consider a positive constant $\alpha > 0$. If $\mathcal{G}$ is 2-rooted, then for almost all $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, a diagonal and invertible matrix $D$ exists to assign the eigenvalues of $-DL$ in the half-plane $\{s : \operatorname{Re}(s) < -\alpha\}$ in addition to two fixed eigenvalues at the origin.

The proof requires the result related to the multiplicative inverse eigenvalue problem by Friedland in 1975.

*Lemma 4.1 ([39]):* Let $A$ be an $n \times n$ complex-valued matrix. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be an arbitrary set of $n$ complex numbers. If all the principal minors of $A$ are distinct from zero, then there exists a diagonal complex valued matrix $D$ such that the spectrum of $DA$ is the set $\sigma$. Moreover, the number of different matrices $D$ is at most $n!$.

*Proof of Theorem 4.1:* Since $\mathcal{G}$ is 2-rooted by assumption, without loss of generality, we label two roots of $\mathcal{G}$ as $n-1$ and $n$, and denote $R = \{n-1, n\}$. By Lemma 3.2, it follows that for almost all $L \in \{L \in \mathcal{L}(\mathcal{G}) : L\xi = 0\}$, all the principal minors of $L_R$ are distinct from zero. Thus, by Lemma 4.1, there exists a diagonal complex matrix $D_1$ that can arbitrarily assign the eigenvalues of $D_1 L_R$. Suppose $D_1$ is chosen to assign the eigenvalue of $D_1 L_R$ in the half-plane $\{s : \mathrm{Re}(s) > \alpha\}$. Note that with $D = \mathrm{diag}(D_1, 0_{2\times 2})$, the eigenvalues of $DL$ are the union of the eigenvalues of $D_1 L_R$ and the two fixed zero eigenvalues. Then by the continuity property of eigenvalues, a diagonal and invertible complex matrix $D$ exists to assign the eigenvalues of $DL$ in the half-plane $\{s : \mathrm{Re}(s) > \alpha\}$ in addition to the two fixed zero eigenvalues. ■

## V. Formation Stabilization Under Temporary Node Failures

In this section, we investigate formation shape control under time-varying topology due to node failures. From the preceding section we know that a generic formation shape can be achieved under a directed sensing graph if and only if the graph is 2-rooted. Therefore, we assume in this section that the nominal directed graph is 2-rooted and is denoted by $\mathcal{G}_0$. When one or more agents temporarily fail to get the relative state measurements of all its in-neighbors at some time, say for example agent $i$, the in-neighbor set $\mathcal{G}_i$ becomes an empty set at that time and the graph switches to the graph generated from $\mathcal{G}_0$ by removing the edges to node $i$. This is called *node failure*. On the other hand, *link failures* may occur in the sense that agent $i$ becomes not able to get the relative state measurement of one in-neighbor temporarily at some time. In this case, the measurements about other in-neighbors by agent $i$ may not be useful in helping achieve the desired formation shape, so these measurements are also disregarded by agent $i$. Hence, link failures can be treated as node failures in our formulation of the formation shape control problem. We denote by $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}_p)$, $p = 1, \ldots m$, the subgraphs of $\mathcal{G}_0$ due to different nodes failures. Let $\mathcal{P} = \{0, 1, \ldots, m\}$ and denote by $\mathcal{G}_{\sigma(t)}$ the switching sensing graph with $\sigma$: $[0, \infty) \to \mathcal{P}$ being a piecewise constant switching signal due to node failures. Denote by $\mathcal{N}_i(t)$ the set of in-neighbors at $t$. It should be noted that $\mathcal{N}_i(t)$ is either empty or the in-neighbor set $\mathcal{N}_i$ of the nominal graph $\mathcal{G}_0$.

We define the node failure rate as

$$r = \lim_{T_0 \to \infty} \sup_{T > T_0} \left[ \frac{T_{\text{fail}}([0, T])}{T} \right]$$

where $T_{\text{fail}}([0, T])$ denotes the total failure time during the time interval $[0, T]$.

