Control Under Stochastic Multiplicative Uncertainties: Part II, Optimal Design for Performance

Weizhou Su, *Senior Member, IEEE*, Jie Chen, *Fellow, IEEE*, Minyue Fu, *Fellow, IEEE*, and Tian Qi, *Member, IEEE*

***Abstract*—This paper studies the optimal control design problem for linear discrete-time systems with stochastic multiplicative uncertainties. These uncertainties are assumed to be present in the control inputs and modeled as independent and identically distributed (i.i.d.) random processes. The optimal performance under study is defined in the mean-square sense, referred to as the mean-square optimal $\mathcal{H}_2$ performance. It is shown that the mean-square optimal $\mathcal{H}_2$ control problem via state feedback can be solved using a mean-square stabilizing solution to a modified algebraic Riccati equation (MARE). A necessary and sufficient condition for the existence of this solution is presented, which constitutes a generalization of the solution to the classic optimal $\mathcal{H}_2$ state feedback design problem, whereas the latter can be obtained by solving an algebraic Riccati equation (ARE). It is also proven that the optimal control design problem can be cast as an eigenvalue problem (EVP). For the output feedback case with possible input delays, we show that the mean-square optimal $\mathcal{H}_2$ control problem also amounts to solving an MARE, when the plant has no nonminimum phase zeros from the inputs to the measurement outputs. That is, the global optimal solution is obtained by solving an MARE incorporating the delays. The implication is that in this case a separation principle still holds.**

***Index Terms*—Algebraic Riccati equation, optimal control, output feedback, stochastic multiplicative uncertainties.**

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W. Su and T. Qi are with the School of Automation Science and Engineering, Key Laboratory of Autonomous Systems and Networked Control, Ministry of Education, South China University of Technology, Guangzhou 510641, China (e-mail: wzhsu@scut.edu.cn; auqt@scut.edu.cn).

J. Chen is with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: jichen@cityu.edu.hk).

M. Fu is with the School of Electrical and Computer Engineering, University of Newcastle, NSW 2308, Australia, and also with the Zhejiang University, Zhejiang 310027, China (e-mail: minyue.fu@newcastle.edu.au).

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I. Introduction

OVER the last four decades, optimal control of linear time-invariant (LTI) systems with stochastic multiplicative uncertainties has attracted a great deal of research interest (see, e.g., [2], [6], [9], [13], [17], [18], [25], [34], [40]). Several applications which appear in economic systems ([1], [8]), aerospace engineering [20], floating point numerical calculation errors in digital systems [34], etc., also motivate the research in this area. Recent development in networked control shows that stochastic multiplicative uncertainties can be used to effectively model certain transmission losses in communication channels such as packet drop-outs ([9], [10], [26], [35], [37]). This observation has consequently motivated active research in both networked control and stochastic control. The major technical difficulty with optimal control under multiplicative uncertainties is that the conventional tools for linear optimal control are inadequate to cope with the inherent nonlinearities yielded by such uncertainties. Many fundamental issues facing this problem are still open.

Studies into LTI systems with stochastic multiplicative uncertainties, typically modeled as independent and identically distributed random processes, can be traced back to the 1960's (see e.g., [16], [19], [32]), in which one of the major research issues is mean-square stability analysis. In [15], [16], [19], the criterion of mean-square stability was presented for systems given in state space models. On the other hand, Willems and Blankenship [34] studied the mean-square stability problem from a frequency-domain, input-output formulation for single-input single-output (SISO) LTI systems. A necessary and sufficient condition for mean-square stability, refered to as the *Mean-Square Small Gain Theorem* in the literature (see, e.g., [18] and [24]), was obtained for SISO systems. This result was extended to a more general version for multi-input multi-output (MIMO) systems in [18], which formed a primary machinery in the authors' development of mean-square stabilizability conditions in Part 1 of this paper [24].

In parallel, Wonham [32], [33] formulated an optimal linear quadratic regulation (LQR) problem for LTI systems subject to stochastic multiplicative uncertainties, and proved that the optimal state feedback can be constructed by obtaining a positive semidefinite solution to a modified algebraic Riccati equation (MARE). Using a numerical example, however, Wonham also showed that the optimal state feedback may not stabilize the system in the mean-square sense, when the noise variance becomes excessively large [33]. Willems and Willems [36] subsequently employed the LQR approach to study the mean-square stabilization problem for these stochastic uncertain systems. They found that when in the quadratic cost the state and control input weighting matrices are both full-rank, a system will be mean-square stabilizable via state feedback if and only if the MARE admits a positive definite solution. Moreover, the resulting optimal state feedback stabilizes the system in the mean-square sense. Despite this significant discovery, a fundamental question has remained elusive and longstanding: In the general case without the aforementioned full rank condition, what is the necessary and sufficient condition for the existence of the mean-square stabilizing solution to the MARE? Endeavor on this fundamental problem has been met with only limited successes. In [25], for a class of LTI systems with the stochastic multiplicative uncertainties, the mean-square stabilizing solution to the MARE was obtained by solving a linear matrix inequality (LMI). In [7] and [39], the problem was studied in terms of mean-square stabilizability and a generalized observability of the stochastic uncertain systems, to which sufficient conditions were presented in several special cases.

In the networked feedback control setting, Elia [9] revisited the mean-square stabilization problem for MIMO LTI systems, in which stochastic multiplicative noises are used to model communication channel uncertainties. Sinopoli *et. al.* [26], [27] studied Kalman filtering problems with packet loss in communication channels, where the channel uncertainty caused by packet loss is modeled as a stochastic multiplicative uncertainty, and a filter design approach based on an MARE is presented. Xiao *et. al.* [37] studied the stabilization problem for networked systems with packet loss and presented an explicit connection between mean-square stabilizability and signal-to-noise ratios of communication channels. In a similar networked control setting, the authors [23], [29] studied optimal tracking problems with stochastic multiplicative uncertainties. In particular, in Part 1 of this paper [24], the authors studied in great depth the stabilization problem with this stochastic multiplicative uncertainty model and developed necessary and sufficient conditions for mean-square stabilizability.

