Control Under Stochastic Multiplicative Uncertainties: Part I, Fundamental Conditions of Stabilizability

Tian Qi, *Member, IEEE*, Jie Chen, *Fellow, IEEE*, Weizhou Su, *Senior Member, IEEE*, and Minyue Fu, *Fellow, IEEE*

***Abstract*—In this two-part paper we study stabilization and optimal control of linear time-invariant systems with stochastic multiplicative uncertainties. We consider structured multiplicative perturbations, which, unlike in robust control theory, consist of static, zero-mean stochastic processes, and we assess the stability and performance of such systems using mean-square measures. While Part 2 of this paper tackles and solves optimal control problems under the mean-square criterion, Part 1 is devoted to the stabilizability problem. We develop fundamental conditions of mean-square stabilizability which ensure that an open-loop unstable system can be stabilized by output feedback in the mean-square sense. For single-input single-output systems, a general, explicit stabilizability condition is obtained. This condition, both necessary and sufficient, provides a fundamental limit imposed by the system's unstable poles, nonminimum phase zeros and time delay. For multi-input multi-output systems, we provide a complete, computationally efficient solution for minimum phase systems possibly containing time delays, in the form of a generalized eigenvalue problem readily solvable by means of linear matrix inequality optimization. Limiting cases and nonminimum phase plants are analyzed in depth for conceptual insights, revealing, among other things, how the directions of unstable poles and nonminimum phase zeros may affect mean-square stabilizability in MIMO systems. Other than their independent interest, stochastic multiplicative uncertainties have found utilities in modeling networked control systems pertaining to, e.g., packet drops, network delays, and fading. Our results herein lend solutions applicable to networked control problems addressing these issues.**

Manuscript received April 7, 2015; revised January 15, 2016 and February 7, 2016; accepted June 6, 2016. Date of publication June 28, 2016; date of current version February 24, 2017. This research was supported in part by the Hong Kong RGC under Projects CityU 111810, CityU 111511, in part by the City University of Hong Kong under Project 9380054, and in part by the Natural Science Foundation of China under Grants 61273109, 60834003, in part by the National Program on Key Basic Research Project (973 Program) under Grant 2014CB845302. Recommended by Associate Editor C. De Persis.

T. Qi and W. Su are with the School of Automation Science and Engineering, Key Laboratory of Autonomous Systems and Networked Control, Ministry of Education, South China University of Technology, Guangzhou 510640, China (e-mail: auqt@scut.edu.cn; wzhsu@scut.edu.cn).

J. Chen is with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: jichen@cityu.edu.hk).

M. Fu is with the School of Electrical and Computer Engineering, University of Newcastle, Callaghan, NSW 2308, Australia, and also with Zhejiang University, Zhejiang 310027, China (e-mail: minyue.fu@newcastle.edu.au).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2016.2585919

***Index Terms*—Erasure channel, mean-square stability and stabilization, networked control, stochastic multiplicative uncertainty.**

I. Introduction

In this two-part series we study stabilization and performance problems for linear time-invariant (LTI) systems subject to stochastic multiplicative uncertainties. Apart from the longstanding interest in this class of systems by themselves, a direct impetus motivating our study is the newly found relevance of multiplicative channel noises to networked control systems. Recent studies by Elia [10] show that random multiplicative noises provide a suitable framework to model communication errors and uncertainties for classes of erasure channels, which can be used to describe a number of data-loss network phenomena, such as packet drops and network delays. Similarly, in their study of estimation problems, Sinopoli *et al.* [32], [35] adopted a multiplicative noise model to account for the communication uncertainty due to packet loss. Thus, while addressing an age-old problem of fundamental interest, the study of control under stochastic multiplicative uncertainties has a close bearing on networked control problems, which, undoubtedly, has seen an extraordinary level of research activities in the recent years (see, e.g., [4], [15], [24], [25], [28], [32], [48] and the references therein).

Throughout this series we model the system uncertainty as a structured multiplicative stochastic perturbation, which, unlike in robust control theory (see, e.g., [49]), consists of static, zero-mean stochastic processes. Under this formulation, which is standard in the classical stochastic control setting, the uncertainty can be interpreted as state-or input-dependent random noises [22], [45], while in the networked control setting, as parallel memoryless noisy communication channels [10], [46]. In doing so, we assess the system's stability and performance based on *mean-square* criteria; in other words, the stability and performance are to be evaluated statistically using such second-order statistics as variance. In Part 1 of this paper we focus on the stabilization problem. One fundamental question we attempt to answer dwells on stabilizability: With given uncertainty variances, can an unstable plant be stabilized despite the presence of such stochastic uncertainties? We seek to develop fundamental conditions that guarantee the stabilizability of LTI systems in the mean-square sense.

Mean-square stability and stabilization, in retrospect, have been long studied for LTI systems under stochastic multiplicative uncertainty formulation. Willems and Blankenship [45] studied the closed loop stability of single-input, single-output (SISO) systems and obtained a necessary and sufficient condition for mean-square stability. Later, Hinrichsen and Pritchard [13], and Lu and Skelton [22] formulated the mean-square stability problem as one of robust stability against stochastic multiplicative uncertainties, which allowed them to obtain necessary and sufficient mean-square stability conditions for multi-input, multi-output (MIMO) systems. In much the same spirit, Elia [10], and Xiao *et al.* [46] developed similar conditions for networked control problems. With the distinctive feature of a frequency-domain, input-output based approach, these developments share much in common with robust stability analysis and lead to stability results reminiscent of small gain conditions, herein dubbed as *mean-square small gain theorems*. Equally noteworthy nonetheless, there has also been a considerable amount of work built on time-domain analysis contingent upon solving certain modified algebraic Ricatti equations (cf. Part 2 of this paper), which seeks to address optimal control problems under random multiplicative noise assumptions. For the most part this latter line of work is more pertinent to our performance studies, and for this reason a summary review is relegated to Part 2 of this paper.

We shall employ a frequency-domain approach enabled by the mean-square small gain theorem alluded to above. Our contributions in Part 1 can be summarized as follows. For SISO systems, the mean-square stabilizability condition is shown to be equivalent to solving an $\mathcal{H}_2$ optimal control problem. We solve this problem explicitly and provide an explicit bound on the noise variance, which furnishes a fundamental limit, both necessary and sufficient, for a system to be stabilizable under the mean-square criterion. Similar to stabilizability results found in the networked control literature, the system's degree of instability is seen to play an essential role in this condition. More generally, with output feedback under consideration, the system's nonminimum phase zeros may also couple with unstable poles to render the condition more stringent. When interpreted in the networked control setting, the result is rather reminiscent of prior work on networked feedback stabilization under various channel descriptions, including, e.g., the minimum data rate of a communication channel required to stabilize an unstable plant [23], [24], [34], [41], [42], [47], [48], stabilization over erasure channels [10], [28], [29], [32], stabilization subject to channel signal-to-noise ratio (SNR) constraints [4], [19], [20], [30], [31], [33], effect of quantization on stabilization [5], [11], [12], [16], [26] and channel delay effect on stabilization [29], [39].

MIMO systems, on the other hand, prove far more intricate and indeed pose a formidable challenge: the mean-square stabilizability condition generally requires solving an optimization problem involving the spectral radius of a certain closed loop transfer function matrix, which is unlikely to be convex. We explore this problem to a greater depth. Among our main contributions, we show that for a MIMO minimum phase plant, this problem is solvable in the general case of output feedback as a generalized eigenvalue problem (GEVP), which can be solved using linear matrix inequality (LMI) optimization methods, thus resulting in a necessary and sufficient condition for mean-square stabilizability. It is useful to point out that with state feedback, a plant is effectively rendered minimum phase and as such the mean-square stabilizability problem is fully resolved in the case of state feedback. Further investigation into limiting cases shows that the stabilizability condition not only depends on the locations of the plant unstable poles, but also the directions associated with the poles.

The mean-square stabilizability for general MIMO nonminimum phase plants remains to be an open problem. Advances, however, are made in this paper on two important accounts. For systems with time delays, which manifest themselves as the system's relative degrees, we develop analogously necessary and sufficient stabilizability conditions. Here the delays may result from the plant itself, or be considered network delays in the networked control setting, which both can be viewed as an extreme nonminimum phase behavior. In the same spirit, the stabilizability condition amounts to solving a GEVP problem, and the delays can be seen to have a direct impact on the mean-square stabilizability. For more generic nonminimum phase behavior, we examine plants containing one unstable pole and one nonminimum phase zero, and show that the mean-square stabilizability condition depends on the mutual orientation of the pole and zero directions. This orientation is measured by the principal angle between the two directions. The result sheds light into the complication brought about by nonminimum phase zeros on the mean-square stabilizability problem, which in general does not admit a convex program, as shown by an illustrative example.

It is worth highlighting the fact that the mean-square stabilization problem being considered herein is in essence one of robust optimal control synthesis with respect to structured uncertainties, albeit with stochastic uncertainties. Similar problems of minimizing the spectral radius have been widely known in robust synthesis [17], [49], and are also found in networked feedback stabilization problems [10], [44]. Problems in this category are by and large unresolved; they are generally known to be nonconvex and as such, only approximate solutions are available based on e.g., numerical algorithms resembling to the *D-K iteration* for μ-synthesis [10], [49]. By obtaining the mean-square stabilizability condition, our result goes significantly beyond the mean-square small gain condition and provides perhaps, to the best of the authors' knowledge, the first solution to the stabilizability problem with structured stochastic multiplicative uncertainties, and more broadly, an exact solution to this variant of robust synthesis problem. Accordingly, by solving this general problem, our result lends solutions readily applicable to related networked control problems, concerning, e.g., erasure channels [10], [46] and SNR-constrained channels [44]. Moreover, since we consider output feedback stabilization, which is markedly more difficult than its state-feedback counterpart, our development brings to light useful insight into how directions of the plant unstable poles and nonminimum phase zeros may affect the stabilizability. In the networked control setting, the latter has a direct implication on how such directions may be aligned with communication channels to hamper feedback stabilization.

We now close this section by briefly commenting on our contributions presented in Part 2. In this companion sequel, we study optimal control problems under the stochastic multiplicative uncertainty formulation, which are formulated and solved in a general, unified framework. While our interest in Part 1 is mainly concentrated on developing fundamental understanding in stabilization and stabilizability, Part 2 reinforces Part 1 by developing computationally efficient methods for optimal synthesis. Thus, unlike in the present paper, Part 2 tackles optimal performance problems from a time-domain, state-space perspective, which leads to optimal solutions cast in the form of readily solvable GEVPs.

Partial results of this paper have been previously presented in [26], [39], and [40].

II. Problem Formulation and Preliminary Results

A. Notation

We collect here the notation used throughout this two-part series. For any complex number z, any vector u, and any matrix A, we denote by z^*, u^*, and A^* their conjugate and conjugate transposes, respectively. For a pair of unitary vectors u, v of the same dimension, we denote the principal angle between their directions by $\angle(u,v) \in [0, \pi/2]$, defined by $\cos\angle(u,v) := |u^* v|$. We say that the two directions are orthogonal if $\angle(u,v) = \pi/2$, and that they are parallel if $\angle(u,v) = 0$. For any square matrix A, we denote its spectral radius by $\rho(A)$ and trace by $\mathrm{Tr}(A)$. The Hölder ℓ_2, ℓ_1, and ℓ_∞ induced norms of a matrix $A = [a_{ij}]$ are denoted by $\|A\|$, $\|A\|_1$, and $\|A\|_\infty$, respectively, i.e.,

$$\|A\|_1 = \max_j \sum_i |a_{ij}|, \quad \|A\|_\infty = \max_i \sum_j |a_{ij}|$$

while its Frobenius norm is denoted by $\|A\|_F = \sqrt{\mathrm{Tr}(A^* A)}$. We write $A \geq 0$ if A is nonnegative definite, and $A > 0$ if it is positive definite. For any transfer function matrix $G(z)$, we represent a state-space realization of $G(z)$ by $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Let the open unit disc be denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the closed unit disc by $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, the unit circle by $\partial\mathbb{D}$, and the complements of $\mathbb{D}$ and $\bar{\mathbb{D}}$ by $\mathbb{D}^c$ and $\bar{\mathbb{D}}^c$, respectively. With respect to $\partial\mathbb{D}$, we define for matrix functions the Hilbert space $\mathcal{L}_2$ by

$$\mathcal{L}_2 := \left\{ F : F(z) \text{ measurable in } \partial\mathbb{D}, \|F\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\|F(e^{j\theta})\right\|_F^2 d\theta \right)^{\frac{1}{2}} < \infty \right\}$$

which is endowed with the inner product

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Tr}\left(F^H(e^{j\theta}) G(e^{j\theta})\right) d\theta.$$

It is well-known that $\mathcal{L}_2$ admits an orthogonal decomposition into the subspaces

$$\mathcal{H}_2 := \left\{ F : F(z) \text{ analytic in } \bar{\mathbb{D}}^c, \|F\|_2 = \left(\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\|F(re^{j\theta})\right\|_F^2 d\theta \right)^{\frac{1}{2}} < \infty \right\}$$

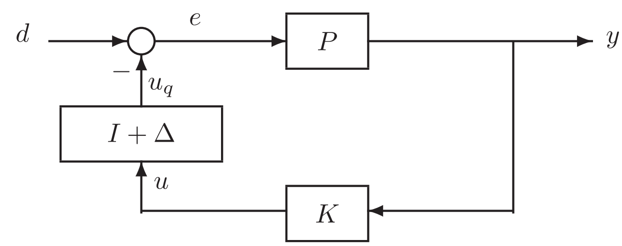


Fig. 1. LTI system with stochastic multiplicative uncertainty.

$$= \left\{ F : F(z) \text{ analytic in } \mathbb{D}, F(0) = 0, \|F\|_2 = \left(\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\|F(re^{j\theta})\right\|^2 d\theta \right)^{\frac{1}{2}} < \infty \right\}.$$

Note that for any $F \in \mathcal{H}_2^\perp$ and $G \in \mathcal{H}_2$, $\langle F, G \rangle = 0$. Define also the Hardy space $\mathcal{H}_\infty = \{F : F(z) \text{ bounded and analytic in } \mathbb{D}^c\}$. A subset of $\mathcal{H}_\infty$, $\mathbb{RH}_\infty$, is the set of all proper stable rational transfer function matrices. Finally, we denote the expectation operator by $E\{\cdot\}$.