For a generic formation vector $\xi \in \mathbb{C}^n$, we let $w_{ij}$'s and $d_i$'s for the nominal graph $\mathcal{G}_0$ be designed as shown in Section IV so as to make the eigenvalues of $DL$ strictly in the right complex plane in addition to two fixed zero eigenvalues. Then it is clear that there exist scalars $\mu_0 > 0$ and $\alpha_s > 0$ such that

$$\left\| e^{-QDL_0 Q^T t} \right\| \leq \mu_0 e^{-\alpha_s t} \tag{14}$$

where $Q$ is the matrix defined in (4). The scalars $\mu_0$ and $\alpha_s$ are easy to compute ([40]).

We consider the following linear control law for the case with possible node failures:

$$u_i = k d_i \sum_{j \in \mathcal{N}_i(t)} w_{ij}(z_j - z_i), \quad i = 1, \ldots, n \tag{15}$$

where $k \in \mathbb{R}$ is a positive scalar.

Under the distributed control law (15), the closed-loop dynamics for the entire network is

$$\dot{z} = -kDL_{\sigma(t)} z \tag{16}$$

where $z \in \mathbb{C}^n$ is the aggregate state vector of $n$ agents and $L_{\sigma(t)}$ is the complex Laplacian matrix associated to $\mathcal{G}_{\sigma(t)}$ with the weights $w_{ij}$'s.

For $p = 1, \ldots, m$, there exist scalars $\mu_p > 0$ and $\alpha_p$ such that

$$\| e^{-QDL_p Q^T t} \| \leq \mu_p e^{-\alpha_p t} \tag{17}$$

where $Q$ is the matrix defined in (4). It should be noted that $\alpha_p$ may or may not be positive. We define

$$\mu := \max\nolimits_{0 \leq p \leq m} \mu_p$$
$$\alpha_u := \min\nolimits_{1 \leq p \leq m} \alpha_p.$$

In the following, we are going to explore the stability conditions. To avoid chattering, the switching signals can not change arbitrarily fast in practice. Therefore, we consider a dwell time assumption and an average dwell time assumption for the switching signals [41], [42]. Suppose the switching signal $\sigma(t)$ switches its value at time instants $t_1, t_2, \ldots$. We say the switching signal has *dwell time* $\tau_D$ if $t_{i+1} - t_i \geq \tau_D$ for all $i$'s. We denote by $\Pi(\tau_D)$ the set of all switching signals with dwell time $\tau_D$, i.e.,

$$\Pi(\tau_D) = \{\sigma(t) : t_{i+1} - t_i \geq \tau_d \text{ for all } i\}.$$

However, in certain situations, the switching signals may occasionally have consecutive discontinuities separated by less than $\tau_D$, but for which the average interval between consecutive discontinuities is no less than $\tau_D$. This leads to the concept of average dwell time. For a switching signal $\sigma(t)$, we let $N_\sigma(t_0, t)$ denote the number of discontinuities of $\sigma(t)$ in the interval $[t_0, t)$. Then the set of all switch signals with *average dwell time* $\tau_D$ and *chatter bound* $N_0$ is denoted as

$$\Pi_{ave}(\tau_D, N_0) = \left\{ \sigma(t) : N_\sigma(t_0, t) \leq N_0 + \frac{t - t_0}{\tau_D} \text{ for any } t > t_0 \geq 0 \right\}.$$

Roughly speaking, when the time interval is long enough, the average dwell time is approximately $\tau_D$ considering the upper-bound of the number of discontinuities in the interval. Clearly, $\Pi(\tau_D)$ is a subset of $\Pi_{ave}(\tau_D, 1)$.

Under practical dwell time or average dwell time conditions, we show in the main result below that the $n$ agents can globally exponentially converge to the desired formation shape.

*Theorem 5.1:* Suppose $\sigma(t) \in \Pi(\tau_D)$ or $\sigma(t) \in \Pi_{ave}(\tau_D, N_0)$ with a constant $\tau_D > 0$ and arbitrary chatter bound $N_0 > 0$. The $n$ agents globally exponentially reach a similar formation of $\xi$ with the exponential convergence rate $\alpha > 0$ under the distributed control law (15) if

$$k\left[(1-r)\alpha_s + r\alpha_u\right] > \alpha + \frac{\ln\mu}{\tau_D}. \tag{18}$$