In Part 2 of this paper we expand our inquiry even further, to more challenging optimal performance problems for LTI systems subject to stochastic multiplicative uncertainties, which, as stated above, can be considered both as optimal stochastic control problems in the classical sense and as optimal control problems in a networked control setting. Our development thus serves a dual purpose: on one hand we develop optimal feedback laws to achieve optimal performance for systems with communication channels operating under stochastic multiplicative noises, while on the other hand, we attempt to solve a longstanding optimal stochastic control problem subject to stochastic multiplicative uncertainties. These two problems are unified under the general framework of *mean-square optimal* $\mathcal{H}_2$ *control*, where a generalized $\mathcal{H}_2$ norm of an LTI system with stochastic multiplicative uncertainties is defined in the mean-square sense and minimization of the generalized $\mathcal{H}_2$ norm via feedback control is considered.

The contributions of this paper can be summarized as follows. We first formulate the mean-square optimal $\mathcal{H}_2$ control problem under stochastic multiplicative uncertainties and present a complete state feedback solution to this problem. More specifically, we show that the optimal state feedback law can be designed based on the mean-square stabilizing solution to an MARE. Subsequently, a necessary and sufficient condition for the existence of the mean-square stabilizing solution to the MARE is obtained explicitly. This result thus resolves the longstanding open issue concerning the existence condition for the mean-square stabilizing solution alluded to above. Interestingly, when the stochastic multiplicative uncertainties are void, the condition reduces to the well-known solution to the classical optimal $\mathcal{H}_2$ state feedback problem (see, e.g., [38]). Moreover, we show that the optimal state feedback design amounts to solving an eigenvalue problem (EVP), which can be solved using any LMI solver [3].

Another main result of this paper is a solution to the mean-square optimal $\mathcal{H}_2$ control problem via output feedback. In its full generality, the output feedback optimal design can be done based on the solutions of two coupled algebraic Riccati equations. Serious technical difficulties arise due to this coupling. However, under the assumption that the plant transfer function from the inputs to the measurements is minimum phase, we show that this coupling does not present a technical difficulty, even when the plant may contain input delays, which can be regarded as a special nonminimum phase behavior. In this case, it turns out that the optimal output feedback controller is fully determined by a delay-dependent MARE for state feedback design. Accordingly, the optimal output feedback design can be accomplished by constructing equivalently an optimal state feedback design based on an auxiliary plant generated from the original plant combined with the delay information. Stated alternatively, a separation principle holds in the sense that the optimal output feedback controller is an observer-like state feedback controller with the state feedback gain fully determined by the solution to an optimal state feedback problem, and the "observer" gain is given by a separate design.

The remainder of this paper is organized as follows. In Section II, we formulate the mean-square optimal $\mathcal{H}_2$ control problem. In Section III, we solve the mean-square optimal $\mathcal{H}_2$ control via state feedback. Section IV studies the mean-square optimal $\mathcal{H}_2$ control via output feedback. The optimal output design problem is solved for a minimum phase plant with possible input delays. Section V concludes the paper.

Partial results of this paper have been previously presented in [23], [29], and [30]. The notation of Part 2 follows that of Part 1 [24]. For ease of readability, Part 2 of this paper is attempted to be sufficiently self-contained.

Notation: We denote a real n-dimensional vector space by R^n; the transpose, inverse, Moore-Penrose generalized inverse of a matrix by $(\cdot)^T$, $(\cdot)^{-1}$, $(\cdot)^\dagger$, respectively; the spectral radius of a matrix by $\rho(\cdot)$. For symmetric matrices X and Y, $X > Y$ (respectively, $X \geq Y$) mean that $X - Y$ is positive definite (respectively, positive semidefinite). Denote the l_1 norm (or the sum norm) of a vector by $\|\cdot\|_1$ (see [14] for details), and the expectation of a random variable by $E\{\cdot\}$. $\mathbb{RH}_\infty$ is the set of all proper stable rational function matrices. Denote the $\mathcal{H}_2$ norm of a proper stable rational function matrix by $\|\cdot\|_2$ (see [5] and [38] for details).

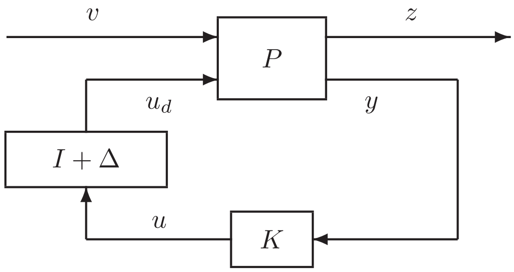

Fig. 1. An LTI system with stochastic multiplicative uncertainty at the control input.

II. Problem Formulation

This paper studies an optimal control problem for discrete-time LTI systems with stochastic multiplicative uncertainties. The feedback system under study is shown in Fig. 1, where P is the plant, K is a feedback controller, I is an identity matrix, and $\Delta = \operatorname{diag}\{\Delta_1, \ldots, \Delta_m\}$ represents stochastic multiplicative uncertainty with a diagonal structure. The control signal u is "corrupted" by the stochastic uncertainty Δ and the "corrupted" control signal u_d is used to drive the plant. The signals v, y and z are the input, the measurement output, and the controlled output, respectively. The state space model of the plant P is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 v(k) + B_2 u_d(k), \quad x(0) = 0 \\ z(k) &= C_1 x(k) \\ y(k) &= C_2 x(k) \end{aligned} \tag{1}$$

where $x(k)$ is the state of the plant. Denote the error in the control signal by $d(k)$, which results from the stochastic multiplicative uncertainties $\Delta_1, \ldots, \Delta_m$, i.e.,

$$d(k) = u_d(k) - u(k) = \Delta(k)u(k). \tag{2}$$

We shall consider both state feedback controller $u(k) = Fx(k)$ and output feedback controller

$$\begin{aligned} \hat{x}(k+1) &= A_c \hat{x}(k) + B_c y(k) \\ u(k) &= C_c \hat{x}(k) + D_c y(k) \end{aligned} \tag{3}$$

respectively. We impose the following assumption throughout the paper.