B. Structured Multiplicative Uncertainty

For consideration of stabilization problems, we focus on the uncertain system depicted in Fig. 1; a more general setup appropriate for addressing performance issues will be given in Part 2. In this configuration, P represents the plant and K the controller, both of which are assumed to be LTI systems. The plant uncertainty is represented by the static component Δ, such that

$$\begin{aligned} u^q(k) &= (I + \Delta(k))\, u(k) \\ \Delta(k) &= \mathrm{diag}\left(\Delta_1(k), \ldots, \Delta_m(k)\right). \end{aligned} \tag{1}$$

In other words, the plant contains a diagonally structured, static multiplicative uncertainty at its input. Throughout this paper we make the following assumptions:

Assumption 1: $\{\Delta_i(k)\}$, $i = 1, \ldots, m$, is a white noise process with variance σ_i^2.

Assumption 2: $\{\Delta_i(k)\}$ and $\{\Delta_j(k)\}$ are uncorrelated processes for $i \neq j$, i.e.,

$$E\left\{\Delta_i(k_1)\Delta_j(k_2)\right\} = 0 \quad \forall\, k_1, k_2 \text{ and } i \neq j.$$

Assumption 3: $\{\Delta_i(k)\}, i = 1, \ldots, m$, is uncorrelated with $\{d(k)\}$.

It is worth pointing out that these assumptions are standard in the earlier studies of random multiplicative noises (see, e.g., [22]).

The structured uncertainty described by (1), together with Assumptions 1–3, renders the system in Fig. 1 as one subject to a structured stochastic multiplicative uncertainty. In the classical stochastic control setting, this uncertainty may result from state-and input-dependent random noises [13], [22]. In addition, it can also be applied to model uncertainties of communication channels. In [10], Elia refers to this uncertainty as structured bounded variance uncertainty, and showed that it provides a

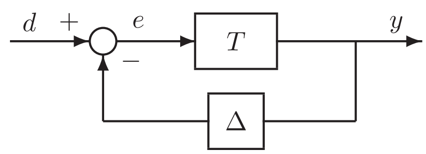


Fig. 2. Mean-square small gain theorem vs stochastic multiplicative uncertainty.

rather general description for modeling erasure and possibly fading communication channels. Independent lossy, memoryless channels with i.i.d. stochastic processes, such as those with packet drops and random delays modeled by Bernoulli processes [10], [35], can be described using this uncertainty description subject to a possible bias in the "nominal" plant, which is induced by possible nonzero means of the processes.

C. Mean-Square Small Gain Theorem

By mean-square stability of the system in Fig. 1, we mean that for any bounded initial states of the plant and controller, the variances of these states will converge asymptotically to the zero matrix when $k \to \infty$. This notion of internal stability can be characterized equivalently from an input-output perspective via the following definition, appropriately tailored from [22], [45].

Definition 1: The system in Fig. 2 is said to be mean-square input-output stable if for any input sequence $\{d(k)\}$ with bounded variance $E\{d(k)d^*(k)\} < \infty$, the variances of the error and output sequences $\{e(k)\}$, $\{y(k)\}$ are also bounded, i.e., $E\{e(k)e^*(k)\} < \infty$ and $E\{y(k)y^*(k)\} < \infty$.

The following result, herein referred to as the mean-square small gain theorem, is adapted from [22] (see also [13] and [45]), which provides a necessary and sufficient condition for mean-square input-output stability. This result will play a pivotal role in our subsequent development.

Lemma 1 (Mean-Square Small Gain Theorem): Let T be a stable LTI system, and $\Delta(k)$ be given by (1). Then under Assumptions 1–3, the system in Fig. 2 is mean-square stable if and only if

$$\rho(W) < 1 \tag{2}$$

where

$$W = \begin{bmatrix} \|T_{11}\|_2^2 & \cdots & \|T_{1m}\|_2^2 \\ \vdots & \cdots & \vdots \\ \|T_{m1}\|_2^2 & \cdots & \|T_{mm}\|_2^2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}.$$

Note that for a SISO system, the mean-square stability condition reduces to

$$\sigma^2 \|T\|_2^2 < 1 \tag{3}$$

where σ^2 is the variance of $\Delta(k)$.

With the uncertainty model (1), it is straightforward to show via direct manipulation that the system in Fig. 1 can be rearranged to that in Fig. 2, with the transfer function matrix $T(z)$ given by the system's complementary sensitivity function

$$T(z) = K(z)P(z)\left[I + K(z)P(z)\right]^{-1}.$$

Thus, under Assumptions 1–3, Lemma 1 can be applied at once to determine the mean-square stability of the system. Let a right and left coprime factorization of the plant transfer function matrix $P(z)$ be given by

$$P = LM^{-1} = \tilde{M}^{-1}\tilde{L}$$

where L, M, $\tilde{L}$, $\tilde{M} \in \mathbb{RH}_\infty$ satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{L} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ L & X \end{bmatrix} = \begin{bmatrix} M & Y \\ L & X \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{L} & \tilde{M} \end{bmatrix} = I \tag{4}$$

for some X, Y, $\tilde{X}$, $\tilde{Y} \in \mathbb{RH}_\infty$. It is well-known that every stabilizing controller K can be parameterized as [49]

$$\begin{aligned} K &= (Y - MR)(LR - X)^{-1} \\ &= (R\tilde{L} - \tilde{X})^{-1}(\tilde{Y} - R\tilde{M}), \quad R \in \mathbb{RH}_\infty. \end{aligned} \tag{5}$$

In turn, every stable complementary sensitivity function $T(z)$ can be found as

$$T = -(Y - MR)\tilde{L}, \qquad R \in \mathbb{RH}_\infty. \tag{6}$$

In light of Lemma 1, the following condition for mean-square stabilizability is immediate.

Lemma 2: Let Δ be given by (1). Then under Assumptions 1–3, the system in Fig. 1 is mean-square stabilizable if and only if

$$\rho_{\min} = \inf_{R \in \mathcal{RH}_\infty} \rho\left(W(R)\right) < 1 \tag{7}$$

where

$$W(R) = \begin{bmatrix} \sigma_1^2 \left\| \left[(Y - MR)\tilde{L}\right]_{11} \right\|_2^2 & \cdots & \sigma_m^2 \left\| \left[(Y - MR)\tilde{L}\right]_{1m} \right\|_2^2 \\ \vdots & \ddots & \vdots \\ \sigma_1^2 \left\| \left[(Y - MR)\tilde{L}\right]_{m1} \right\|_2^2 & \cdots & \sigma_m^2 \left\| \left[(Y - MR)\tilde{L}\right]_{mm} \right\|_2^2 \end{bmatrix}.$$

We note that the condition (7) is found in [10] and [44] as well. As such, the solution to the minimization problem in (7) also gives a necessary and sufficient condition for feedback stabilization over the erasure channel under consideration in [10], and the SNR-constrained channel in [44]. The solution to this problem, however, requires synthesizing an optimal $R \in \mathbb{RH}_\infty$, which is currently unavailable and is a principal objective of this paper.

III. SISO Systems

For a SISO system, the multiplicative uncertainty Δ given in (1) is a scalar white noise process with variance σ^2. It follows from Lemma 2 that the closed-loop system in Fig. 1 is mean-square stabilizable if and only if:

$$\sigma^2 \inf_{R \in \mathbb{RH}_\infty} \left\| (Y - MR)\tilde{L} \right\|_2^2 < 1. \tag{8}$$

Thus, the mean-square stabilizability amounts to solving a standard $\mathcal{H}_2$ problem. In this section, we solve this problem explicitly and derive explicit conditions for mean-square stabilizability.

A. Mean-Square Stabilizability Bounds

We consider general nonminimum phase plants with a time delay. A discrete-time system contains a time delay of length $\tau \geq 0$ if its relative degree is τ.

Theorem 1: Suppose that $P(z)$ has a relative degree $\tau \geq 0$. Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \ldots, n$ and $s_k \in \bar{\mathbb{D}}^c$, $k = 1, \ldots, l$ be the unstable poles and nonminimum phase zeros of $P(z)$, respectively. Then under Assumptions 1–3, the system in Fig. 1 is mean-square stabilizable if and only if

$$\sigma^2 f_{ps}^* \left(D_p^*\right)^{\tau-1} Q_p (D_p)^{\tau-1} f_{ps} < 1 \tag{9}$$

where f_{ps}, Q_p and D_p are given by

$$f_{ps} = \begin{bmatrix} f_1(p_1)\tilde{L}_{\text{in}}^{-1}(p_1) \\ \vdots \\ f_n(p_n)\tilde{L}_{\text{in}}^{-1}(p_n) \end{bmatrix}, \quad D_p = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix}$$

$$Q_p = \left[\frac{\left(|p_i|^2 - 1\right)\left(|p_j|^2 - 1\right)}{p_i^* p_j - 1}\right]$$

with

$$\tilde{L}_{\text{in}}(z) = \prod_{k=1}^{l} \frac{z - s_k}{1 - s_k^* z}$$

$$f_i(z) = \prod_{j \neq i}^{n} \frac{1 - p_j^* z}{z - p_j}, \quad i = 1, \ldots, n.$$

Proof: We first conduct an all-pass factorization $M(z) = M_{\text{in}}(z) M_{\text{out}}(z)$, such that

$$\begin{aligned} M_{\text{in}}^{-1}(z) &= \prod_{i=1}^{n} \frac{1 - p_i^* z}{z - p_i} \\ &= \sum_{i=1}^{n} f_i(p_i) \frac{1 - |p_i|^2}{z - p_i} + M_{\text{in}}^{-1}(\infty). \end{aligned} \tag{10}$$

Next, factorize $\tilde{L}(z)$ as $\tilde{L} = \tilde{L}_{\text{out}} \tilde{L}_{\text{in}}$. Let $\hat{L}_{\text{out}} = \tilde{L}_{\text{out}} z^{\tau}$. Then, $\hat{L}_{\text{out}}^{-1} \in \mathbb{RH}_\infty$. In view of the Bezout identity (4), we have $M\tilde{X} - Y\hat{L}_{\text{out}}\tilde{L}_{\text{in}} z^{-\tau} = 1$, which gives rise to $Y(p_i)\hat{L}_{\text{out}}(p_i) = -p_i^{\tau} \tilde{L}_{\text{in}}^{-1}(p_i)$. By conducting a partial fraction, we obtain

$$\begin{aligned} M_{\text{in}}^{-1}(z) Y(z) \hat{L}_{\text{out}}(z) &= \sum_{i=1}^{n} f_i(p_i) \frac{|p_i|^2 - 1}{z - p_i} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} + Z(z) \\ &= \sum_{i=1}^{n} \left(f_i(p_i) \frac{|p_i|^2 - 1}{z - p_i} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} \right. \\ &\qquad \left. + f_i(p_i) \frac{|p_i|^2 - 1}{p_i} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} \right) \\ &\quad + \left(Z(z) - \sum_{i=1}^{n} f_i(p_i) \right. \\ &\qquad \left. \times \frac{|p_i|^2 - 1}{p_i} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} \right) \\ &= \sum_{i=1}^{n} f_i(p_i) \frac{\left(|p_i|^2 - 1\right) z}{p_i (z - p_i)} \\ &\quad \times \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} + \hat{Z}(z). \end{aligned}$$

Here $Z, \hat{Z} \in \mathbb{RH}_\infty$. It is clear that

$$\hat{Z} - M_{\text{out}} R \hat{L}_{\text{out}} \in \mathcal{H}_2$$

$$\sum_{i=1}^{n} f_i(p_i) \frac{\left(|p_i|^2 - 1\right) z}{p_i (z - p_i)} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau} \in \mathcal{H}_2^{\perp}.$$

This leads us to

$$\begin{aligned} \left\|(Y - MR)\tilde{L}\right\|_2^2 &= \left\|(Y - MR)\hat{L}_{\text{out}}\right\|_2^2 \\ &= \|M_{\text{in}}^{-1} Y \hat{L}_{\text{out}} - M_{\text{out}} R \hat{L}_{\text{out}}\|_2^2 \\ &= \left\|\sum_{i=1}^{n} f_i(p_i) \frac{\left(|p_i|^2 - 1\right) z}{p_i (z - p_i)} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau}\right\|_2^2 \\ &\quad + \|\hat{Z} - M_{\text{out}} R \hat{L}_{\text{out}}\|_2^2. \end{aligned}$$

Since $\hat{L}_{\text{out}}^{-1} \in \mathbb{RH}_\infty$, we have

$$\inf_{R \in \mathbb{RH}_\infty} \|\tilde{Z} - M_{\text{out}} R \tilde{L}_{\text{out}}\|_2^2 = 0.$$

Consequently

$$\begin{aligned} &\inf_{R \in \mathbb{RH}_\infty} \left\|(Y - MR)\tilde{L}\right\|_2^2 \\ &\quad = \left\|\sum_{i=1}^{n} f_i(p_i) \frac{\left(|p_i|^2 - 1\right) z}{p_i (z - p_i)} \tilde{L}_{\text{in}}^{-1}(p_i) p_i^{\tau}\right\|_2^2. \end{aligned}$$

The proof is then completed by noting that

$$\left\langle \frac{1}{z - p_i}, \frac{1}{z - p_j} \right\rangle = \frac{1}{p_i^* p_j - 1}.$$

■

Theorem 1 provides a complete solution to the mean-square stabilizability problem in the SISO case. Similar to earlier results addressing different performance problem (see, e.g., [8], [9], [18]), the theorem characterizes the minimal requirement for mean-square stabilization in terms of the plant's unstable poles, nonminimum phase zeros, and relative degree. It is clear that other than the effect by the unstable poles alone, the nonminimum phase zeros may couple with the unstable poles to aggravate the stabilizability condition, and that close proximity of unstable poles and nonminimum phase zeros imposes a stringent limit on the uncertainty variance. This result exhibits a rather intricate dependence of the stabilizability condition on the plant unstable poles and nonminimum phase zeros, which can be partly attributed to the curse of output feedback. As to be shown below, in the case of state feedback, which is void of nonminimum phase effect, the condition can be simplified considerably, similar to most of the networked feedback stabilization results.