*Proof:* Consider the linear transformation $x = Qz$ with $Q$ defined in (4). Note that $I_n - Q^TQ$ is an orthogonal projection matrix onto the span of $\mathbf{1}_n$ and $\xi$, and that $\mathbf{1}_n$ and $\xi$ are in the null space of $L_p$ for any $p \in \mathcal{P}$, which together imply $L_{\sigma(t)}(I - Q^TQ) = 0$ or equivalently $L_{\sigma(t)} = L_{\sigma(t)}Q^TQ$. Thus, under the distributed control law (15) we obtain that

$$\begin{aligned}\dot{x} &= Q\dot{z} = -kQDL_{\sigma(t)}z = -kQDL_{\sigma(t)}Q^TQz\\ &= -kQDL_{\sigma(t)}Q^Tx.\end{aligned} \tag{19}$$

For any $p \in \mathcal{P}$, we denote $M_p = -kQDL_pQ^T$. Then it follows from (14) and (17) that

$$\begin{aligned}&\left\|e^{M_pt}\right\| \le \mu e^{-k\alpha_s t} \quad \text{for } p = 0,\\ &\left\|e^{M_pt}\right\| \le \mu e^{-k\alpha_u t} \quad \text{for any } p \neq 0.\end{aligned} \tag{20}$$

Denote by $t_1, t_2, \ldots$ the time instants, at which the switching occurs, and suppose $\sigma(t) = p_i$ for $t \in [t_{i-1}, t_i)$. Then for any $t > 0$ lying in the interval $[t_i, t_{i+1})$, the solution to (19) with its initial condition $x_0 = Qz_0$ is of the following form:

$$x(t) = e^{M_{p_{i+1}}(t-t_i)}e^{M_{p_i}(t_i-t_{i-1})}\cdots e^{M_{p_1}t_1}x_0.$$

Using (20), we have

$$\|x(t)\| \le \mu\mu^{N_\sigma(0,t)}e^{-k\alpha_sT_s(t)-k\alpha_uT_u(t)}\|x_0\|$$

where $N_\sigma(0,t)$ is the number of switching occurred in $[0,t)$, and $T_s(t)$ and $T_u(t)$ are the total activation time of the system without any node failure and the systems with some node failures during $[0,t)$.

From the condition (18), we know there exists a $\delta > 0$ such that

$$k\left[(1-r)\alpha_s + r\alpha_u - \delta\right] > \alpha + \frac{\ln\mu}{\tau_D}. \tag{21}$$

Let $\epsilon = \delta/(\alpha_s - \alpha_u)$, which is positive. Then (21) can be rewritten as

$$k\left[(1-(r+\epsilon))\,\alpha_s + (r+\epsilon)\alpha_u\right] > \alpha + \frac{\ln\mu}{\tau_D}. \tag{22}$$

On the other hand, for this $\epsilon$, we know that there exists a $T_0 > 0$ such that

$$\sup_{T>T_0}\left[\frac{T_{\text{fail}}\left([0,t]\right)}{T}\right] < r + \epsilon. \tag{23}$$

Now without loss of generality, suppose $t = T_0$. Thus, utilizing (23) first and then (22), we obtain that

$$\begin{aligned}\|x(t)\| &\le \mu\mu^{N_\sigma(0,t)}e^{-k[(1-(r+\epsilon))\alpha_s+(r+\epsilon)\alpha_u]t}\|x_0\|\\ &\le \mu\mu^{N_\sigma(0,t)}e^{-\left[\alpha+\frac{\ln\mu}{\tau_D}\right]t}\|x_0\|.\end{aligned}$$

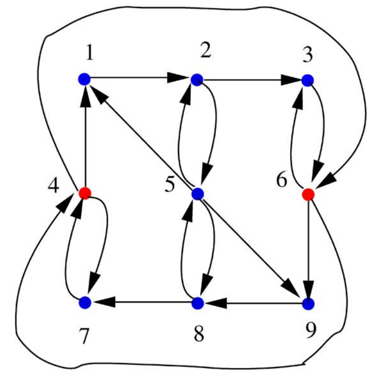


Fig. 4. The sensing graph $\mathcal{G}$ used in our simulations.