Assumption 1: The uncertainties $\{\Delta_i(k)\}, i = 1, 2, \ldots, m$, are independent and identically distributed (i.i.d) processes and mutually uncorrelated with

$$E\{\Delta_i(k)\} = 0, E\{\Delta_i(k_1)\Delta_i(k_2)\} = \begin{cases} \sigma_i^2, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}$$

$$E\{\Delta_i(k_1)\Delta_j(k_2)\} = 0 \quad \forall i \neq j.$$

In addition, $\{\Delta_i(k)\}, i = 1, 2, \ldots, m$, are uncorrelated with the input signal $\{v(k)\}$.

For simplicity, we also assume that the plant P is strictly proper. Our results, nevertheless, can be extended to the case where z contains feed-through terms u_d and v. On the other hand, when the measurement y in the plant (1) includes the term u_d, extra assumptions are needed to avoid certain technical difficulties (see [9] and Remark 1 for details).

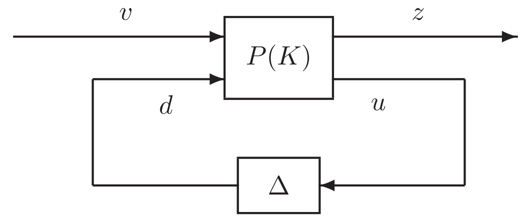

Fig. 2. The closed-loop system with stochastic multiplicative uncertainty.

A. Mean-Square Stability

In this subsection, we give the definitions of mean-square stability and mean-square stabilizability as well as a preliminary result on mean-square stability. To this end, we reconfigure the system in Fig. 1 as the system in Fig. 2. The signals v, z and u are the same as in Fig. 1 and d is defined in (2). Here Δ represents the diagonal stochastic multiplicative uncertainty, and $P(K)$ is the nominal closed-loop system, i.e., the closed-loop system of the plant P with a feedback controller K in Fig. 1 when the stochastic uncertainty Δ is void. With the error signal given in (2), we may rewrite the plant model as

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 v(k) + B_2 d(k) + B_2 u(k) \\ z(k) &= C_1 x(k) \\ y(k) &= C_2 x(k). \end{aligned} \tag{4}$$

Denote the transfer function of $P(K)$ from $\{v, d\}$ to $\{z, u\}$ by G_e and partition it as follows:

$$G_e = \begin{bmatrix} G_{z0} & G_{z1} & \cdots & G_{zm} \\ G_{10} & G_{11} & \cdots & G_{1m} \\ & \cdots & \cdots & \\ G_{m0} & G_{m1} & \cdots & G_{mm} \end{bmatrix} \tag{5}$$

where G_{z0} maps v to z, G_{zi} maps the ith component d_i of d to z, G_{i0} maps v to the ith component of u and G_{ij} maps the jth component d_j of d to the ith component of u, $i = 1, 2, \ldots, m$. For convenience, denote the ith block-column of G_e by G_{i-1}, i.e., $G_e = [G_0 \ G_1 \ \cdots \ G_m]$.

Definition 1: The closed-loop system in Fig. 2 is said to be *mean-square stable* if K stabilizes P and in addition, for zero input and any initial state, the covariance of the state $E\{x(k)x(k)^T\}$ is bounded for all k and converges to zero asymptotically.

Definition 2: The plant (1), (2) is said to be *mean-square stabilizable via state feedback (or via output feedback)* if there exists a state feedback law $u(k) = Fx(k)$ [respectively, an output feedback law (3)] such that the resulting closed-loop system is mean-square stable.

Denote

$$G = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ & \cdots & \\ G_{m1} & \cdots & G_{mm} \end{bmatrix} \tag{6}$$

$$\hat{G} = \begin{bmatrix} \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \ddots & \\ \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix} \tag{7}$$

and $\Sigma = \text{diag}\{\sigma_1^2, \ldots, \sigma_m^2\}$.

Lemma 1—(see [18]): Suppose the nominal closed-loop system G_e of the system shown in Fig. 2 is stable. Then, the system is mean-square stable if and only if the spectral radius of the matrix $\hat{G}\Sigma$ is less than one, i.e., $\rho(\hat{G}\Sigma) < 1$.

B. Mean-Square $\mathcal{H}_2$ Norm and Optimal $\mathcal{H}_2$ Control

Suppose the feedback controller K stabilizes the plant (1), (2) in the mean-square sense, i.e., the closed-loop system in Fig. 2 is mean-square stable. Assume that the input sequence $\{v(k)\}$ is an i.i.d. process with zero mean and that the covariance matrix is an identity matrix. In addition, assume that the initial state of the plant is zero. The *mean-square $\mathcal{H}_2$ norm* of the system is defined as the square root of the averaged power of the output signal z. Denote this averaged power by J_{H_2}, i.e.,

$$J_{H_2} = E\left\{\lim_{k\to\infty} \frac{1}{k+1}\sum_{i=0}^{k} z(i)^T z(i)\right\}. \quad (8)$$

Note that the controlled output z is related to the input noise v and the stochastic uncertainty Δ. The expectation in (8) thus operates jointly over the distributions of v and Δ. In the classic optimal $\mathcal{H}_2$ theory, J_{H_2} given by (8) is the square of the standard $\mathcal{H}_2$ norm of an LTI system in the absence of the uncertainty Δ (see, e.g., [38]). In that case, the expectation in (8) is taken solely over v.

Let $\mathcal{K}$ be the set of all possible controllers stabilizing the closed-loop system in Fig. 2 in the mean-square sense. For any given $K \in \mathcal{K}$, denote the function J_{H_2} by $J_{H_2}(K)$. Then the optimal performance of interest in this paper is given by

$$J_{\text{opt}} = \inf_{K\in\mathcal{K}} J_{H_2}(K) = \inf_{K\in\mathcal{K}}\inf_{\sigma_0}\left\{\frac{1}{\sigma_0^2} : J_{H_2}(K) < \frac{1}{\sigma_0^2}\right\}. \quad (9)$$

This ushers in the mean-square optimal $\mathcal{H}_2$ control problem, the central problem to be tackled in the sequel.

Mean-Square Optimal $\mathcal{H}_2$ Control Problem: The objective of this problem is to find an optimal controller K_{opt} to stabilize the plant (1), (2) in the mean-square sense and to minimize the performance cost J_{H_2}, i.e.,

$$K_{\text{opt}} = \arg\inf_{K\in\mathcal{K}} J_{H_2}(K).$$

We have the following characterization of the mean-square $\mathcal{H}_2$ norm.