The following theorem addresses individually the pole and zero effects, allowing more explicit insights to be gained.

Theorem 2: Suppose that $P(z)$ has relative degree $\tau \geq 0$.

(i) Suppose that $P(z)$ has no nonminimum phase zero, and let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \ldots, n$ be its unstable poles. Then for $\tau = 0, 1$, the system in Fig. 1 is mean-square stabilizable if and only if

$$\frac{1}{\sigma^2} > \left(\prod_{i=1}^{n} |p_i|^{2(\tau-1)}\right)\left(\prod_{i=1}^{n} |p_i|^2 - 1\right). \tag{11}$$

(ii) Let $p \in \bar{\mathbb{D}}^c$ be the only unstable pole of $P(z)$, and $s_k \in \bar{\mathbb{D}}^c$, $k = 1, \ldots, l$ be the nonminimum phase zeros of $P(z)$. Then the system in Fig. 1 is mean-square stabilizable if and only if

$$\frac{1}{\sigma^2} > |p|^{2(\tau-1)}(|p|^2-1)\prod_{k=1}^{l}|\frac{1-s_k^* p}{p-s_k}|^2. \quad (12)$$

(iii) Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \ldots, n$ and $s_k \in \bar{\mathbb{D}}^c$, $k = 1, \ldots, l$ be the unstable poles and nonminimum phase zeros of $P(z)$, respectively. Then for the system in Fig. 1 to be mean-square stabilizable, it is necessary that for all $i = 1, \ldots, n$

$$\frac{1}{\sigma^2} > |p_i|^{2(\tau-1)}\left(|p_i|^2-1\right)\prod_{k=1}^{l}\left|\frac{1-s_k^* p_i}{p_i-s_k}\right|^2. \quad (13)$$

Proof: For Case (i), we first consider $\tau = 0$. Under the assumption that $P(z)$ has no nonminimum phase zero, we have $\tilde{L}^{-1} \in \mathbb{RH}_\infty$. It follows from the Bezout identity (4) that $M_{\text{in}}^{-1} Y \tilde{L} = M_{\text{out}}\tilde{X} - M_{\text{in}}^{-1}$, which leads to

$$\begin{aligned}\left\|(Y-MR)\tilde{L}\right\|_2^2 &= \left\|M_{\text{in}}^{-1}Y\tilde{L} - M_{\text{out}}R\tilde{L}\right\|_2^2\\ &= \left\|M_{\text{in}}^{-1} - M_{\text{out}}\tilde{X} + M_{\text{out}}R\tilde{L}\right\|_2^2\\ &= \left\|M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(0)\right\|_2^2\\ &\quad + \left\|M_{\text{in}}^{-1}(0) - M_{\text{out}}\tilde{X} + M_{\text{out}}R\tilde{L}\right\|_2^2.\end{aligned}$$

As a result

$$\inf_{R\in\mathbb{RH}_\infty}\left\|(Y-MR)\tilde{L}\right\|_2^2 = \left\|M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(0)\right\|_2^2.$$

Write

$$M_{\text{in}}(z) = \prod_{i=1}^{n} B_i(z), \qquad B_i(z) = \frac{z-p_i}{1-p_i^* z}.$$

It follows that:

$$\begin{aligned}&\left\|M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(0)\right\|_2^2\\ &= \left\|\prod_{i=1}^{n}B_i^{-1}(z) - \prod_{i=1}^{n}B_i^{-1}(0)\right\|_2^2\\ &= \left\|B_n^{-1}(z) - B_n^{-1}(0)\right\|_2^2\\ &\quad + \left|B_n^{-1}(0)\right|^2\left\|\prod_{i=1}^{n-1}B_i^{-1}(z) - \prod_{i=1}^{n-1}B_i^{-1}(0)\right\|_2^2\\ &= \left(1-\frac{1}{|p_n|^2}\right) + \frac{1}{|p_n|^2}\left\|\prod_{i=1}^{n-1}B_i^{-1}(z) - \prod_{i=1}^{n-1}B_i^{-1}(0)\right\|_2^2\\ &= 1 - \prod_{i=1}^{n}\frac{1}{|p_i|^2}.\end{aligned}$$

This establishes the case for $\tau = 0$. For $\tau \geq 1$, we have $\tilde{L}(\infty) = 0$, and hence from the Bezout identity (4), $M_{\text{in}}^{-1}(\infty) - M_{\text{out}}(\infty)\tilde{X}(\infty) = 0$. Thus

$$\begin{aligned}z\left(M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(\infty)\right) &\in \mathcal{H}_2^\perp\\ z\left(M_{\text{in}}^{-1}(\infty) - M_{\text{out}}\tilde{X}\right) &\in \mathcal{H}_2.\end{aligned}$$

This recognition leads to

$$\begin{aligned}\left\|(Y-MR)\tilde{L}\right\|_2^2 &= \left\|M_{\text{in}}^{-1} - M_{\text{out}}\tilde{X} + M_{\text{out}}R\tilde{L}\right\|_2^2\\ &= \left\|\left(M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(\infty)\right)\right.\\ &\quad \left. + \left(M_{\text{in}}^{-1}(\infty) - M_{\text{out}}\tilde{X} + M_{\text{out}}R\tilde{L}\right)\right\|_2^2\\ &= \left\|z\left(M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(\infty)\right)\right.\\ &\quad \left. + z\left(M_{\text{in}}^{-1}(\infty) - M_{\text{out}}\tilde{X}\right) + M_{\text{out}}R\tilde{L}z\right\|_2^2\\ &= \left\|z\left(M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(\infty)\right)\right\|_2^2\\ &\quad + \left\|z\left(M_{\text{in}}^{-1}(\infty) - M_{\text{out}}\tilde{X}\right) + M_{\text{out}}R\tilde{L}z\right\|_2^2.\end{aligned}$$

Since for $\tau = 1$, $\tilde{L}z$ is invertible in $\mathbb{RH}_\infty$, it is immediate that

$$\begin{aligned}&\inf_{R\in\mathbb{RH}_\infty}\left\|(Y-MR)\tilde{L}\right\|_2^2\\ &= \left\|z\left(M_{\text{in}}^{-1}(z) - M_{\text{in}}^{-1}(\infty)\right)\right\|_2^2\\ &= \left\|z\left(\prod_{i=1}^{n}B_i^{-1}(z) - \prod_{i=1}^{n}B_i^{-1}(\infty)\right)\right\|_2^2\\ &= \left\|z\left(\prod_{i=1}^{n-1}B_i^{-1}(z) - \prod_{i=1}^{n-1}B_i^{-1}(\infty)\right)\right\|_2^2\\ &\quad + \left|\prod_{i=1}^{n-1}B_i^{-1}(\infty)\right|^2\\ &\quad \times \left\|z\left(B_n^{-1}(z) - B_n^{-1}(\infty)\right)\right\|_2^2\\ &= \prod_{i=1}^{n-1}|p_i|^2 - 1 + \prod_{i=1}^{n-1}|p_i|^2\left(|p_n|^2-1\right)\\ &= \prod_{i=1}^{n}|p_i|^2 - 1\end{aligned}$$

where the last equality was also established previously in [27]. Case (ii) follows directly from (9). To establish the necessary condition in Case (iii), it suffices to observe, analogously as in the proof for Theorem 1, that for any $i = 1, \ldots, n$

$$\begin{aligned}\inf_{R\in\mathbb{RH}_\infty}\left\|(Y-MR)\tilde{L}\right\|_2^2 &\geq \left\|f_i(p_i)\frac{\left(|p_i|^2-1\right)z}{p_i(z-p_i)}\tilde{L}_{\text{in}}^{-1}(p_i)p_i^\tau\right\|_2^2\\ &= |p_i|^{2(\tau-1)}\left(|p_i|^2-1\right)\prod_{k=1}^{l}\left|\frac{1-s_k^* p_i}{p_i-s_k}\right|^2.\end{aligned}$$

The proof is thus completed. ■

Of the above bounds, it is interesting to see that (11) replicates a number of previous results [4], [10], [12], [28], [42], [46] obtained under different channel model assumptions, which provide fundamental bounds on, e.g., channel data rate, channel capacity, and channel SNR required for stabilization in a state feedback configuration. This condition thus reinforces the previous results, by providing yet another viable case for which the lower bound (11) is fundamental; the degree of instability imposes a fundamental limit on the noise level allowable. The bounds (12) and (13) strengthen this result further, which demonstrate explicitly the effect due to nonminimum phase zeros and time delays. It is seen herein that the difficulty of stabilization increases exponentially with the delay length. Furthermore, when in the extreme the plant undergoes nearly unstable pole-zero cancelation, stabilization will become impossible.

B. An Illustrative Example

We next use an example to illustrate our preceding results.

Example 1: The system

$$P(z) = \frac{z-a}{z^r(z-1.1)(z-1.4)}, \qquad r \geq 0.$$

has a relative degree $\tau = r+1$, and contains a variable zero at $z = a$. We use the MATLAB command $randn(\cdot)$ to generate a random Gaussian signal and normalize it to a zero-mean white process $\{d(k)\}$ such that $E\{d^2(k)\} = 0.05$. The uncertainty Δ is generated similarly. We compute $E\{y^2(k)\}$. At each k, the computation takes 20 000 iterations.

We first fix $a = 0.5$ and $r = 0$. In this case, the maximal variance allowed for mean-square stabilization is found as $\sigma^2_{\max} = 0.7291$, and the optimal controller attaining this bound is obtained as

$$K(z) = \frac{0.8766z - 0.8906}{z - 0.5}.$$

It can be demonstrated that for σ^2 marginally less and marginally greater than 0.7291, the output variance $E\{y^2(k)\}$ converges and diverges, respectively; in other words, the closed-loop system is and is not mean-square stable. For σ^2 significantly less ($\sigma^2 = 0.5$) and greater ($\sigma^2 = 1$) than 0.7291, Fig. 3(a) and (b) show that this convergence and divergence becomes more dramatic. Note that $E\{e^2(k)\}$ displays the same convergence and divergence behavior, but is smaller in orders of magnitude and is so omitted. With the same controller, we then change $r = 0$ to $r = 1$. Fig. 4 shows that the closed-loop mean-square stability is rather sensitive to this change, in spite of a low noise variance ($\sigma^2 = 0.05$). Indeed, for $r = 1$, $E\{y^2(k)\}$ is seen to diverge rapidly. This is expected since the delay τ has an exponential effect on $\sigma^2_{\max}$.

Next, we allow a to vary from $a = 0$ to $a > 1$, so that the zero $z = a$ can be minimum phase or nonminimum phase. This case is meant to illustrate how a nonminimum phase zero may couple with unstable poles to affect mean-square stability. Fig. 5 shows how $\sigma^2_{\max}$ varies with a, for $\tau = 1, 2$, respectively. Note that any value for σ^2 below the $\sigma^2_{\max}$ curve can guarantee closed-loop mean-square stability, and hence is indicated as mean-square stability (MSS) region. One can see that when the zero $z = a$ becomes nonminimum phase, $\sigma^2_{\max}$ decreases sharply. The zoom-in plot shows in particular that $\sigma^2_{\max}$ drops to zero at $a = 1.1, 1.4$, where the two unstable poles lie. On the other hand, when the zero moves farther away from the poles, $\sigma^2_{\max}$ increases.

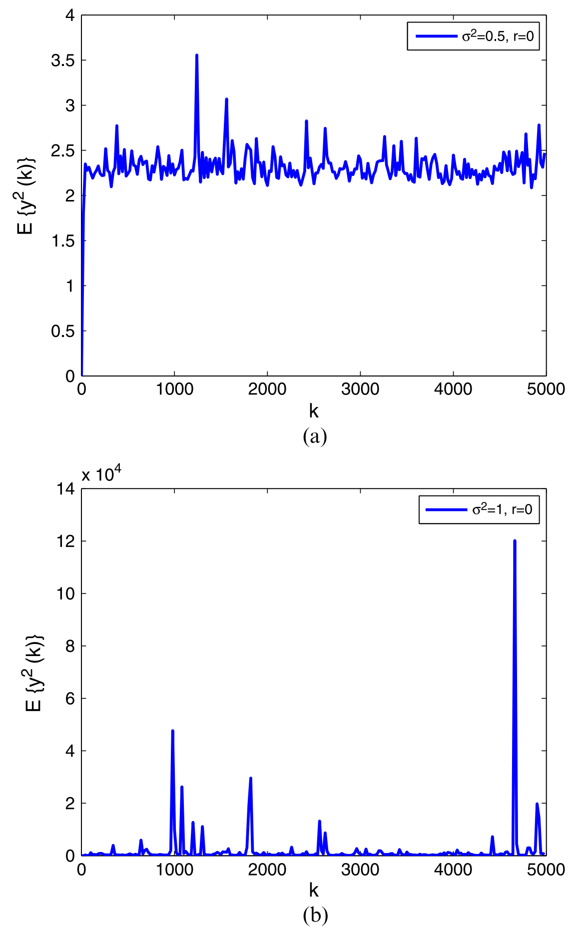


Fig. 3. Pole effect and sensitivity of stabilizability bound. (a) Mean-square stable closed-loop system with $\sigma^2 = 0.5$, $r = 0$. (b) Mean-square unstable closed-loop system with $\sigma^2 = 1$, $r = 0$.