Suppose $\sigma(t) \in \Pi_{ave}(\tau_D, N_0)$. Then we know that

$$\mu^{N_\sigma(0,t)} \le \mu^{N_0+\frac{t}{\tau_D}}$$

and thus

$$\|x(t)\| \le \mu^{N_0+1}e^{-\alpha t}\|x_0\|.$$

Repeating this argument shows that the sub-sequence of state at $T_0, 2T_0, \ldots$ decays exponentially with the rate $\alpha$. This further implies that $\|x(t)\|$ converges to 0 with the exponential decay rate $\alpha$. Therefore, by Definition 2.4, it is concluded that the $n$ agents globally exponentially reach a similar formation of $\xi$ with the exponential convergence rate $\alpha$.

Moreover, it can be seen that $\Pi(\tau_D) \subset \Pi_{ave}(\tau_D, 1)$ from their definitions. Hence, if $\sigma(t) \in \Pi(\tau_D)$ and (18) holds, then the $n$ agents also globally exponentially reach a similar formation of $\xi$ with the exponential convergence rate $\alpha$. ■

*Remark 5.1:* Note that the left-hand side of (18) in Theorem 5.1 is equivalent to the negative real part of the averaged dominant eigenvalues for formation shape control in some sense. The condition (18) indicates that this averaged dominant eigenvalue should be able to generate an exponential convergence rate $\alpha$ after neutralizing possible growth due to switching. The scalar $k$ can be chosen big enough to fulfil the condition (18).

*Remark 5.2:* It is worth to mention that when the sensing graph is fixed, the dwell time $\tau_D$ is infinity and the failure rate $r = 0$. Then the condition (18) in Theorem 5.1 degenerates to the same condition for exponential stability with the ensured convergence rate $\alpha$ under a fixed topology.

## VI. Simulation Results

In this section, we present several simulations to illustrate our results. The example we consider consists of 9 agents. Suppose the sensing graph $\mathcal{G}$ is given in Fig. 4, which is 2-rooted with roots {4, 6}, namely, every node other than 4 and 6 has two disjoint paths from {4, 6}.

The desired formation shape is described by the target configuration

$$\xi = [-1+\iota,\ \iota,\ 1+\iota, -1,\ 0,\ 1, -1-\iota, -\iota,\ 1-\iota]^T$$

in a global reference frame $\Sigma$. This is a 3-by-3 grid. It should be pointed out that this target configuration is not generic. However, note that a generic property, in topology and algebraic geometry, is one that holds on a dense open set, so a generic configuration assumption is only sufficient and that is why the simulation example can be still successful as we will see.

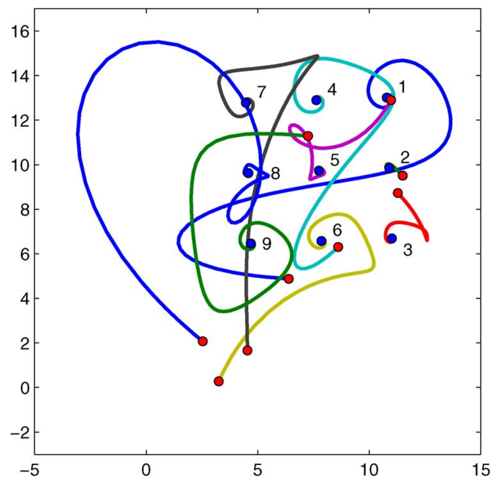


Fig. 5. The solution trajectories. (Fixed topology).

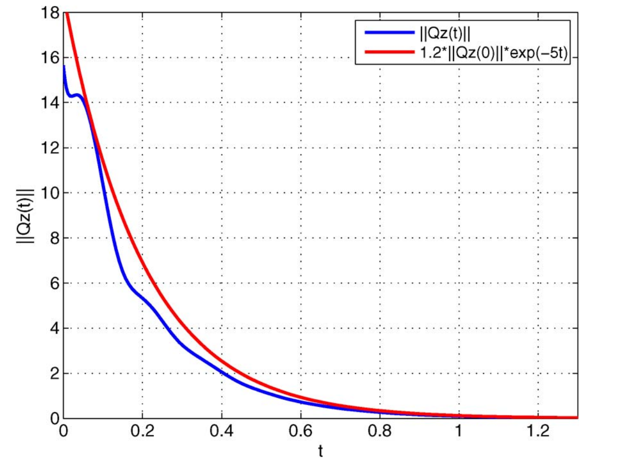


Fig. 6. The norm of $Qz(t)$. (The sensing graph $\mathcal{G}$ is fixed and the third biggest real part of all the eigenvalues of $-DL$ is $-5$).