Lemma 2: Suppose the plant in (1) and (2) satisfies Assumption 1. Let the controller K stabilize the plant P in the mean-square sense. Then

$$J_{H_2} = \|G_{z0}\|_2^2 + \begin{bmatrix}\|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2\end{bmatrix}\Sigma(I-\hat{G}\Sigma)^{-1} \times \begin{bmatrix}\|G_{10}\|_2^2 \\ \vdots \\ \|G_{m0}\|_2^2\end{bmatrix}. \quad (10)$$

Proof: See Appendix A. ■

It is clear that when the uncertainties are absent (equivalently, $\sigma_i = 0$, $i = 1, \ldots, m$), J_{H_2} reduces to the standard $\mathcal{H}_2$ cost $\|G_{z0}\|_2^2$ in the conventional $\mathcal{H}_2$ control problem (see, e.g., [38]).

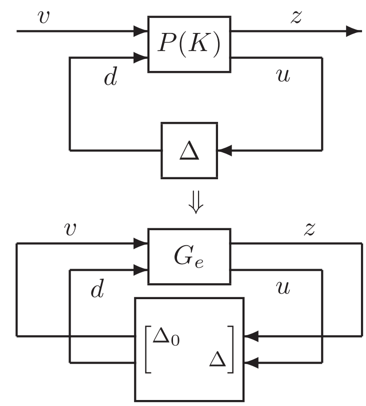

Fig. 3. Mean-square performance problem transformed to mean-square stabilization problem.

Remark 1: The key feature of the closed-loop system in Fig. 2 which is used in the proof of Lemma 2 is that $\Delta_i(k)$ is uncorrelated with $u_i(k)$ for $i = 1, \ldots, m$, $k = 1, 2, \ldots, \infty$. This holds when the parameter matrix of u_d in the measurement output equation is equal to zero matrix, and is a strict upper or lower triangular matrix (more details see [9]). Once Lemma 2 holds, the results in this work can be extended to these cases when z and y include feed-through terms u_d and v, without technical difficulties.

Now, let

$$\hat{G}_e = \begin{bmatrix} \|G_{z0}\|_2^2 & \|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2 \\ \|G_{10}\|_2^2 & \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \cdots & \cdots & \\ \|G_{m0}\|_2^2 & \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix}. \quad (11)$$

The lemma below follows from straightforward algebraic manipulations, which can also be established following [18].

Lemma 3: Suppose that the plant in (1) and (2) satisfies Assumption 1 and K stabilizes the plant P. Then for any $\sigma_0 > 0$

$$J_{H_2} < \frac{1}{\sigma_0^2} \quad \text{and} \quad \rho(\hat{G}\Sigma) < 1 \quad (12)$$

if and only if

$$\rho(\hat{G}_e\Sigma_e) < 1$$

where $\Sigma_e \triangleq \text{diag}\{\sigma_0^2, \sigma_1^2, \ldots, \sigma_m^2\}$.

Thus in the same spirit as in [18], Lemma 3 shows that the mean-square performance problem can be reformulated as a mean-square stabilization problem for the augmented system G_e, with the mean-square stability determined by the condition $\rho(\hat{G}_e\Sigma_e) < 1$, in light of Lemma 1. This recognition underlies our essential technical approach in solving the optimal mean-square $\mathcal{H}_2$ control problem. Fig. 3 illustrates this transformation.

III. Mean-Square Optimal $\mathcal{H}_2$ Control via State Feedback

The section studies the mean-square optimal $\mathcal{H}_2$ control via state feedback. Our objective is to find an optimal state feedback law for the plant (1), (2) such that the resulting closed-loop system is mean-square stable and its mean-square $\mathcal{H}_2$ norm (or the performance cost J_{H_2}) is minimized. From Lemma 3 and the characterization (9), it is clear that this amounts to finding an optimal state feedback such that it stabilizes the plant in the mean-square sense and additionally

$$\begin{aligned} K_{\text{opt}} &= \arg \inf_{K\in\mathcal{K}} \inf_{\sigma_0} \left\{ \frac{1}{\sigma_0^2} : J_{H_2}(K) < \frac{1}{\sigma_0^2} \right\} \\ &= \arg \inf_{K\in\mathcal{K}} \inf_{\sigma_0} \left\{ \frac{1}{\sigma_0^2} : \rho(\hat{G}_e\Sigma_e) < 1 \right\}. \end{aligned} \tag{13}$$

In this optimal controller design problem, the spectral radius $\rho(\hat{G}_e\Sigma_e)$ of the matrix $\hat{G}_e\Sigma_e$ plays a key role. To study this, we note that $\hat{G}_e\Sigma_e$ is a positive matrix (i.e., all of its entries are nonnegative) and give a preliminary result on positive matrices.

Lemma 4—(see [14]): For any square positive matrix T, its spectral radius is given by

$$\rho(T) = \inf_{\Gamma} \max_{j} \sum_i \gamma_i^2 t_{ij} \frac{1}{\gamma_j^2}$$

where $\Gamma = \operatorname{diag}\{\gamma_1^2, \ldots, \gamma_m^2\} > 0$ and t_{ij} is the $\{i,j\}$th entry of the matrix T.

Applying Lemma 4 to the optimal design constraint

$$\rho(\hat{G}_e\Sigma_e) < 1$$

we obtain that for any σ_0 satisfying $\inf_K J_{H_2} < 1/\sigma_0^2$, there exists $\hat{\Gamma}_e = \{1, \Gamma\}, \Gamma = \{\gamma_1^2, \ldots, \gamma_m^2\} > 0$ such that

$$\left\| \hat{\Gamma}_e \hat{G}_e \Sigma_e \hat{\Gamma}_e^{-1} e_i \right\|_1 < 1,\ i = 1, \ldots, m+1 \tag{14}$$

where e_i is the ith column of the $(m+1)\times(m+1)$ identity matrix.