IV. MIMO Systems

Lemma 2 makes it clear that to ascertain mean-square stabilizability of a MIMO system, one generally must solve a minimization problem for the spectral radius $\rho(W(R))$ over the set of $R \in \mathbb{RH}_\infty$. This poses a highly nontrivial task. While for SISO systems the spectral radius reduces to the $\mathcal{H}_2$ norm of the complementary sensitivity function, which defines a convex function and hence is duly solved, this attribute is not preserved to MIMO systems. Indeed, as our subsequent case study shows, the MIMO case in general constitutes a nonconvex optimization problem.

In this section we extend our preceding results to MIMO systems. For a MIMO plant $P(z)$, a complex number $p \in \bar{\mathbb{D}}^c$ is a plant unstable pole with an output direction vector η, $\|\eta\| = 1$ if $\eta^* M(p) = 0$. Similarly, a complex number $s \in \bar{\mathbb{D}}^c$ is a nonminimum phase zero of $P(z)$ with an input direction vector ζ, $\|\zeta\| = 1$ if $\tilde{L}(s)\zeta = 0$. In the sequel, for an all-pass transfer function matrix $M_{\text{in}}(z) = \left[\begin{array}{c|c} A_{\text{in}} & B_{\text{in}} \\ \hline C_{\text{in}} & D_{\text{in}} \end{array}\right]$, we use the familiar realization of

$$M_{\text{in}}^{-1}(z) = \left[\begin{array}{c|c} A_{\text{in}} - B_{\text{in}}D_{\text{in}}^{-1}C_{\text{in}} & -B_{\text{in}}D_{\text{in}}^{-1} \\ \hline D_{\text{in}}^{-1}C_{\text{in}} & D_{\text{in}}^{-1} \end{array}\right]. \quad (14)$$

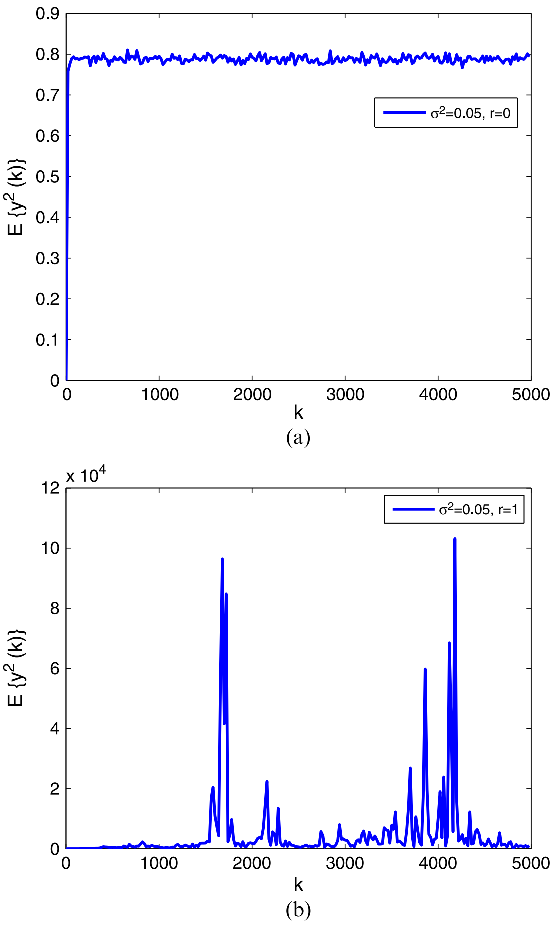


Fig. 4. Delay effect on mean-square stabilizability. (a) Mean-square stable closed-loop system with $\sigma^2 = 0.05,\ r = 0$. (b) Mean-square unstable closed-loop system with $\sigma^2 = 0.05$, $r = 1$.

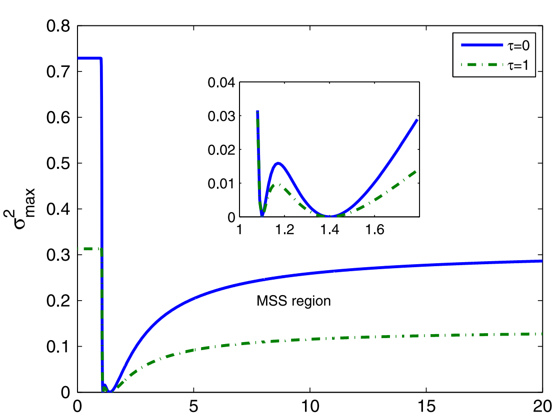


Fig. 5. Zero effect on mean-square stabilizability.

We shall also write

$$\hat{A} = A_{\text{in}} - B_{\text{in}} D_{\text{in}}^{-1} C_{\text{in}}. \tag{15}$$

It is useful to point out that the eigenvalues of $\hat{A}$ coincide with the zeros of $M_{\text{in}}(z)$.

The following lemma (see, e.g., [1], [38]) will serve as the technical basis in our subsequent developments. This alternative characterization of $\rho(W)$ is rather similar to robust control synthesis problems which also resort to scaled norm minimization [17], [49].

Lemma 3: For any nonnegative matrix W

$$\rho(W) = \inf_{\Gamma} \|\Gamma W \Gamma^{-1}\|_1 = \inf_{\Gamma} \|\Gamma W \Gamma^{-1}\|_\infty \tag{16}$$

where the infimum is taken over the set of positive diagonal matrices $\Gamma = \text{diag}(\gamma_1^2, \ldots, \gamma_m^2)$.

We shall first present a general stabilizability condition for minimum phase systems and next address delay and nonminimum phase systems. This organization will facilitate the presentation since the results require rather different proofs.

A. Minimum Phase Plants

In this section we present a computationally efficient solution to the mean-square stabilizability problem for minimum phase plants. This result casts the necessary and sufficient stabilizability condition as the solution to a parameterized algebraic Riccati equation (ARE), which is solvable as a GEVP problem.

Theorem 3: Suppose that $P(z)$ is minimum phase and has relative degree zero. Let $M = M_{\text{in}} M_{\text{out}}$ be an inner-outer factorization of M, with $M_{\text{in}}(z) = \left[\begin{array}{c|c} A_{\text{in}} & B_{\text{in}} \\ \hline C_{\text{in}} & D_{\text{in}} \end{array}\right]$ being an inner of $M(z)$. Then under Assumptions 1–3

$$\rho_{\min} = \inf \left\{ \mu : \sigma_i^2 e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* \hat{A}^{*-1} X \hat{A}^{-1} B_{\text{in}} D_{\text{in}}^{-1} e_i < \mu e_i^* \Gamma e_i, \quad i = 1, \ldots, m \right\} \tag{17}$$

where $X > 0$ is the solution to the ARE

$$A_{\text{in}}^* X A_{\text{in}} - X + C_{\text{in}}^* \Gamma C_{\text{in}} - (A_{\text{in}}^* X B_{\text{in}} + C_{\text{in}}^* \Gamma D_{\text{in}}) \times (B_{\text{in}}^* X B_{\text{in}} + D_{\text{in}}^* \Gamma D_{\text{in}})^{-1} (B_{\text{in}}^* X A_{\text{in}} + D_{\text{in}}^* \Gamma C_{\text{in}}) = 0. \tag{18}$$

Furthermore

$$\rho_{\min} = \inf \left\{ \mu : \left(\sum_{i=1}^{m} \gamma_i X_i \right) - \frac{1}{\mu} \gamma_i \sigma_i^2 \hat{A}^{-1} B_{\text{in}} D_{\text{in}}^{-1} e_i \cdot e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* \hat{A}^{*-1} > 0, \gamma_i > 0,\ i = 1, \ldots, m \right\} \tag{19}$$

where $X_i \geq 0$, $i = 1, \ldots, m$ is the solution to the Lyapunov equation

$$X_i - \hat{A} X_i \hat{A}^* + B_{\text{in}} D_{\text{in}}^{-1} e_i e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* = 0. \tag{20}$$

The system in Fig. 1 is mean-square stabilizable if and only if $\rho_{\min} < 1$.

Proof: We first note from Lemmas 2 and 3 that

$$\rho(W(R)) = \inf_{\Gamma} \left\| \Gamma W(R) \Gamma^{-1} \right\|_1 = \inf_{\Gamma} \max_{i} \sigma_i^2 \left\| \Gamma^{\frac{1}{2}} (Y - MR) \tilde{L} e_i \right\|_2^2 \gamma_i^{-2}. \tag{21}$$

Then for any $\Gamma = \text{diag}(\gamma_1^2, \ldots, \gamma_m^2)$, $\gamma_i > 0$, $i = 1, \ldots, m$, the transfer function matrix $\Gamma^{1/2} M_{\text{in}}$ can be factorized as

$\Gamma^{1/2}M_{\rm in} = M_{\Gamma{\rm in}}M_{\Gamma{\rm out}}$, where $M_{\Gamma{\rm in}} = \left[\begin{array}{c|c} A_{\Gamma{\rm in}} & B_{\Gamma{\rm in}} \\ \hline C_{\Gamma{\rm in}} & D_{\Gamma{\rm in}}\end{array}\right]$ is inner with the realization [21]

$$M_{\Gamma{\rm in}} = \left[\begin{array}{c|c} A_{\rm in}+B_{\rm in}F & B_{\rm in}\left(D_{\rm in}^*\Gamma D_{\rm in}+B_{\rm in}^*XB_{\rm in}\right)^{-\frac{1}{2}} \\ \hline \Gamma^{\frac{1}{2}}C_{\rm in}+\Gamma^{\frac{1}{2}}D_{\rm in}F & \Gamma^{\frac{1}{2}}D_{\rm in}\left(D_{\rm in}^*\Gamma D_{\rm in}+B_{\rm in}^*XB_{\rm in}\right)^{-\frac{1}{2}}\end{array}\right]$$

with $X > 0$ being the solution to the ARE (18) and

$$F = -\left(B_{\rm in}^*XB_{\rm in} + D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}\left(B_{\rm in}^*XA_{\rm in} + D_{\rm in}^*\Gamma C_{\rm in}\right).$$

Using the Bezout identity (4), it follows that for any $i = 1, \ldots, m$:

$$\begin{aligned}&\left\|\Gamma^{\frac{1}{2}}(Y-MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} \\ &= \left\|M_{\Gamma{\rm in}}^{-1}e_i - (M_{\Gamma{\rm out}}M_{\rm out}\tilde{X} - M_{\Gamma{\rm out}}M_{\rm out}R\tilde{L})e_i\gamma_i^{-1}\right\|_2^2 \\ &= \left\|(M_{\Gamma{\rm in}}^{-1}(z) - M_{\Gamma{\rm in}}^{-1}(0))e_i\right\|_2^2 \\ &\quad + \left\|(\gamma_i M_{\Gamma{\rm in}}^{-1}(0) - M_{\Gamma{\rm out}}M_{\rm out}\tilde{X} + M_{\Gamma{\rm out}}M_{\rm out}R\tilde{L})e_i\right\|_2^2\gamma_i^{-2}.\end{aligned}$$

We proceed to calculate the $\mathcal{L}_2$ norm of $(M_{\Gamma{\rm in}}^{-1}(z) - M_{\Gamma{\rm in}}^{-1}(0))e_i$, which, according to (14), has the realization as shown at the bottom of the page. It follows from an standard exercise [49] that:

$$\left\|\left(M_{\Gamma{\rm in}}^{-1}(z) - M_{\Gamma{\rm in}}^{-1}(0)\right)e_i\right\|_2^2 = e_i^*D_{\rm in}^{*-1}B_{\rm in}^*\hat{A}^{*-1}X\hat{A}^{-1}B_{\rm in}D_{\rm in}^{-1}e_i\gamma_i^{-2}$$

where X is the solution to the Lyapunov equation

$$X - \hat{A}^*X\hat{A} + C_{\Gamma{\rm in}}^*D_{\Gamma{\rm in}}^{*-1}D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}} = 0. \tag{22}$$

It is readily recognized that this equation coincides with the ARE (18). Next, we note that

$$\gamma_i M_{\Gamma{\rm in}}^{-1}(0)e_i = -D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}}\hat{A}^{-1}B_{\rm in}D_{\rm in}^{-1}e_i + \left(D_{\rm in}^*\Gamma D_{\rm in} + B_{\rm in}^*XB_{\rm in}\right)^{\frac{1}{2}}D_{\rm in}^{-1}e_i.$$

Since $\tilde{L}$ has relative degree zero, this implies that for all $i = 1, \ldots, m$

$$\inf_{R\in\mathbb{RH}_\infty}\left\|\left(\gamma_i M_{\Gamma{\rm in}}^{-1}(0) - M_{\Gamma{\rm out}}M_{\rm out}\tilde{X} + M_{\Gamma{\rm out}}M_{\rm out}R\tilde{L}\right)e_i\right\|_2 = 0$$

holds for the same optimal $R \in \mathbb{RH}_\infty$. As a consequence

$$\begin{aligned}\inf_{R\in\mathbb{RH}_\infty}\rho\left(W(R)\right) &= \inf_{R\in\mathbb{RH}_\infty}\inf_{\Gamma}\max_i \sigma_i^2\left\|\Gamma^{\frac{1}{2}}(Y-MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} \\ &= \inf_{\Gamma}\inf_{R\in\mathbb{RH}_\infty}\max_i \sigma_i^2\left\|\Gamma^{\frac{1}{2}}(Y-MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} \\ &= \inf_{\Gamma}\max_i \sigma_i^2\left\|\left(M_{\Gamma{\rm in}}^{-1}(z) - M_{\Gamma{\rm in}}^{-1}(0)\right)e_i\right\|_2^2.\end{aligned}$$