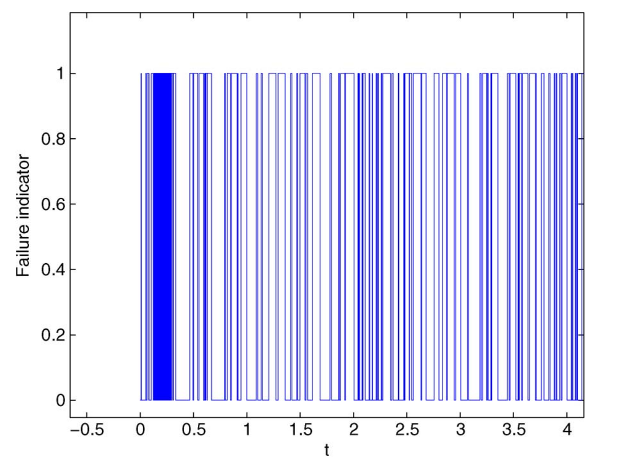


Fig. 7. The failure indicator function (1 indicates the occurrence of one or more agent failures and 0 indicates no failure).

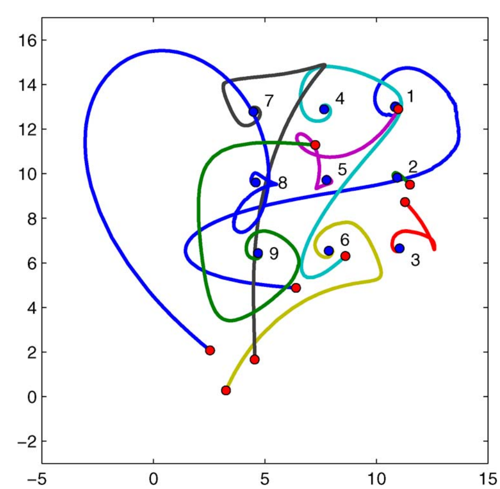


Fig. 8. The solution trajectories. (The sensing graph is time-varying due to temporary node failures).

According to the sensing graph $\mathcal{G}$ given in Fig. 4 and the target configuration $\xi$ defined above, the weights are solved from the formula (7). For example, we choose

$$
\begin{aligned}
w_{14} &= 6.7016 - 0.9582\iota, & w_{15} &= -2.8717 + 3.8299\iota\\
w_{21} &= -0.2767 + 1.1172\iota, & w_{25} &= -1.1172 - 0.2767\iota\\
w_{32} &= 5.5818 - 0.6142\iota, & w_{36} &= 0.6142 + 5.5818\iota\\
w_{46} &= 1.7128 + 2.1726\iota, & w_{47} &= 4.3452 - 3.4257\iota\\
w_{52} &= -2.6901, & w_{58} &= -2.6901\\
w_{63} &= 6.0798 + 0.0326i\iota, & w_{64} &= -0.0163 + 3.0399\iota\\
w_{74} &= -3.9112 - 2.6227\iota, & w_{78} &= -2.6227 + 3.9112\iota\\
w_{85} &= -1.8900 + 3.0397\iota, & w_{89} &= 3.0397 + 1.8900\iota\\
w_{95} &= 1.7471 + 6.1059\iota, & w_{96} &= 4.3587 - 7.8530\iota.
\end{aligned}
$$

The next step is to find $d_i$, $i = 1, \ldots, 9$, such that the eigenvalues of $-DL$ lie in the half-plane $\{s : \mathrm{Re}(s) \leq -5\}$ in addition to two fixed eigenvalues at 0 if we expect the exponential convergence rate to be $\alpha = 5$. For this objective, the following $d_i$'s are used:

$$
\begin{aligned}
d_1 &= 9, & d_2 &= -0.5, & d_3 &= 2\\
d_4 &= 4, & d_5 &= -3, & d_6 &= 4\\
d_7 &= -4, & d_8 &= 5, & d_9 &= 3.
\end{aligned}
$$

First, we consider $k = 1$ and suppose the sensing graph $\mathcal{G}$ is fixed (no node failures). The solution trajectories in this simulation are plotted in Fig. 5, which demonstrate that the agents converge to a similar formation of 3-by-3 grid. Moreover, the norm of $Qz(t)$ with $Q$ defined in (4) is plotted in Fig. 6, from which we can see that it is exponentially convergent with the rate $\alpha = 5$.