From the definition of the $\mathcal{H}_2$ norm (see [38]), the constraint (14) in the optimal feedback design can be written as

$$\left\| \Gamma_e^{\frac{1}{2}} G_i \right\|_2^2 \frac{\sigma_i^2}{\gamma_i^2} < 1,\ i = 0, 1, \ldots, m,\ \gamma_0 = 1 \tag{15}$$

where $\Gamma_e = \operatorname{diag}\{I, \Gamma\}$ and I is an identity matrix with a compatible size.

In the state feedback case, according to (15), we obtain the following lemma:

Lemma 5: A state feedback law

$$u(k) = Fx(k) \tag{16}$$

stabilizes the plant (1), (2) in the mean-square sense with $J_{H_2} < 1/\sigma_0^2$ for a given $\sigma_0 > 0$ if and only if there exists $\Gamma > 0$ such that

$$\begin{aligned} &\left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F) B_1 \right\|_2^2 \sigma_0^2 < 1 \\ &\left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F) B_{2i} \right\|_2^2 \frac{\sigma_i^2}{\gamma_i^2} < 1,\ i = 1, \ldots, m \end{aligned} \tag{17}$$

where $A_F = A + B_2F$ and B_{2i} is the ith column of B_2.

Lemma 5 shows that in the optimal state feedback design, the design constraint $\rho(\hat{G}_e\Sigma_e) < 1$ is equivalent to (17). Hence, we re-formulate the optimal state feedback design problem in (13). The mean-square optimal $\mathcal{H}_2$ control via state feedback is to find

$$F_{\text{opt}} = \arg \inf_{F,\sigma_0} \left\{ \frac{1}{\sigma_0^2} : \text{subject to (17)} \right\}.$$

Next, we review the standard optimal $\mathcal{H}_2$ state feedback design for the following auxiliary plant (without stochastic multiplicative uncertainties):

$$\begin{aligned} x_\Gamma(k+1) &= Ax_\Gamma(k) + B_1 v(k) + B_2 u(k) \\ z_\Gamma(k) &= \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x_\Gamma(k) + \begin{bmatrix} 0 \\ \Gamma^{\frac{1}{2}} \end{bmatrix} u(k) + Dv(k). \end{aligned} \tag{18}$$

Lemma 6: For the plant (18), the discrete algebraic Riccati equation (DARE)

$$X_\Gamma = A^T X_\Gamma A - A^T X_\Gamma B_2 \left(\Gamma + B_2^T X_\Gamma B_2\right)^{-1} B_2^T X_\Gamma A + C_1^T C_1 \tag{19}$$

has a stabilizing solution[1] X_Γ if and only if (A, B_2) is stabilizable and (A, C_1) has no unobservable poles on the unit circle. Moreover, X_Γ is the unique stabilizing solution and the largest positive semidefinite solution to the DARE (19). This stabilizing solution X_Γ is a positive definite matrix if (A, C_1) is observable. The minimum $\mathcal{H}_2$ norm of the closed-loop transfer function from v to z via state feedback $u(k) = Fx_\Gamma(k)$ is given by

$$\begin{aligned} \min_F &\left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A - B_2F)^{-1} B_1 + D \right\|_2^2 \\ &= \operatorname{tr}\left\{ B_1^T X_\Gamma B_1 \right\} + \operatorname{tr}\{D^T D\} \end{aligned} \tag{20}$$

and the associated optimal state feedback gain is given by

$$F_\Gamma = -\left(\Gamma + B_2^T X_\Gamma B_2\right)^{-1} B_2 X_\Gamma A. \tag{21}$$

The first part of the lemma above follows [5], [38] and the second part comes from Theorem 6.4.1 in [5].

To generalize this lemma to the mean-square optimal $\mathcal{H}_2$ state feedback design for the plant (1), (2), consider the following MARE:

$$X = A^T XA - A^T XB_2 \left[\Phi(X)\Sigma + B_2^T XB_2\right]^{-1} B_2^T XA + C_1^T C_1 \tag{22}$$

with $\Phi(X) = \operatorname{diag}\{B_{21}^T XB_{21}, \ldots, B_{2m}^T XB_{2m}\}$. A solution X to the MARE (22) is called a *mean-square stabilizing solution* if the closed-loop system of the plant (1), (2) with the state feedback (16) is mean-square stable, where

$$F = -\left[\Phi(X)\Sigma + B_2^T XB_2\right]^{-1} B_2^T XA. \tag{23}$$

We have the following result.

Lemma 7: The plant (1), (2) is mean-square stabilizable via state feedback and the closed-loop system has $J_{H_2} < 1/\sigma_0^2$ for a given $\sigma_0 > 0$ if and only if there exist $X_\Gamma > 0, \Gamma > 0$, F_Γ such that

$$X_\Gamma > (A + B_2F_\Gamma)^T X_\Gamma (A + B_2F_\Gamma) + C_1^T C_1 + F_\Gamma^T \Gamma F_\Gamma \tag{24}$$

$$\operatorname{tr}\left\{B_1^T X_\Gamma B_1\right\} < \frac{1}{\sigma_0^2},\ B_{2i}^T X_\Gamma B_{2i} < \frac{1}{\sigma_i^2} e_i^T \Gamma e_i, i = 1, \ldots, m. \tag{25}$$

[1]The solution X_Γ to (19) is said to be a stabilizing solution if $A + B_2F_\Gamma$ is stable with F_Γ in (21).

Proof: See Appendix B. ■

We are now ready to state the first two main results of this paper, one on mean-square stabilisability and one on mean-square $\mathcal{H}_2$ performance control.

Theorem 1: Suppose the plant (1), (2) satisfies Assumption 1. The MARE (22) has a mean-square stabilizing solution X if and only if the plant (1), (2) is mean-square stabilizable and (A, C_1) has no unobservable poles on the unit circle. In addition, this solution X is the unique mean-square stabilizing solution and the largest positive semidefinite solution to the MARE.