Alternatively, we may write the last equality as

$$\begin{aligned}\rho_{\min} &= \inf_{\Gamma}\Big\{\mu : \sigma_i^2 e_i^*D_{\rm in}^{*-1}B_{\rm in}^*\hat{A}^{*-1}X\hat{A}^{-1}B_{\rm in}D_{\rm in}^{-1}e_i\gamma_i^{-2} \le \mu, \\ &\qquad i = 1, \ldots, m\Big\} \\ &= \inf_{\Gamma}\Big\{\mu : \sigma_i^2 e_i^*D_{\rm in}^{*-1}B_{\rm in}^*\hat{A}^{*-1}X\hat{A}^{-1}B_{\rm in}D_{\rm in}^{-1}e_i \le \mu e_i^*\Gamma e_i, \\ &\qquad i = 1, \ldots, m\Big\}.\end{aligned}$$

This establishes (17). To prove (19), we calculate $D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}}$, which is found to be

$$D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}} = -\left(B_{\rm in}^*XB_{\rm in} + D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1/2}B_{\rm in}^*X\hat{A}.$$

Thus, the Lyapunov equation (22) can be rewritten as

$$X - \hat{A}^*X\hat{A} + \hat{A}^*XB_{\rm in} \times \left(B_{\rm in}^*XB_{\rm in} + D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}B_{\rm in}^*X\hat{A} = 0$$

and further as

$$X - \hat{A}^*\Big(X - XB_{\rm in}\left(B_{\rm in}^*XB_{\rm in} + D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}B_{\rm in}^*X\Big)\hat{A} = 0.$$

Employing the *Sherman-Morrison-Woodbury formula* [14], we have

$$X - XB_{\rm in}\left(B_{\rm in}^*XB_{\rm in} + D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}B_{\rm in}^*X = \left(X^{-1} + B_{\rm in}\left(D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}B_{\rm in}^*\right)^{-1}.$$

Consequently, the Lyapunov equation (22) can be written as

$$X - \hat{A}^*\left(X^{-1} + B_{\rm in}\left(D_{\rm in}^*\Gamma D_{\rm in}\right)^{-1}B_{\rm in}^*\right)^{-1}\hat{A} = 0$$

or equivalently

$$X^{-1} - \hat{A}X^{-1}\hat{A}^* + B_{\rm in}D_{\rm in}^{-1}\Gamma^{-1}D_{\rm in}^{*-1}B_{\rm in}^* = 0. \tag{23}$$

Let X_i be the solution to (20). Then it is readily seen that the solution to (23), and equivalently that to the ARE (18), is given by

$$X = \left(\sum_{i=1}^m \gamma_i^{-2}X_i\right)^{-1}.$$

$$\begin{aligned}\left(M_{\Gamma{\rm in}}^{-1}(z) - M_{\Gamma{\rm in}}^{-1}(0)\right)e_i &= \left[\begin{array}{c|c} A_{\Gamma{\rm in}} - B_{\Gamma{\rm in}}D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}} & -(A_{\Gamma{\rm in}} - B_{\Gamma{\rm in}}D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}})^{-1}B_{\Gamma{\rm in}}D_{\Gamma{\rm in}}^{-1}e_i \\ \hline D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}} & 0\end{array}\right] \\ &= \left[\begin{array}{c|c} \hat{A} & -\hat{A}^{-1}B_{\rm in}D_{\rm in}^{-1}\Gamma^{-\frac{1}{2}}e_i \\ \hline D_{\Gamma{\rm in}}^{-1}C_{\Gamma{\rm in}} & 0\end{array}\right]\end{aligned}$$

Substitute X into the inequalities in (17). Then by a repeated use of *Schur complement* [14], the inequalities in (17) are found to be equivalent to

$$\left(\sum_{i=1}^{m} \gamma_i^{-2} X_i\right) - \frac{1}{\mu}\gamma_i^{-2}\sigma_i^2 \hat{A}^{-1} B_{\text{in}} D_{\text{in}}^{-1} e_i e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* \hat{A}^{*-1} > 0.$$

The proof is now completed by setting γ_i^{-2} to γ_i. ■

Further analysis and discussion on Theorem 3 are deferred to the next subsection, together with those for Theorem 4.

B. Delay Effects

We now generalize Theorem 3 to delay systems. For this purpose, we consider plants with input delays, which can be described by delays in the transfer function matrix $\tilde{L}(z)$

$$\tilde{L}(z) = \tilde{L}_{\text{out}}(z)\begin{bmatrix} z^{-\tau_1} & & \\ & \ddots & \\ & & z^{-\tau_m}\end{bmatrix}$$
$$\tau_i \geq 0,\ i = 1, \ldots, m. \tag{24}$$

Here we assume that $\tilde{L}_{\text{out}}^{-1}(z) \in \mathbb{RH}_\infty$.

Theorem 4: Suppose that $P(z)$ has no zero in $\bar{\mathbb{D}}^c$ except at the point of infinity, and that $\tilde{L}$ is given in (24). Let $M = M_{\text{in}} M_{\text{out}}$ be an inner-outer factorization of M, with $M_{\text{in}}(z) = \left[\begin{array}{c|c} A_{\text{in}} & B_{\text{in}} \\ \hline C_{\text{in}} & D_{\text{in}}\end{array}\right]$ being an inner of $M(z)$. Then under Assumptions 1–3

$$\rho_{\min} = \inf\Big\{\mu : \sigma_i^2 e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* (\hat{A}^*)^{\tau_i - 1} X \hat{A}^{\tau_i - 1} B_{\text{in}} D_{\text{in}}^{-1} e_i < \mu e_i^* \Gamma e_i,\ i = 1, \ldots, m\Big\} \tag{25}$$

where $X > 0$ is the solution to the ARE (18). Furthermore

$$\rho_{\min} = \inf\Bigg\{\mu : \left(\sum_{i=1}^{m} \gamma_i X_i\right) - \frac{1}{\mu}\gamma_i \sigma_i^2 \hat{A}^{\tau_i - 1} B_{\text{in}} D_{\text{in}}^{-1} e_i \cdot e_i^* D_{\text{in}}^{*-1} B_{\text{in}}^* (\hat{A}^*)^{\tau_i - 1} > 0,\ \gamma_i > 0,\ i = 1, \ldots, m\Bigg\} \tag{26}$$

where X_i is the solution to the Lyapunov equation (20). The system in Fig. 1 is mean-square stabilizable if and only if $\rho_{\min} < 1$.

Proof: See Appendix A. ■

It is important to note that the stabilizability conditions in Theorem 3 and Theorem 4 are both a GEVP problem, which can be efficiently solved using LMI optimization techniques [2], [3] with a line search method. Thus, while unable to provide an analytical solution as in the case of SISO systems, these results nonetheless furnish necessary and sufficient conditions for mean-square stabilizability, and from a computational standpoint, resolve the output feedback mean-square stabilizability problem for MIMO systems with no finite nonminimum phase zeros. Inadvertently, this also solves the state feedback mean-square stabilizability problem, as implicated by the fact that the solutions depend only on the matrices $A_{\text{in}}, B_{\text{in}}, D_{\text{in}}$. Indeed, in using state feedback, the controlled plant is effectively rendered minimum phase. It is worth noting that since the eigenvalues of $\hat{A}$ coincide with the plant unstable poles, Theorem 4 shows that the mean-square stabilizability condition becomes proportionally more demanding as the delays in the plant increase. It is also useful to point out that in addition to the locations of the plant unstable poles, the realizations of the inner factor M_{in} also depend on their directions, and as such, so do the stabilizability conditions. In particular, when $B_{\text{in}} D_{\text{in}}^{-1} e_i = 0$, the Lyapunov equation (20) yields the trivial solution $X_i = 0$, and hence the ith inequality in (19) is rendered moot; in other words, σ_i^2 can be arbitrary and the uncertainty Δ_i has no effect on the closed-loop stability.

In what follows we analyze in further depth the dependence of the stabilizability on the pole directions, by resorting to limiting case studies. We shall consider first the case that $P(z)$ has a single unstable pole $p \in \bar{\mathbb{D}}^c$ with output direction vector η. For a given η, we introduce the index set

$$I = \{1 \leq i \leq m :\ \eta^* e_i \neq 0\}.$$

More generally, in the case of multiple poles $p_j \in \bar{\mathbb{D}}^c$ with pole direction vectors η_j, $j = 1, \ldots, n$, we define with respect to η_j the set

$$I_j = \left\{1 \leq i \leq m :\ \eta_j^* e_i \neq 0\right\}$$

and with respect to e_i

$$J_i = \left\{1 \leq j \leq n :\ \eta_j^* e_i \neq 0\right\}.$$

Corollary 1: Suppose that $P(z)$ has no zero in $\mathbb{D}^c$ and that $\tilde{L}$ is given in (24). Suppose also that $P(z)$ has a single unstable pole $p \in \bar{\mathbb{D}}^c$ with output direction vector η. Then under Assumptions 1–3, the system in Fig. 1 is mean-square stabilizable if and only if

$$\sum_{i \in I} \frac{1}{|p|^{2(\tau_i - 1)}\sigma_i^2} > |p|^2 - 1. \tag{27}$$

Proof: It follows from [7] that an all-pass factor M_{in} can be constructed as:

$$M_{\text{in}}(z) = \left[\begin{array}{c|c} \frac{1}{p^*} & \frac{\sqrt{|p|^2 - 1}}{p^*}\eta^* \\ \hline \frac{\sqrt{|p|^2 - 1}}{p^*}\eta & I - \left(1 + \frac{1}{p^*}\right)\eta\eta^* \end{array}\right].$$

With this realization, it can be verified by a direct calculation that the ARE (18) admits the solution

$$X = \frac{1}{\eta^* \Gamma^{-1} \eta}.$$

Thus, by invoking Theorem 4, we obtain

$$\begin{aligned}\rho_{\min} &= \inf_{\Gamma} \max_{i} \left(|p|^2 - 1\right)\sigma_i^2 \frac{|p|^{2(\tau_i - 1)}|\eta^* e_i|^2 \gamma_i^{-2}}{\eta^* \Gamma^{-1} \eta} \\ &= \left(|p|^2 - 1\right) \inf_{\Gamma} \max_{i \in I} \sigma_i^2 \frac{|p|^{2(\tau_i - 1)}|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i \in I} |\eta^* e_i|^2 \gamma_i^{-2}}.\end{aligned}$$

The infimum is found at such γ_i, $i \in I$ that for $i \neq j$, $i, j \in I$

$$\sigma_i^2 \frac{|p|^{2(\tau_i - 1)}|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i \in I} |\eta^* e_i|^2 \gamma_i^{-2}} = \sigma_j^2 \frac{|p|^{2(\tau_j - 1)}|\eta^* e_j|^2 \gamma_j^{-2}}{\sum_{i \in I} |\eta^* e_i|^2 \gamma_i^{-2}}$$

which gives rise to the solution

$$\rho_{\min} = \left(|p|^2 - 1\right) \frac{1}{\sum_{i\in I} \frac{1}{|p|^{2(\tau_i-1)}\sigma_i^2}}.$$

This completes the proof. ■

Corollary 1 gives a glimpse into the relevance of pole directions on mean-square stabilizability, which is seen to depend on the alignment between the pole direction and the Euclidean basis. Evidently, if $\eta^* e = 0$ for some i, then the uncertainty Δ_i has no influence on the stabilizability condition (27), reaffirming the above observation on the scenario $B_{\text{in}} D_{\text{in}}^{-1} e_i = 0$. When in the networked control setting Δ_i is interpreted as the noise in the i-th channel, this means that the pole direction is orthogonal to the channel and the noise in that channel casts no effect.

The following two limiting cases of multiple poles provide additional insight. For simplicity, we restrict our attention to plants with relative degree zero and one.

Corollary 2: Suppose that $P(z)$ has no zero in $\mathbb{D}^c$ and has relative degree $\tau = 0, 1$.

(i) Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \ldots, n$, be the unstable poles of $P(z)$ with parallel directions spanned by a pole direction vector η. Then

$$\rho_{\min} = \frac{1}{\sum_{i\in I} \frac{1}{\sigma_i^2}} \left(\prod_{i=1}^{n} |p_i|^{2(\tau-1)} \right) \left(\prod_{i=1}^{n} |p_i|^2 - 1 \right).$$

(ii) Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \ldots, n$, $n \le m$, be the unstable poles of $P(z)$ with orthogonal directions spanned by η_i. Then

$$\rho_{\min} = \max_i \frac{\sum_{j\in J_i} |p_j|^{2(\tau-1)} \left(|p_j|^2 - 1\right)}{\sum_{i\in I_j} \frac{1}{\sigma_i^2}}. \tag{28}$$

Proof: We prove the corollary for $\tau = 1$; the case $\tau = 0$ follows analogously and hence is omitted. Without loss of generality, for the n parallel pole directions, we may assume that the pole direction vectors $\eta_i = \eta$, for some unitary vector η such that $\eta^* M(p_i) = 0$, $i = 1, \ldots, n$. In this case, an all-pass factor of $\Gamma^{1/2} M(z)$ can be found as [6], [43]

$$M_{\Gamma\text{in}}(z) = [\eta_\Gamma \quad U_\Gamma] \begin{bmatrix} \prod_{i=1}^{n} \frac{z-p_i}{1-p_i^* z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_\Gamma^* \\ U_\Gamma^* \end{bmatrix} \tag{29}$$

where $\eta_\Gamma = \Gamma^{-(1/2)}\eta / \|\Gamma^{-(1/2)}\eta\|$, and $[\eta_\Gamma \; U_\Gamma]$ is a unitary matrix. For a plant with relative degree one, it follows from the proof of Theorem 3 that:

$$\begin{aligned} \rho_{\min} &= \inf_\Gamma \max_i \sigma_i^2 \left\| \left(M_{\Gamma\text{in}}^{-1}(z) - M_{\Gamma\text{in}}^{-1}(\infty)\right) e_i \right\|_2^2 \\ &= \inf_\Gamma \max_i \sigma_i^2 \left|\eta_\Gamma^* e_i\right|^2 \left\| \prod_{i=1}^{n} \frac{1-p_i^* z}{z-p_i} - \prod_{i=1}^{n} (-p_i^*) \right\|_2^2 \\ &= \inf_\Gamma \max_i \left(\prod_{i=1}^{n} |p_i|^2 - 1 \right) \sigma_i^2 \frac{|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i=1}^{n} |\eta^* e_i|^2 \gamma_i^{-2}}. \end{aligned}$$