Second, we consider $k = 1$ but the sensing graph is time-varying due to temporary node failures. In this simulation, the failure of every agent is assumed to occur randomly. The indicator function in Fig. 7 shows the occurrence of failures (one or more agents) when the function value is 1. It is counted that the average dwell time is $\tau_D = 0.0135$ for this simulation. The corresponding solution trajectories are presented in Fig. 8 and the norm of $Qz(t)$ is plotted in Fig. 9. From the simulation result, we can see that the norm of $Qz(t)$ (in Fig. 9) overshoots the same upper-bound for the fixed topology case (in Fig. 6).

Third, we consider $k = 1.2$ to shift the eigenvalues of $-DL$ into the half-place $\{s : \mathrm{Re}(s) \leq -7\}$ and adopt the same time-varying graph as for the second simulation. The norm of $Qz(t)$ is given in Fig. 10. Now it is seen that the 9 agents are able to reach a similar formation with the exponential convergence rate $\alpha = 5$ even in the presence of node failures over time.

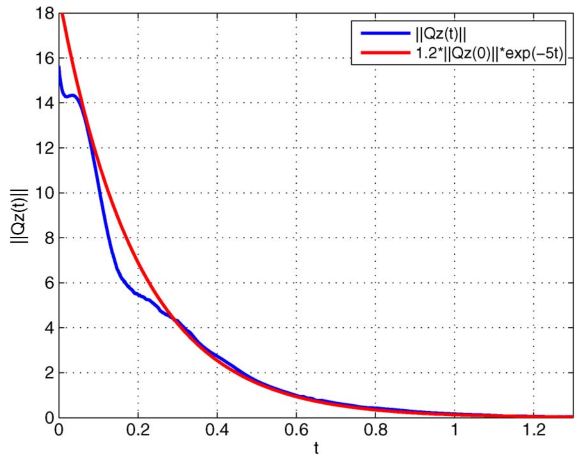


Fig. 9. The norm of $Qz(t)$. (The sensing graph is time-varying due to temporary node failures and the third biggest real part of all the eigenvalues of $-DL$ is $-5$).

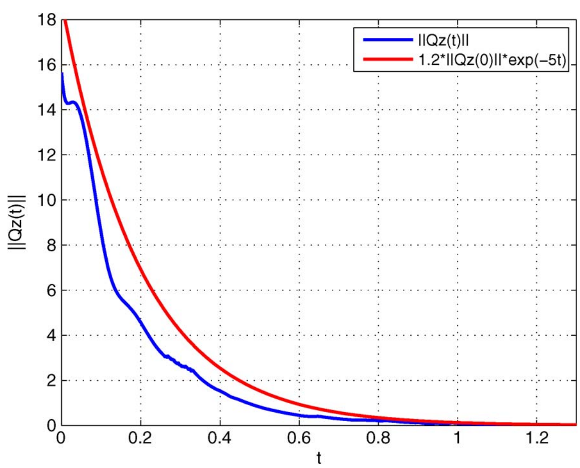


Fig. 10. The norm of $Qz(t)$. (The sensing graph is time-varying due to temporary node failures and the third biggest real part of all the eigenvalues of $-DL$ is $-7$).

## VII. Conclusion

In this paper, we consider a network of agents governed only by a local control and address the formation shape control problem using complex-weight based local interaction strategies. The more general and more challenging setup of directed networks and time-varying networks due to temporary node failures is considered. The technical challenge in establishing the connection between the complex-valued Laplacian and the connectivity pattern of directed graphs is overcome under the assumption of generic target configurations for a network of agents. Then a necessary and sufficient graphical condition is developed, under which a formation shape subject to rotations, translations, and dilations can be characterized in terms of complex weights over a directed graph. This answers the fundamental question whether a formation shape can be formed and maintained under a distributed control law for multiple agents using unidirectional and relative-position information in local frames. Moreover, exponential convergence towards the desired formation shape is analyzed under fixed topology and time-varying topology due to temporary node failures. It is shown that exponential convergence with specific decay rate is always possible by tuning the diagonal scaling parameters for both fixed topology and time-varying topology.