Proof: See Appendix C. ■

Theorem 2: Suppose the plant (1), (2) satisfies Assumption 1, (A, B_2) is stabilizable and (A, C_1) has no unobservable poles on the unit circle. Then the plant is mean-square stabilizable if and only if the largest solution[2] X to the MARE (22) is positive semidefinite. When the above holds, the minimum performance cost J_{H_2} of the closed-loop achievable via linear state feedback is given by

$$\inf_F J_{H_2} = \text{tr}\left\{B_1^T X B_1\right\}$$

and the optimal state feedback gain F_{opt} is given by (23). ■

Proof: See Appendix D. ■

In the case when the matrix $C_1^T C_1$ has full rank, the optimal state feedback design problem was solved in [32], [33] and [36]. However, there appears a peculiarity in this design problem, which is illustrated in Example 1 later, in the case when the matrix $C_1^T C_1$ is not a full rank matrix. This has attracted many research efforts (see, e.g., [39] and references therein). Theorem 1 presents a complete answer to this problem, i.e., the necessary and sufficient condition for which the largest positive semidefinite solution to the MARE (22) can be used to design the mean-square optimal $\mathcal{H}_2$ state feedback law in the latter case. It would be interesting to notice that, when the stochastic uncertainties in the plant are void, the mean-square optimal $\mathcal{H}_2$ control problem reduces to a standard optimal $\mathcal{H}_2$ control problem. In this case, the necessary and sufficient condition for the existence of the mean-square stabilizing solution to the MARE (22), which degenerates to a DARE, becomes that (A, B_2) is stabilizable and (A, C_1) has no unobservable poles on the unit circle. This is a well-known result in the classic optimal $\mathcal{H}_2$ control (see for example [5] and [38]).

Compared with the existing results (see for example [36] and [39]), Theorem 2 gives a more precise description for the mean-square stabilizability of the plant (1), (2) in terms of the plant model and the solution to the MARE (22).

Now we offer a numerically efficient algorithm for solving the mean-square optimal state feedback gain.

Theorem 3: Suppose the plant (1), (2) satisfies Assumption 1 and (A, C_1) has no unobservable poles on the unit circle. Then, the MARE (22) has a mean-square stabilizing solution X if and only if the following linear matrix inequality problem, called the EVP problem (see [3] for definition), has a positive semidefinite solution S:

$$\inf_{S,Z,V,\pi_1,\ldots,\pi_m} \text{tr}\{Z\} \tag{26}$$

subject to

$$\begin{bmatrix} S & SA^T + V^T B_2^T & V^T & SC_1^T \\ AS + B_2 V & S & 0 & 0 \\ V & 0 & \Pi & 0 \\ C_1 S & 0 & 0 & I \end{bmatrix} > 0 \tag{27}$$

$$\begin{bmatrix} Z & B_1^T \\ B_1 & S \end{bmatrix} > 0, \ \begin{bmatrix} \frac{\pi_i}{\sigma_i^2} & \pi_i B_{2i}^T \\ \pi_i B_{2i} & S \end{bmatrix} > 0, \ i = 1, \ldots, m$$

$$\Pi = \text{diag}\{\pi_1, \pi_2, \ldots, \pi_m\} > 0. \tag{28}$$

Moreover, when the solutions exist, the minimum performance cost J_{H_2}, the mean-square stabilizing solution X to the MARE (22) and the optimal state feedback gain are given by

$$\inf_F J_{H_2} = \text{tr}\left\{B_1^T S_{\text{opt}}^\dagger B_1\right\}, \ X = S_{\text{opt}}^\dagger, \ F = V_{\text{opt}} S_{\text{opt}}^\dagger$$

respectively, where S_{opt} and V_{opt} are the optimal solution to the EVP problem (26)–(28).

Proof: The proof is the latter part of that for Theorem 1. ■

The following example illustrates the peculiarity of unobservable poles on the unit circle in the mean-square optimal $\mathcal{H}_2$ control problem. The algorithm in minimizing the performance cost J_{H_2} is also illustrated using this example.

Example 1: Consider the plant (1), (2) with parameters as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

The variances σ_1^2 and σ_2^2 of the stochastic multiplicative uncertainties Δ_1 and Δ_2 are equal to 0.01. The largest positive semidefinite solution to MARE (22) and the optimal state feedback (23) are given by, respectively

$$X = \begin{bmatrix} 1.0082 & 0 & 1.0038 \\ 0 & 0 & 0 \\ 1.0038 & 0 & 1.0335 \end{bmatrix}$$

$$F = \begin{bmatrix} -0.8096 & 0 & -0.3755 \\ -0.1831 & 0 & -1.6192 \end{bmatrix}.$$

The poles of the resulting nominal closed-loop system are -1, 0.0066, 0.5646. Note that the unobservable pole of (A, C_1) is -1. The state feedback gain F designed by the solution X can not relocate this pole.

On the other hand, when $C_1 = [1, 1, 1]$, (A, C_1) has no unobservable poles on the unit circle. The largest solution to

[2] X being the largest solution means that $X \geq \tilde{X}$ for any other solution $\tilde{X}$.

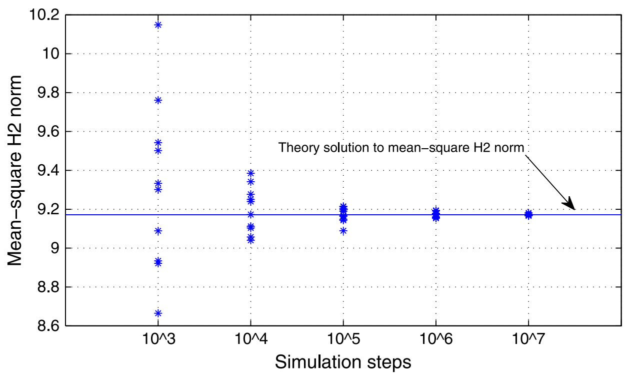

Fig. 4. The numerical solutions of the minimum performance cost J_{H_2}.

the MARE (22) and the optimal state feedback (23) are

$$X = \begin{bmatrix} 1.0333 & 0.9876 & 1.0014 \\ 0.9876 & 1.0214 & 1.0058 \\ 1.0014 & 1.0058 & 1.1277 \end{bmatrix}$$
$$F = \begin{bmatrix} -0.8262 & 0.3086 & -0.0343 \\ -0.1661 & 0.1874 & -1.465 \end{bmatrix}.$$

The poles of the resulting nominal closed-loop system are −0.3439, 0.0189, 0.5298. Moreover, according to Theorem 3, we obtain that the minimum performance cost $\inf_F J_{H_2}$ of the resulting closed-loop system is 9.1719. On the other hand, according to (8), the performance cost J_{H_2} is calculated by using the output z of the resulting closed-loop system. Fig. 4 shows the simulation results with steps from 10^3 to 10^7, respectively. The numerical results are convergent to the theoretic result when simulation steps are increasing.