The rest of the proof for (i) then follows as in that for Corollary 1. To establish Corollary 2-(ii), it suffices to note that with mutually orthogonal pole directions η_i, $i = 1, \ldots, n$, an all-pass factor of $\Gamma^{1/2} M(z)$ can be constructed as [6]

$$M_{\Gamma\text{in}}(z) = \begin{bmatrix} \hat{\eta}^* \\ \vdots \\ \hat{\eta}_n^* \\ \hat{U}^* \end{bmatrix}^* \begin{bmatrix} \frac{z-p_1}{1-p_1^* z} & & & \\ & \ddots & & \\ & & \frac{z-p_n}{1-p_n^* z} & \\ & & & I \end{bmatrix} \begin{bmatrix} \hat{\eta}^* \\ \vdots \\ \hat{\eta}_n^* \\ \hat{U}^* \end{bmatrix}$$

where $\hat{\eta}_i = \Gamma^{-(1/2)}\eta_i / \|\Gamma^{-(1/2)}\eta_i\|$, and $[\hat{\eta}_1 \; \cdots \; \hat{\eta}_n \; \hat{U}]$ is a unitary matrix. Similarly, we obtain

$$\begin{aligned} \rho_{\min} &= \inf_\Gamma \max_i \sigma_i^2 \left\| \left(M_{\Gamma\text{in}}^{-1}(z) - M_{\Gamma\text{in}}^{-1}(\infty)\right) e_i \right\|_2^2 \\ &= \inf_\Gamma \max_i \sigma_i^2 \sum_{j=1}^{n} \left|\hat{\eta}_j^* e_i\right|^2 \left\| \frac{1-p_j^* z}{z-p_j} + p_j^* \right\|_2^2 \\ &= \inf_\Gamma \max_i \sigma_i^2 \sum_{j\in J_i} \left(|p_j|^2 - 1\right) \frac{\left|\eta_j^* e_i\right|^2 \gamma_i^{-2}}{\sum_{i\in I_j} \left|\eta_j^* e_i\right|^2 \gamma_i^{-2}}. \end{aligned} \tag{30}$$

Define

$$x_i = \sigma_i^2 \sum_{j\in J_i} \left(|p_j|^2 - 1\right) \frac{\left|\eta_j^* e_i\right|^2 \gamma_i^{-2}}{\sum_{i\in I_j} \left|\eta_j^* e_i\right|^2 \gamma_i^{-2}}.$$

It follows that the minimax problem in (30) achieves the minimum at $x_i = x_k$ for $i \ne k$, $i, k \in I_j$, subject to the constraint:

$$\sum_{i\in I_j} \frac{x_i}{\sigma_i^2} = \sum_{j\in J_i} \left(|p_j|^2 - 1\right).$$

This leads to the solution (28), thus completing the proof. ■

Hence, when in the extreme their directions are parallel, the unstable poles contribute to the difficulty to stabilization *collectively* as if in a SISO system. In contrast, when the directions are orthogonal, the poles tend to affect the stabilizability *individually* in an additive manner.

C. Nonminimum Phase Zeros

Unlike for minimum phase plants, stabilization of nonminimum phase plants via output feedback proves fundamentally more difficult. In this section we provide a case study which helps illustrate this difficulty. Our first result is a characterization of mean-square stabilizability for plants containing one nonminimum phase zero.

Theorem 5: Suppose that $P(z)$ has relative degree $\tau = 0, 1$. Suppose also that $P(z)$ has one unstable pole $p \in \bar{\mathbb{D}}^c$ with output direction vector η, and one nonminimum phase zero $s \in \bar{\mathbb{D}}^c$ with input direction vector ζ. Then

$$\rho_{\min} = |p|^{2(\tau-1)} \left(|p|^2 - 1\right) \times \inf_\Gamma \left(\frac{|\eta^*\zeta|^2 \left(\left|\frac{1-p^* s}{p-s}\right|^2 - 1 \right)}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2 \left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2} + \frac{\left\|\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2} \right). \tag{31}$$

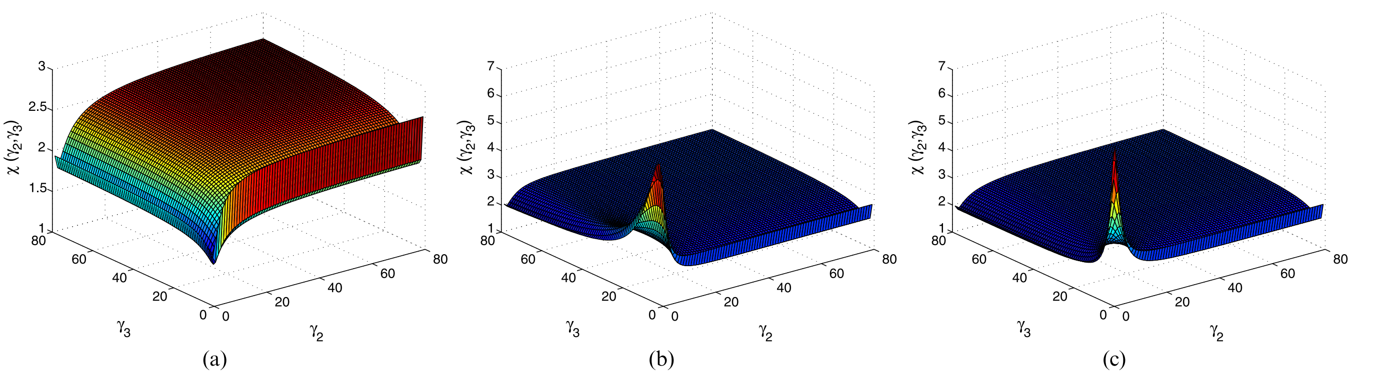


Fig. 6. The minimum $\chi(\gamma_2,\gamma_3) = \inf_{R\in\mathbb{RH}_\infty} \rho(W(R))$. (a) $\angle(\eta,\zeta) = \pi/2$. (b) $\angle(\eta,\zeta = \pi/4$. (c) $\angle(\eta,\zeta) = 0$.

Furthermore

$$\frac{1}{\sum_{i\in I}\frac{1}{\sigma_i^2}}|p|^{2(\tau-1)}\left(|p|^2-1\right) \le \rho_{\min}$$

$$\le \left|\frac{1-p^*s}{p-s}\right|^2 \frac{1}{\sum_{i\in I}\frac{1}{\sigma_i^2}}|p|^{2(\tau-1)}\left(|p|^2-1\right) \quad (32)$$

and the lower bound is achieved when $\angle(\eta,\zeta) = \pi/2$.

Proof: See Appendix B. ■

It is clear from Theorem 5 that the presence of nonminimum phase zeros will generally worsen a plant's mean-square stabilizability. The nonminimum phase effect, while invariably present in SISO systems, will however vanish when the zero direction is perpendicular to the pole direction. Such is also the case, for example, when the pole direction vector is an Euclidean basis vector, regardless of the zero direction vector. This characteristic sheds light into the intricacy in how nonminimum phase zeros may affect stabilizability, a complication one does not observe in the case of state feedback. The example given in the next subsection reinforces further this observation.

D. A MIMO Example

The following example demonstrates that in the presence of nonminimum phase zeros, the mean-square stabilizability problem is in general not a convex optimization problem, thus underlying the difficulty to find a necessary and sufficient stabilizability condition.

Example 2: Consider a plant with relative degree one, given as

$$P(z) = \begin{bmatrix} \frac{50+a}{z} & 0 & -\frac{26+25a}{z} \\ \frac{1}{z} & -\frac{1}{z} & 0 \\ 0 & 0 & \frac{z-3.5}{z^2} \end{bmatrix}$$

$$\times \left(\mathbf{1}_{3\times 3} \otimes \frac{1}{3}\frac{1-3/2z}{z-3/2} + I - \mathbf{1}_{3\times 3} \otimes \frac{1}{3}\right) \quad (33)$$

where $\mathbf{1}_{3\times 3}$ is a 3×3 matrix with all elements equal to one. This plant has an unstable pole $p = 1.5$ with output direction vector $\eta = (1/\sqrt{3})[1,1,1]^*$, and a nonminimum phase zero $s = 3.5$ with input direction vector $\zeta = (1/\sqrt{2+a^2})[1,1,a]^*$.

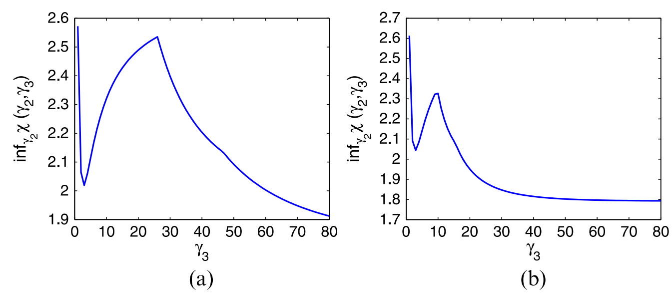


Fig. 7. $\inf_{\gamma_2} \chi(\gamma_2,\gamma_3)$. (a) $\angle(\eta,\zeta) = \pi/4$. (b) $\angle(\eta,\zeta) = 0$.

As such, $\angle(\eta,\zeta) = (a+2)/\sqrt{3a^2+6}$. Let $\sigma_1^2 = 9/4$, $\sigma_2^2 = 4$, $\sigma_3^2 = 25/4$. With no loss of generality, take $\gamma_1 = 1$ and define

$$\chi(\gamma_2,\gamma_3) = |p|^{2(\tau-1)}\left(|p|^2-1\right) \times \left(\frac{|\eta^*\zeta|^2\left(\left|\frac{1-p^*s}{p-s}\right|^2-1\right)}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2\left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2} + \frac{\left\|\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2}\right).$$

Fig. 6 shows how $\chi(\gamma_2,\gamma_3)$ varies with γ_2,γ_3 for the different pole and zero direction alignments corresponding to $\angle(\eta,\zeta) = \pi/2,\ \pi/4,\ 0$. For the case $\angle(\eta,\zeta) = \pi/2$, it is shown in the proof of Theorem 5 that the minimization of $\chi(\gamma_2,\gamma_3)$ can be converted, via a one-to-one transformation, into one of convex minimization. It is thus unsurprising that $\chi(\gamma_2,\gamma_3)$ has a unique minimum. On the other hand, for $\angle(\eta,\zeta) = \pi/4,\ 0$, $\chi(\gamma_2,\gamma_3)$ has more than one local minimum, which is seen even more clearly in Fig. 7, by focusing on the dependence of $\chi(\gamma_2,\gamma_3)$ on γ_3. This partly explains the difficulty to compute $\rho_{\min}$. Note also that as a function of $\angle(\eta,\zeta)$, the example shows that $\rho_{\min}$ is not a monotone function, contrary to one's intuition.

V. Conclusion

In Part 1 of this paper we have studied the stabilizability of LTI systems in the presence of stochastic multiplicative uncertainties. We derived fundamental conditions for mean-square stabilizability, which provide necessary and sufficient conditions for a system to be stabilizable via output feedback. For SISO systems, an explicit, analytical bound is derived on the uncertainty variance. This bound yields the fundamental limit required for stabilization, imposed by the plant's unstable poles, nonminimum phase zeros, as well as time delays. For MIMO systems, we derived a computationally efficient necessary and sufficient condition for minimum phase systems with possible delays, which coincidentally furnishes a solution to state-feedback mean-square stabilizability problem as well. This condition amounts to solving a GEVP problem which is readily solvable using LMI optimization techniques. Further investigation into limiting cases shows that in a MIMO system, the stabilizability condition is sensitive to the directions of the unstable poles and nonminimum phase zeros and may vary widely depending on the alignment between these directions.

In Part 2 we aim to develop synthesis methods for optimal control subject to stochastic multiplicative uncertainties, wherein the stabilization and optimal performance problems are formulated and tackled in a general framework unified under the mean-square small gain theorem. With a design-oriented perspective, computationally efficient solutions are particularly sought after and indeed obtained by making use of state-space descriptions. Hence, together with the stabilizability conditions given herein, this two-part paper provides a rather full exposition of optimal control under structured multiplicative uncertainties. When interpreted in the networked control setting, our results in Part 2 solve a number of optimal performance problems with channels operating under random multiplicative noises. This should be especially welcomed, for performance issues of networked control systems have seldom been investigated.