In this paper, however, we have not accounted for internal dynamics of the agents, uncertain environment forces, and other external forces. It will be interesting to generalize our methodology to deal with formation control of agents with internal dynamics, uncertain environment forces, or nonholonomic constraints such as those in [6], [10], [43] and [44]. Moreover, except for node failures, the general time-dependence aspects of multi-agent formation has not been considered. However, further investigation will be possible along the avenue developed in this paper. Also, based on the formation control law developed in this paper, it will be possible to take into account other control specifications such as group path following and collision avoidance during motion.

In addition to multi-vehicle formations, this work also has applicability in network localization such as those in [45]–[47], for which a few nodes, called anchors, know their absolute positions while the other nodes have only relative position measurements and also lack the common sense of direction. The fundamental problem of whether all the nodes in such a network are localizable can then be transformed to finding the topological properties of the sensing directed graphs, for which there exists a unique realization.

## Appendix

*Proof of Lemma 3.1:* We prove this lemma in the iterative way.

1) We consider a rooted graph $\mathcal{G}$, that is, there exists a node, from which every other node is reachable.
   - (1a) It is certain that a rooted graph $\mathcal{G}$ has a spanning tree. Without loss of generality, we denote it as $\mathcal{T}$, with the root set $R_1 = \{r_i\}$. Then for the Laplacian $L'$ of $\mathcal{T}$ with almost all weights $w_{ij}$'s, $\det(L'_{R_1}) \neq 0$. Compose another set $R_2$ by adding any one other node to $R_1$ and then it is clear that the nodes not in $R_2$ are all reachable from $R_2$. So $\det(L'_{R_2}) \neq 0$. Repeating this argument, it can be concluded that all the principal minors of $L'_{R_1}$ are distinct from zero for the Laplacian $L'$ of $\mathcal{T}$ with almost all weights $w_{ij}$'s. Note that comparing with the Laplacian $L'$ of $\mathcal{T}$, the Laplacian $L$ of $\mathcal{G}$ has possibly more non-zero entries. Hence, by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that all the principal minors of $L$ are distinct from zero for almost all weights $w_{ij}$'s.
   - (1b) We show in the following that for the Laplacian $L'$ of $\mathcal{T}$ with almost all weights $w_{ij}$'s, $\det(M') \neq 0$ where $M'$ is the sub-matrix of $L'$ by deleting the row corresponding to the root $r_i$ and the column corresponding to any other node, say $v_j \in \mathcal{V} - R_1$. Denote $l'_i$ and $l'_j$ the row vectors of $L'$ corresponding to node $r_i$ and $v_j$ respectively. Clearly, $l'_i = 0$ since $r_i$ is the root of $\mathcal{T}$. We take the following elementary row transformation:

$$L' = \begin{bmatrix} \vdots \\ l'_i \\ \vdots \\ l'_j \\ \vdots \end{bmatrix} \Rightarrow \bar{L}' = \begin{bmatrix} \vdots \\ l'_i + l'_j \\ \vdots \\ l'_j \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ l'_j \\ \vdots \\ l'_j \\ \vdots \end{bmatrix}.$$

Denote by $\bar{\mathcal{T}}$ the graph associated with $\bar{L}'$. It is certain that node $r_i$ is also a root of $\bar{\mathcal{T}}$. Moreover, notice that $\bar{L}'(i,j) = L'(j,j) \neq 0$ (where $\bar{L}'(i,j)$ and $L'(j,j)$ stand for the corresponding entry of $\bar{L}'$ and $L'$). So there is an edge from $v_j$ to $r_i$, which means $v_j$ is also a root of $\bar{\mathcal{T}}$. Thus, according to (1a), $\det(\bar{L}'_{\{v_j\}}) \neq 0$. Notice that $M'$ can be obtained from $\bar{L}'_{\{v_j\}}$ via elementary row transformations, so $\det(M') \neq 0$, too. By the same argument as given in the end of (1a), the conclusion follows for the Laplacian $L$ of $\mathcal{G}$ with almost all weights $w_{ij}$'s.