Notice in this example that the two plants have the same state space equation (which is mean-square stabilizable) but different output equations. We have obtained the largest positive semi-definite solutions to the MAREs for the two plants, respectively. In the former case, the state feedback designed by the positive semidefinite solution to the MARE associated with the first plant can not stabilize this plant (i.e., the resulting closed-loop system has a pole on the unit circle) while in the latter case, the state feedback designed by the positive semidefinite solution to the MARE associated with the second plant stabilizes the second plant in the mean-square sense. This difference results from the unobservable pole of the plant on the unit circle.

Before completing this section, we give a generalized version of Theorems 1 and 2 to systems also with state-dependent stochastic multiplicative uncertainties. More specifically, we consider a generalized version of the plant (1) as follows:

$$x(k+1) = \left(A + \sum_{i=1}^{m_s} A_i \Delta_{si}(k)\right) x(k) + B_1 v(k) + B_2 u_q(k)$$
$$z(k) = C_1 x(k) \tag{29}$$

where Δ_{si}, $i = 1, \ldots, m_s$, are the state-dependent stochastic multiplicative uncertainties.

Assumption 2: For the plant (29), the following conditions hold:

1) The stochastic uncertainties Δ_{si}, $i = 1, \ldots, m_s$, are mutually independent i.i.d. processes with zero mean and variances σ_{si}, respectively, and these noises are independent of v and all $\Delta_j, j = 1, \ldots, m$.
2) The matrices A_i $i = 1, \ldots, m_s$ have rank one, $A_i = B_{si} C_{si}$, B_{si} and C_{si} are a column vector and a row vector, respectively, $i = 1, \ldots, m_s$.

Theorem 4: Consider the generalized plant (29) with (2) satisfying Assumption 1 and 2. The MARE (30) has a unique mean-square stabilizing solution X

$$X = A^T X A + \sum_{i=1}^{m_s} \sigma_{si}^2 A_i^T X A_i + C_1^T C_1 - A^T X B_2 \left[\Phi(X)\Sigma + B_2^T X B_2\right]^{-1} B_2^T X A \tag{30}$$

with $\Phi(X) = \mathrm{diag}\{B_{21}^T X B_{21}, \ldots, B_{2m}^T X B_{2m}\}$ if and only if the plant (29) is mean-square stabilizable and $(A, [C_1^T \; C_{s1}^T \; \cdots \; C_{sm_s}^T]^T)$ has no unobservable poles on the unit circle. In addition, X is a positive semidefinite matrix and the largest solution to the MARE (30). When the above holds, the mean-square optimal $\mathcal{H}_2$ control via state feedback (16) is given by (23) with X being the unique mean-square stabilizing solution to (30), and the minimum performance cost is given by

$$\inf_F J_{H_2} = \mathrm{tr}\left\{B_1^T X B_1\right\}.$$

Proof: See Appendix E. ■

IV. Mean-Square Optimal $\mathcal{H}_2$ Control via Output Feedback

We now generalize the optimal design approach in Theorems 1–3 to output feedback design with possible input delays. It turns out that the general case is very difficult to deal with. For this reason, we restrict ourselves to minimum-phase plants with fixed input delay steps.

Suppose that the plant (1), (2) has input delay steps $r_0 - 1$, $r_1 - 1, \ldots, r_m - 1$ in the input channels for the input signal v and the control signals $u_{d1}, \ldots, u_{dm}$, respectively, where $r_i \geq 1$, $i = 0, 1, \ldots, m$. The transfer function G_y of the plant from (v, u_d) to y is cascaded by the original plant $C_2(zI - A)^{-1}[B_1 \; B_2]$ and the channel with fixed delay steps, i.e.,

$$G_y = C_2(zI - A)^{-1}[B_1 \quad B_2] \times \mathrm{diag}\{z^{-r_0+1} I, z^{-r_1+1}, \ldots, z^{-r_m+1}\}. \tag{31}$$

It is clear that the transfer function G_y has all the zeros of the original plant and that the input delay steps change the relative degrees of the columns in the transfer function or the zeros in infinite only. We add the following assumption:

Assumption 3: The transfer function of the original plant $C_2(zI - A)^{-1}[B_1 \; B_2]$ is minimum phase. The transfer function from v to y has relative degree $r_0 \geq 1$ and the transfer function from the i-th component of u_d to y has relative degree $r_i \geq 1$, i.e., $C_2 A^j B_1 = 0, j = 0, \ldots, r_0 - 2$, the matrix $C_2 A^{r_0 - 1} B_1$ has full column rank, $C_2 A^j B_{2i} = 0$, $j = 0, \ldots, r_i - 2$ and $C_2 A^{r_i - 1} B_{2i} \neq 0$, $i = 1, \ldots, m$, respectively, and

$$\Psi = [A^{r_0 - 1} B_1 \quad A^{r_1 - 1} B_{21} \quad \cdots \quad A^{r_m - 1} B_{2m}] \tag{32}$$

has full column rank.

Remark 2: Assumption 3 means that the transfer function $G_y \text{diag}\{z^{r_0}I, z^{r_1}, \ldots, z^{r_m}\}$ is left invertible in $\mathbb{RH}_\infty$. But we do not impose any constraint on the transfer function from (v, u_d) to z for the plant. Moreover, the plants which satisfy Assumption 3 are slightly more general than those plants given by (31) since these plants have given relative degrees but may not have the component $\text{diag}\{z^{-r_0+1}I, z^{-r_1+1}, \ldots, z^{-r_m+1}\}$.

Suppose that Assumption 1 holds for the plant (1), (2). We consider all possible output feedback controllers in the form of (3) which can stabilize this plant in the mean-square sense. Following the discussion in Section III, the mean-square optimal $\mathcal{H}_2$ control design is to find the optimal output feedback law K_{opt} in (13) among all possible mean-square stabilizing output feedback controllers (3) subject to the constraint (15).