Appendix A
Proof of Theorem 4

We begin with (16) and subsequently (21), whereas the latter gives rise to

$$\begin{aligned}\left\|\Gamma^{\frac{1}{2}}(Y-MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} &= \left\|z\left(M_{\Gamma\text{in}}^{-1}(z)-M_{\Gamma\text{in}}^{-1}(\infty)\right)e_i\right\|_2^2+\Big\|\Big(\gamma_i M_{\Gamma\text{in}}^{-1}(\infty)\\ &\quad - M_{\Gamma\text{out}}M_{\text{out}}\tilde{X} + M_{\Gamma\text{out}}M_{\text{out}}R\tilde{L}\Big)e_i\Big\|_2^2\gamma_i^{-2}.\end{aligned}$$

Following analogously the proof of Theorem 3, we find that:

$$\begin{aligned}&\left\|z\left(M_{\Gamma\text{in}}^{-1}(z)-M_{\Gamma\text{in}}^{-1}(\infty)\right)e_i\right\|_2^2\\ &\qquad = e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^* X B_{\text{in}} D_{\text{in}}^{-1} e_i \gamma_i^{-2}\end{aligned}$$

where X is the solution to the ARE (18). Note also that

$$\gamma_i M_{\Gamma\text{in}}^{-1}(\infty)e_i = \left(D_{\text{in}}^*\Gamma D_{\text{in}} + B_{\text{in}}^* X B_{\text{in}}\right)^{\frac{1}{2}} D_{\text{in}}^{-1}e_i.$$

Let the impulse response sequence of $M_{\Gamma\text{out}}M_{\text{out}}\tilde{X}e_i$ be denoted as $\{f_k\}$; that is

$$M_{\Gamma\text{out}}(z)M_{\text{out}}(z)\tilde{X}(z)e_i = \sum_{k=0}^{\infty} f_k z^{-k}.$$

Since $\tilde{L}e_i$ has relative degree τ_i, the impulse response sequence of $M_{\Gamma\text{out}}M_{\text{out}}R\tilde{L}e_i$ is equal to zero for $k = 0, 1, \ldots, \tau_i - 1$. Furthermore, since $\gamma_i M_{\Gamma\text{in}}^{-1}(\infty) - M_{\Gamma\text{out}}(\infty)M_{\text{out}}(\infty)\tilde{X}(\infty) = 0$, we have

$$\begin{aligned}&\left\|\left(\gamma_i M_{\text{in}}^{-1}(\infty) - M_{\Gamma\text{out}}M_{\text{out}}\tilde{X} + M_{\Gamma\text{out}}M_{\text{out}}R\tilde{L}\right)e_i\right\|_2^2\\ &\qquad = \sum_{k=1}^{\tau_i-1}\|f_k\|^2 + \left\|\sum_{k=\tau_i}^{\infty} f_k z^{-k} - M_{\Gamma\text{out}}M_{\text{out}}R\tilde{L}e_i\right\|_2^2.\end{aligned}$$

Evidently

$$\inf_{R\in\mathbb{RH}_\infty}\left\|\sum_{k=\tau_i}^{\infty} f_k z^{-k} - M_{\Gamma\text{out}}M_{\text{out}}R\tilde{L}e_i\right\|_2^2 = 0.$$

As such

$$\begin{aligned}&\inf_{R\in\mathbb{RH}_\infty}\left\|\Gamma^{\frac{1}{2}}(Y-MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2}\\ &\qquad = \sum_{k=1}^{\tau_i-1}\|f_k\|^2\gamma_i^{-2} + e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^* X B_{\text{in}} D_{\text{in}}^{-1}e_i\gamma_i^{-2}. \quad (34)\end{aligned}$$

We next seek to determine the impulse response sequence $\{f_k\}$ for $k = 1, \ldots, \tau_i - 1$. For this purpose, denote the impulse response sequences of $M_{\Gamma\text{in}}(z)$ and $M_{\Gamma\text{in}}^{-1}(z)e_i$ by $\{g_k\}$ and $\{h_k\}$, respectively. From the Bezout identity $M_{\Gamma\text{in}}M_{\Gamma\text{out}}M_{\text{out}}\tilde{X}e_i - \Gamma^{1/2}Y\tilde{L}e_i = \gamma_i e_i$, and the fact that $\tilde{L}e_i$ has relative degree τ_i, it follows at once that:

$$\begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ \vdots & \vdots & & \\ g_{\tau_i-1} & g_{\tau_i-2} & \cdots & g_0\end{bmatrix}\begin{bmatrix} f_0\\ f_1\\ \vdots\\ f_{\tau_i-1}\end{bmatrix} = \begin{bmatrix}\gamma_i e_i\\ 0\\ \vdots\\ 0\end{bmatrix}.$$

On the other hand, since $M_{\Gamma\text{in}}M_{\Gamma\text{in}}^{-1}e_i = e_i$, we have $f_k = \gamma_i h_k,\ k = 0, 1, \ldots, \tau_i - 1$, where $\{h_k\}$, as the impulse response sequence of $M_{\Gamma\text{in}}^{-1}(z)e_i$, is found to be

$$\begin{aligned}h_k &= -D_{\Gamma\text{in}}^{-1}C_{\Gamma\text{in}}\hat{A}^{k-1}B_{\Gamma\text{in}}D_{\Gamma\text{in}}^{-1}e_i\\ &= -D_{\Gamma\text{in}}^{-1}C_{\Gamma\text{in}}\hat{A}^{k-1}B_{\text{in}}D_{\text{in}}^{-1}e_i\gamma_i^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^{\tau_i-1}\|f_k\|^2\gamma_i^{-2} &= \gamma_i^{-2}\sum_{k=1}^{\tau_i-1} e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^*\hat{A}_{\text{in}}^{*k-1}C_{\Gamma\text{in}}^* D_{\Gamma\text{in}}^{*-1}\\ &\qquad \cdot D_{\Gamma\text{in}}^{-1}C_{\Gamma\text{in}}\hat{A}_{\text{in}}^{k-1}B_{\text{in}}D_{\text{in}}^{-1}e_i. \quad (35)\end{aligned}$$

In view of the Lyapunov equation (22), we then obtain

$$\begin{aligned}\sum_{k=1}^{\tau_i-1}\|f_k\|^2 &= \sum_{k=1}^{\tau_i-1} e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^*\hat{A}_{\text{in}}^{*k-1}(\hat{A}^* X\hat{A} - X)\\ &\qquad \cdot \hat{A}_{\text{in}}^{k-1}B_{\text{in}}D_{\text{in}}^{-1}e_i\\ &= \sum_{k=1}^{\tau_i-1} e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^*\hat{A}_{\text{in}}^{*k} X\hat{A}_{\text{in}}^{k}B_{\text{in}}D_{\text{in}}^{-1}e_i\\ &\qquad - \sum_{k=1}^{\tau_i-1} e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^*\hat{A}_{\text{in}}^{*k-1} X\hat{A}_{\text{in}}^{k-1}B_{\text{in}}D_{\text{in}}^{-1}e_i\\ &= e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^*\hat{A}_{\text{in}}^{*\tau_i-1} X\hat{A}_{\text{in}}^{\tau_i-1}B_{\text{in}}D_{\text{in}}^{-1}e_i\\ &\qquad - e_i^* D_{\text{in}}^{*-1}B_{\text{in}}^* X B_{\text{in}}D_{\text{in}}^{-1}e_i.\end{aligned}$$

The proof is completed by combining the last equality with (34). ■

APPENDIX B
PROOF OF THEOREM 5

We examine the characterization

$$\rho(W(R)) = \inf_{\Gamma} \left\|\Gamma W(R)\Gamma^{-1}\right\|_\infty = \inf_{\Gamma} \max_{j} \left\|e_j^* \Gamma^{\frac{1}{2}}(Y - MR_2)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2$$

and prove the case $\tau = 1$. The theorem can be established analogously for $\tau = 0$. Consider the simplicial set

$$\mathbf{\Lambda} = \left\{\mathrm{diag}(\lambda_1, \ldots, \lambda_m) : \sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0,\ i = 1, \ldots, m\right\}.$$

It is easy to see that

$$\inf_{\Gamma} \left\|\Gamma W(R)\Gamma^{-1}\right\|_\infty = \inf_{\Gamma} \max_{\Lambda \in \mathbf{\Lambda}} \sum_{j=1}^{m} \lambda_j \left\|e_j^* \Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2.$$

Accordingly

$$\begin{aligned}\inf_{R\in\mathbb{RH}_\infty} \rho\left(W(R)\right) &= \inf_{R\in\mathbb{RH}_\infty} \inf_{\Gamma} \max_{\Lambda\in\mathbf{\Lambda}} \sum_{j=1}^{m} \lambda_j \left\|e_j^* \Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2 \\ &= \inf_{\Gamma} \inf_{R\in\mathbb{RH}_\infty} \max_{\Lambda\in\mathbf{\Lambda}} \sum_{j=1}^{m} \lambda_j \left\|e_j^* \Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2.\end{aligned}$$

Since $\sum_{j=1}^{m} \lambda_j \|e_j^* \Gamma^{1/2}(Y - MR)\tilde{L}\Gamma^{-(1/2)}\Sigma^{1/2}\|_2^2$ is convex in R and affine in Λ, we may exchange the infimum and the maximization [36], [37], so as to obtain

$$\begin{aligned}\rho_{\min} &= \inf_{\Gamma} \max_{\Lambda\in\mathbf{\Lambda}} \inf_{R\in\mathbb{RH}_\infty} \sum_{j=1}^{m} \lambda_j \left\|e_j^* \Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2 \\ &= \inf_{\Gamma} \max_{\Lambda\in\mathbf{\Lambda}} \inf_{R\in\mathbb{RH}_\infty} \left\|\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2.\end{aligned}$$

We then proceed to solve the minimization problem over $R \in \mathbb{RH}_\infty$. Toward this end, we conduct all-pass factorizations $\Lambda^{1/2}\Gamma^{1/2}M = \hat{M}_{\mathrm{in}}\hat{M}_{\mathrm{out}}$ and $\tilde{L}\Gamma^{-(1/2)}\Sigma^{1/2} = \hat{L}_{\mathrm{out}}\hat{L}_{\mathrm{in}}$, respectively, so that

$$\hat{M}_{\mathrm{in}}(z) = [\eta_\Lambda \quad U_\Lambda] \begin{bmatrix} \frac{z-p}{1-p^*z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_\Lambda^* \\ U_\Lambda^* \end{bmatrix}$$

$$\hat{L}_{\mathrm{in}}(z) = [\zeta_\Sigma \quad V_\Sigma] \begin{bmatrix} \frac{z-s}{1-s^*z} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \zeta_\Sigma^* \\ V_\Sigma^* \end{bmatrix}$$

where

$$\eta_\Lambda = \frac{\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|}, \qquad \zeta_\Sigma = \frac{\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta}{\left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|}$$

and $[\eta_\Lambda\ U_\Lambda]$, $[\zeta_\Sigma\ V_\Sigma]$ are both unitary matrices. It follows that:

$$\begin{aligned}&\left\|\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2 \\ &= \left\|\left(\hat{M}_{\mathrm{in}}^{-1}(z) - \hat{M}_{\mathrm{in}}^{-1}(\infty)\right)\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p)\right\|_2^2 \\ &\quad + \left\|Q - \hat{M}_{\mathrm{out}}R\hat{L}_{\mathrm{out}}\right\|_2^2\end{aligned}$$

form some $Q \in \mathbb{RH}_\infty$. Hence

$$\begin{aligned}&\inf_{R\in\mathbb{RH}_\infty} \left\|\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\right\|_2^2 \\ &= \left\|\left(\hat{M}_{\mathrm{in}}^{-1}(z) - \hat{M}_{\mathrm{in}}^{-1}(\infty)\right)\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p)\right\|_2^2 \\ &= \left\|\frac{1-|p|^2}{z-p}\eta_\Lambda\eta_\Lambda^*\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p)\right\|_2^2 \\ &= \left(|p|^2 - 1\right)\left\|\eta_\Lambda^*\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p)\right\|_2^2.\end{aligned}$$

From the Bezout identity (4), however, we have

$$\eta_\Lambda^*\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p) = -\frac{\eta^*}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|}\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\hat{L}_{\mathrm{in}}^{-1}(p).$$

This leads to

$$\begin{aligned}&\left\|\eta_\Lambda^*\Lambda^{\frac{1}{2}}\Gamma^{\frac{1}{2}}Y(p)\hat{L}_{\mathrm{out}}(p)\right\|_2^2 \\ &= \left\|\frac{\eta^*\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|}\left(\frac{1-s^*p}{p-s}\zeta_\Sigma + V_\Sigma\right)\right\|_2^2 \\ &= \frac{\eta^*\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|}\left(\left(\left|\frac{1-s^*p}{p-s}\right|^2 - 1\right)\zeta_\Sigma\zeta_\Sigma^* + I\right) \\ &\quad \times \frac{\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|} \\ &= \left(\frac{|\eta^*\zeta|^2\left(\left|\frac{1-s^*p}{p-s}\right|^2 - 1\right)}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2\left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2} + \frac{\left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}\right)\end{aligned}$$

and consequently to

$$\rho_{\min} = \inf_{\Gamma} \max_{\Lambda\in\mathbf{\Lambda}} \left(\frac{|\eta^*\zeta|^2\left(\left|\frac{1-s^*p}{p-s}\right|^2 - 1\right)}{\left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2} + \left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2\right) \cdot \frac{|p|^2 - 1}{\left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}. \quad (36)$$

Next, we seek to find $\max_{\Lambda\in\mathbf{\Lambda}}(1/\|\Lambda^{-(1/2)}\Gamma^{-(1/2)}\eta\|^2)$, or equivalently, $1/(\min_{\Lambda\in\mathbf{\Lambda}}\|\Lambda^{-(1/2)}\Gamma^{-(1/2)}\eta\|^2)$. For this purpose, consider the function

$$\begin{aligned}f(\Lambda, \kappa) &= \left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2 + \kappa(\lambda_1 + \cdots + \lambda_m - 1) \\ &= \sum_{i=1}^{m} \frac{|\eta^* e_i|^2}{\lambda_i\gamma_i^2} + \kappa(\lambda_1 + \cdots + \lambda_m - 1).\end{aligned}$$

Setting the partial derivatives of $f(\Lambda, \kappa)$ to zero, we obtain

$$-\left(\frac{|\eta^* e_i|^2}{\gamma_i^2}\right)\frac{1}{\lambda_i^2} + \kappa = 0.$$

Together with the equation $\sum_{i=1}^{m} \lambda_i = 1$, this yields the solution

$$\lambda_i = \frac{\left(\frac{|\eta^* e_i|}{\gamma_i}\right)}{\sum_{i=1}^{m}\left(\frac{|\eta^* e_i|}{\gamma_i}\right)}$$

$$\min_{\Lambda\in\mathbf{\Lambda}} \left\|\Lambda^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2 = \left(\sum_{i=1}^{m} \frac{|\eta^* e_i|}{\gamma_i}\right)^2 = \left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2$$

Thus, (31) is established. To prove the inequalities in (32), we claim that

$$\inf_{\Gamma} \frac{\left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2} = \frac{1}{\sum_{i\in I}\frac{1}{\sigma_i^2}}.$$

Toward this end, let us introduce the variables

$$x_i = \frac{\left(\frac{|\eta^* e_i|}{\gamma_i}\right)}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1}, \qquad i = 1, \ldots, m.$$

It is clear that $x_i = 0$ for $i \notin I$, and that $\sum_{i\in I} x_i = 1$. Furthermore

$$\frac{\left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2} = \sum_{i\in I}\sigma_i^2 x_i^2.$$

Solving this constrained minimization problem, we are led to

$$\inf_{\Gamma} \frac{\left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2} = \min\left\{\sum_{i\in I}\sigma_i^2 x_i^2 : \sum_{i\in I} x_i = 1, x_i \geq 0,\ i\in I\right\} = \frac{1}{\sum_{i\in I}\frac{1}{\sigma_i^2}}.$$

The lower bound in (32) then follows trivially, which is achieved when $\cos\angle(\eta,\zeta) = 0$. To establish the upper bound, we note that

$$\begin{aligned}\left\|\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right\|^2 \left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2 &= \left\|\left(\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right)\left(\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right)^*\right\| \\ &\geq \rho^2\left(\left(\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right)\left(\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\eta\right)^*\right) \\ &= |\eta^*\zeta|^2.\end{aligned}$$

This means that

$$\frac{|\eta^*\zeta|^2\left(\left|\frac{1-p^*s}{p-s}\right|^2 - 1\right)}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2\left\|\Sigma^{-\frac{1}{2}}\Gamma^{\frac{1}{2}}\zeta\right\|^2} + \frac{\left\|\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2} \leq \left|\frac{1-p^*s}{p-s}\right|^2 \frac{\left\|\Gamma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\eta\right\|^2}{\left\|\Gamma^{-\frac{1}{2}}\eta\right\|_1^2}.$$

The proof is now completed. ∎

REFERENCES

[1] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York, NY, USA: Academic, 1979.