2) We consider a 2-rooted graph $\mathcal{G}$, that is, there exists two nodes from which every other node is 2-reachable.

(2a) Without loss of generality, let $R = \{r_1, r_2\}$ be the root set of $\mathcal{G}$ and let $L \in \mathcal{L}(\mathcal{G})$. By removing an arbitrary node in $R$, say $r_2$ without loss of generality, and its incident edges, we denote the resulting sub-graph by $\mathcal{G}_1$. Clearly, $\mathcal{G}_1$ is rooted with the root set $R_1 = \{r_1\}$. Denote by $L_1$ the corresponding Laplacian of $\mathcal{G}_1$. Then it is known from (1) that all the principal minors of $(L_1)_{R_1}$ are nonzero, and $\det(M_1) \neq 0$ where $M_1$ is the sub-matrix of $L_1$ by deleting the row corresponding to $r_1 \in R_1$ and any one column. This implies, all the principal minors of $L_R$ are nonzero and $\det(M) \neq 0$ for $M$ being the sub-matrix of $L$ by deleting the two rows corresponding to nodes in $R$ and two columns corresponding to one node in $R$ and the other not in $R$.

(2b) Then it remains to show that $\det(M) \neq 0$ for $M$ being the sub-matrix of $L$ by deleting the two rows corresponding to nodes in $R$ and any two columns corresponding to nodes neither in $R$. Consider any one node not in $R$, say $v_j$. Without loss of generality, denote $r_1 = v_1$ and $r_2 = v_2$, and denote by $l_i$ the row vector of $L$ corresponding to node $v_i$. We make the following elementary row transformation for $L$, i.e.,

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_j \\ \vdots \end{bmatrix} \Rightarrow \bar{L} = \begin{bmatrix} \sum_{i=1}^{n} k_i l_i \\ l_2 \\ \vdots \\ l_j \\ \vdots \end{bmatrix}.$$

Since $\mathcal{G}$ is 2-rooted, every root in $R$ has at least one out-going edge. Thus, with a proper choice of $k_i$'s, $\bar{L}(1,1)$, $\bar{L}(1,2)$ and $\bar{L}(1,j)$ can all become nonzero. Denote by $\bar{\mathcal{G}}$ the graph associated with $\bar{L}$. It is certain that $\bar{\mathcal{G}}$ is also 2-rooted with the root set $R$. Moreover, there are edges from node $r_2$ and node $v_j$ to node $r_1$. So node $r_1$ is 2-reachable from the set $\bar{R} = \{r_2, v_j\}$.

Now we consider any node $v_k$ in $\mathcal{V} - \{R, v_j\}$ and show that $v_k$ is 2-reachable from $\bar{R}$, too. Certainly, if the two disjoint paths from $R$ to $v_k$ do not pass through node $v_j$, then there exist two disjoint paths from $\bar{R}$ to $v_k$ with one path going from node $v_j$ to $r_1$ followed by the path from $r_1$ to $v_k$. If for the two disjoint paths from $R$ to $v_k$, there exists a path, from node $r_2$ to $v_k$, passing through node $v_j$, then there still exist two disjoint paths from $\bar{R}$ to $v_k$ with two new paths: One is the path from $v_j$ to $v_k$ taken from the path $r_2$ to $v_k$ and the other is the path connecting from $r_2$ to $r_1$ and $r_1$ to $v_k$. If there exists a path, from node $r_1$ to $v_k$, passing through node $v_j$, then there still exist two disjoint paths from $\bar{R}$ to $v_k$ with one new path going from $v_j$ to $v_k$ taken from the path $r_1$ to $v_k$. This means, $\bar{\mathcal{G}}$ is 2-rooted with the root set $\bar{R}$. Hence, according to the argument in (1), we can know that $\det(\bar{L}_{\bar{R}}) \neq 0$. Notice that $M$ can be obtained from $\bar{L}_{\bar{R}}$ via elementary row transformations, so $\det(M) \neq 0$. Thus, the conclusion follows. ■

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