[2] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.

[3] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[4] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1391–1403, 2007.

[5] R. Brockett and D. Liberzon, "Quantized feedback stabilizalion of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1279–1289, 2000.

[6] J. Chen, "Sensitivity integral relations and design tradeoffs in linear multivariable feedback systems," *IEEE Trans. Autom. Control*, vol. 40, no. 10, pp. 1700–1716, 1995.

[7] J. Chen, "Integral constraints and performance limits on complementary sensitivity," *Syst. Control Lett.*, vol. 39, no. 1, pp. 45–53, 2000.

[8] J. Chen, "Logarithmic integrals, interpolation bounds, and performance limitations in mimo systems," *IEEE Trans. Autom. Control*, vol. 45, no. 6, pp. 1098–1115, 2000.

[9] J. Chen, L. Qiu, and O. Toker, "Limitations on maximal tracking accuracy," *IEEE Trans. Autom. Control*, vol. 45, no. 2, pp. 326–331, 2000.

[10] N. Elia, "Remote stabilization over fading channels," *Syst. Control Lett.*, vol. 54, no. 3, pp. 237–249, 2005.

[11] N. Elia and S. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[12] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[13] D. Hinrichsen and A. Pritchard, "Stability radii of systems with stochastic uncertainty and their optimization by output feedback," *SIAM J. Control Optimiz.*, vol. 36, no. 6, pp. 1972–1998, 1996.

[14] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[15] "Special issue on networked control system," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1421–1603, 2004.

[16] H. Ishii and B. Francis, "Quadratic stabilization of sampled-data systems with quantization," *Automatica*, vol. 39, no. 10, pp. 1793–1800, 2003.

[17] M. Khammash, M. Salapaka, and T. V. Voorhis, "Robust synthesis in ℓ_1: A globally optimal solution," *IEEE Trans. Autom. Control*, vol. 46, no. 11, pp. 1744–1754, 2001.

[18] P. P. Khargonekar and A. Tannenbaum, "Noneuclidean metrics and the robust stabilization of systems with parameter uncertainty," *IEEE Trans. Autom. Control*, vol. 30, no. 10, pp. 1005–1013, 1985.

[19] Y. Li, E. Tuncel, and J. Chen, "Optimal tracking over an additive white gaussian noise channel," in *Proc. 2009 Amer. Control Conf.*, Saint Louis, MO, USA, Jun. 2009, pp. 4026–4031.

[20] Y. Li, E. Tuncel, J. Chen, and W. Su, "Optimal tracking performance of discrete-time systems over an additive white noise channel," in *Proc. Joint 48th IEEE Conf. Decision Control, 2009 Chinese Control Conf.*, Shanghai, China, Dec. 2009, pp. 2070–2075.

[21] Z. Lin, B. M. Chen, A. Saberi, and Y. A. Shamash, "Inner-outer factorization of discrete-time transfer function matrices," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 11, pp. 941–945, 1996.

[22] J. Lu and R. E. Skelton, "Mean-square small gain theorem for stochastic control: Discrete-time case," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 490–494, 2002.

[23] G. N. Nair and R. J. Evans, "Stabilization with data rate limited feedback: Tightest attainable bounds," *Syst. Control Lett.*, vol. 41, no. 1, pp. 49–56, 2000.

[24] G. N. Nair, F. Fagnani, S. Sampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[25] "Special issue on technology of networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 1–312, 2007.

[26] T. Qi, W. Su, and J. Chen, "Optimal tracking design and performance analysis for LTI systems with quantization effects," in *Proc. Joint 48th IEEE Conf. Decision Control, 2009 Chinese Control Conf.*, Shanghai, China, Dec. 2009, pp. 4945–4950.

[27] L. Qiu and J. Chen, "Time domain performance limitations of feedback control," in *Proc. 13th Int. Symp. Mathem. Theory Netw. Syst.*, Padova, Italy, Jul. 1998, pp. 369–372.

[28] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 554–568, 2013.

[29] D. E. Quevedo, E. I. Silva, and G. C. Goodwin, "Control over unreliable networks affected by packet erasures and variable transmission delays," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 4, pp. 672–685, 2008.

[30] A. Rojas, J. H. Braslavsky, and R. H. Middleton, "Control over a bandwidth limited signal to noise ratio constrained communication channel," in *Proc. Joint 44th IEEE Conf. Decision Control, 2005 Eur. Control Conf.*, Seville, Spain, Dec. 2005, pp. 197–202.

[31] A. Rojas, J. H. Braslavsky, and R. H. Middleton, "Output feedback stabilisation over bandwidth limited, signal to noise ratio constrained communication channels," in *Proc. 2006 Amer. Control Conf.*, Minneapolis, MN, USA, Jun. 2006, pp. 2789–2794.

[32] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proc. IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[33] E. I. Silva, G. C. Goodwin, and D. E. Quevedo, "Control system design subject to snr constraints," *Automatica*, vol. 46, no. 12, pp. 428–436, 2009.

[34] E. I. Silva, D. E. Quevedo, and G. C. Goodwin, "Optimal coding for bit-rate limited networked control systems in the presence of data loss," in *Proc. 46th IEEE Conf. Decision Control*, New Orleans, LA, USA, Dec. 2007, pp. 665–670.

[35] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, I. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[36] M. Sion, "On general minimax theorems," *Pacific J. Mathem.*, vol. 8, no. 1, pp. 171–176, 1958.

[37] J. Stoer, "Duality in nonlinear programming and the minimax theorem," *Numerische Mathematik*, vol. 5, no. 1, pp. 371–379, 1963.

[38] J. Stoer and C. Witzgall, "Transformations by diagonal matrices in a normed spaces," *Numerische Mathematik*, vol. 4, no. 1, pp. 158–171, 1962.

[39] W. Su and J. Chen, "The mean-square stabilization of networked feedback systems with channel delays," in *Proc. 2013 Chinese Control Conf.*, Xi'an, China, Jul. 2013, pp. 6756–6761.

[40] W. Su, T. Qi, J. Chen, and M. Fu, "Optimal tracking design of a MIMO linear system with quantization effects," in *Proc. 2011 IFAC World Congr.*, Milano, Italy, Aug. 2011, pp. 3268–3273.

[41] S. Tatikonda and S. M. Mitter, "Control over noisy channels," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1196–1201, 2002.

[42] S. Tatikonda and S. M. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[43] O. Toker, J. Chen, and L. Qiu, "Tracking performance limitations in LTI multivariable discrete-time systems," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 49, no. 5, pp. 657–670, 2002.

[44] F. Vargas, J. Chen, and E. Silva, "Necessary and sufficient conditions for mean square stabilization over mimo SNR-constrained channels," in *Proc. 51st IEEE Conf. Decision Control*, Maui, Hawaii, USA, Dec. 2012, pp. 26–31.

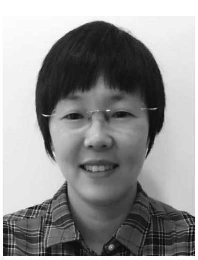
[45] J. Willems and G. Blankenship, "Frequency domain stability criteria for stochastic systems," *IEEE Trans. Autom. Control*, vol. 16, no. 4, pp. 292–299, 1971.

[46] N. Xiao, L. Xie, and L. Qiu, "Mean square stabilization of multi-input systems over stochastic multiplicative channels," in *Proc. Joint 48th IEEE Conf. Decision Control, 2009 Chinese Control Conf.*, Shanghai, China, Dec. 2009, pp. 6893–6898.

[47] K. You and L. Xie, "Minimum data rate for mean square stabilization of discrete LTI systems over lossy channels," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2373–2378, 2010.

[48] K. You and L. Xie, "Minimum data rate for mean square stabilizability of linear systems with markovian packet losses," *IEEE Trans. Autom. Control*, vol. 56, no. 4, pp. 772–785, 2011.

[49] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey, NJ, USA: Prentice-Hall, 1986.



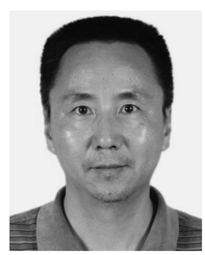
Tian Qi (S'08–M'09) was born in Chaoyang, Liaoning Province, China in 1981. She received the B.S. degree in automation from Hangzhou Dianzi University, Hangzhou, China in 2003, the M.S. degree and the Ph.D. degree in control theory and control engineering, from South China University of Technology, Guangzhou, China in 2006 and 2009, respectively.

Dr. Tian Qi joined the South China University of Technology in 2009, where she is currently a Lecturer. From May 2011 to August 2011, she was with the Department of Electronic Engineering, City University of Hong Kong, as a postdoctoral researcher. From January 2012 to December 2012, she was a Research Associate in the Department of Electrical and Computer Engineering, Hong Kong University of Science and Technology. Her main research interests are in the areas of networked control, multi-agent systems, time-delay systems, robust control and optimization.

Jie Chen (S'87–M'89–SM'98–F'07) is a Chair Professor in the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China. He received the B.S. degree in aerospace engineering from Northwestern Polytechnic University, Xian, China in 1982, the M.S.E. degree in electrical engineering, the M.A. degree in mathematics, and the Ph.D. degree in electrical engineering, all from The University of Michigan, Ann Arbor, Michigan, in 1985, 1987, and 1990, respectively.

Prior to joining City University, he was with School of Aerospace Engineering and School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, Georgia from 1990 to 1993, and with University of California, Riverside, California, from 1994 to 2014, where he was a Professor and served as Professor and Chair for the Department of Electrical Engineering from 2001 to 2006. His main research interests are in the areas of linear multivariable systems theory, system identification, robust control, optimization, time-delay systems, networked control, and multi-agent systems.

An elected Fellow of IEEE, Fellow of AAAS, Fellow of IFAC and a Yangtze Scholar/Chair Professor of China, Dr. Chen was a recipient of 1996 US National Science Foundation CAREER Award, 2004 SICE International Award, and 2006 Natural Science Foundation of China Outstanding Overseas Young Scholar Award. He served on a number of journal editorial boards, as an Associate Editor and a Guest Editor for the IEEE Transactions on Automatic Control, a Guest Editor for IEEE Control Systems Magazine, and an Associate Editor for Automatica, He was also the founding Editor-in-Chief for Journal of Control Science and Engineering. He is currently an IEEE Control Systems Society Distinguished Lecturer and serves as an Associate Editor for SIAM Journal on Control and Optimization.



Weizhou Su (S'96–M'00–SM'05) received the B.Eng. and M.Eng. degrees in automatic control engineering from the Southeast University, Nanjing, Jiangsu, China, in 1983 and 1986, respectively, the M.Eng. degree in electrical and electronic engineering from Nanyang Technological University, in 1996, and the Ph.D. degree in electrical engineering from the University of Newcastle, Newcastle, NSW, Australia, in 2000.

From 2000 to 2004, he held research positions in the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Hong Kong, China; the School of QMMS, University of Western Sydney, Sydney, Australia, respectively. He joined the School of Automation Science and Engineering, South China University of Technology, Guangzhou, China, in 2004 where he is currently a full professor. His research interests include networked control system, robust control, fundamental performance limitation of feedback control, and signal processing.

Minyue Fu (F'03) received the B.Sc. degree in electrical engineering from the University of Science and Technology of China, Hefei, China, in 1982, and the M.S. and Ph.D. degrees in electrical engineering from the University of Wisconsin-Madison, Madison, WI, USA. in 1983 and 1987, respectively.

From 1987 to 1989, he was an Assistant Professor in the Department of Electrical and Computer Engineering, Wayne State University, USA. He joined the Department of Electrical and Computer Engineering at the University of Newcastle, Australia, in 1989. Currently, he is a Chair Professor of Electrical Engineering. He has been a Visiting Associate Professor at the University of Iowa, USA, Nanyang Technological University, Singapore and Tokyo University, Tokyo, Japan. He has held a ChangJiang Visiting Professorship at Shandong University, Jinan, China, a Qian-ren Professorship at Zhejiang University, China, and a Qian-ren Professorship at Guangdong University of Technology, China. He has been an Associate Editor for the IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Signal Processing, and the Journal of Optimization and Engineering. His main research interests include control systems, signal processing, and communications. His current research projects include networked control systems, smart electricity networks, and super-precision positioning control systems.