Note that the set of inequalities in (15) can be expressed as

$$\max_{\Lambda_e} \bar{J}_{H_2} < 1, \text{ with } \bar{J}_{H_2} = \sum_{i=0}^{m} \lambda_i^2 \left\| \Gamma_e^{\frac{1}{2}} G_{i+1} \right\|_2^2 \frac{\sigma_i^2}{\gamma_i^2} < 1 \quad (33)$$

where $\gamma_0 = 1$, $\text{tr}\{\Lambda_e\} = 1$ with $\Lambda_e = \text{diag}\{\lambda_0^2, \lambda_1^2, \ldots, \lambda_m^2\}$.

To solve this optimal design problem, the following auxiliary plant with given parameters Γ_e and Λ_e is considered:

$$\begin{aligned} \bar{x}_\Gamma(k+1) &= A\bar{x}_\Gamma(k) + B_1\sigma_0\lambda_0 v(k) \\ &\quad + B_2\Sigma^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}d(k) + B_2 u(k) \\ \bar{z}_\Gamma(k) &= \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \bar{x}_\Gamma(k) + \begin{bmatrix} 0 \\ \Gamma^{\frac{1}{2}} \end{bmatrix} u(k) \\ \bar{y}_\Gamma(k) &= C_2\bar{x}_\Gamma(k). \end{aligned} \quad (34)$$

The performance cost $\bar{J}_{H_2}$ given in (33) is the square of $\mathcal{H}_2$ norm of the plant (34) with an output feedback controller in the form (3). For the given Γ and Λ_e, the optimal feedback control design in minimizing the performance cost $\bar{J}_{H_2}$ is the standard optimal $\mathcal{H}_2$ control problem via output feedback, which is solved from the stabilizing solutions X_Γ and Y_Γ to the DARE (19) and following DARE (35), respectively

$$\begin{aligned} &AY_\Gamma A^T - Y_\Gamma - AY_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger C_2 Y_\Gamma A^T \\ &\qquad + B_1\sigma_0^2\lambda_0^2 B_1^T + B_2\Gamma^{-1}\Sigma\Lambda B_2^T = 0. \end{aligned} \quad (35)$$

Assumption 4: For the plant (1) (or (34)), it is assumed that

1) (A, B_2) is stabilizable and (A, C_1) has no unobservable poles on the unit circle;
2) (A, C_2) is detectable and $(A, [B_1\ B_2])$ has no unstabilizable poles on the unit circle.

Lemma 8—(see [31]): For the auxiliary plant (34), the following statements are true:

1) The DARE (19) has a unique stabilizing solution X_Γ if and only if Assumption 4.1 holds;
2) The DARE (35) has a unique stabilizing solution Y_Γ if and only if Assumption 4.2 holds.

In addition, X_Γ and Y_Γ are positive semidefinite matrices. The optimal output feedback controller K for the auxiliary plant (34) is given by

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} A + B_2F_\Gamma + L_\Gamma C_2 - B_2 L_{0\Gamma} C_2 & L_\Gamma - B_2 L_{0\Gamma} \\ \hline L_\Gamma C_2 - F_\Gamma & L_{0\Gamma} \end{array}\right] \quad (36)$$

where

$$F_\Gamma = -\left(\Gamma + B_2^T X_\Gamma B_2\right)^{-1} B_2^T X_\Gamma A \quad (37)$$

$$\begin{aligned} L_\Gamma &= -AY_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger \\ L_{0\Gamma} &= -F_\Gamma Y_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger. \end{aligned} \quad (38)$$

The corresponding minimum performance cost for the plant (34) with this output feedback is given by

$$\begin{aligned} \inf_K \bar{J}_{H_2} &= \sigma_0^2\lambda_0^2 \text{tr}\left\{B_1^T X_\Gamma B_1\right\} + \text{tr}\left\{B_2^T X_\Gamma B_2 \Gamma^{-1}\Sigma\Lambda\right\} \\ &\quad + \text{tr}\left\{Z F_\Gamma Y_\Gamma F_\Gamma^T Z\right\} \\ &\quad - \text{tr}\left\{Z F_\Gamma Y_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger C_2 Y_\Gamma F_\Gamma^T Z\right\} \end{aligned} \quad (39)$$

where $Z = (\Gamma + B_2^T X_\Gamma B_2)^{1/2}$.

The lemma above shows that the optimal output feedback K for the given Γ and Λ_e includes three parameters F_Γ, L_Γ and $L_{0\Gamma}$, determined by two equations (19) and (35) which are highly coupled through Γ and Λ_e. This causes a huge difficulty in the mean-square optimal $\mathcal{H}_2$ control design via output feedback. However, we show below that, under Assumption 3, solutions can be simplified. The key observation is that under Assumption 3, the coupling between two DAREs becomes very weak. It turns out that the optimal output feedback for the auxiliary plant (34) is a much simpler than that given by Lemma 8.

Lemma 9: Suppose the plant (1) satisfies Assumptions 3 and 4. Then, for given Γ and Λ_e, the stabilizing solution to the DARE (35) is given by

$$Y_\Gamma = \sum_{i=0}^{r_0-1} A^i B_1 \sigma_0^2 \lambda_0^2 B_1^T A^{T^i} + \sum_{j=1}^{m} \sum_{i=0}^{r_j-1} A^i B_{2j} \sigma_j^2 \lambda_j^2 \gamma_j^{-2} B_{2j}^T A^{T^i}. \quad (40)$$

Moreover, Y_Γ is the unique stabilizing solution to the DARE (35), i.e., $Y_\Gamma \geq 0$ and $A + L_\Gamma C_2$ is stable with $L_\Gamma = -AY_\Gamma C_2^T (C_2 Y_\Gamma C_2^T)^\dagger$.

Proof: The fact that (40) is a positive semidefinite solution to (35) follows by direct substitution. The stabilizing property and uniqueness come from [28] and [31] under the given assumptions. ■

Remark 3: In the case when the plant (1) has relative degrees one, i.e., without input delays, the stabilizing solution to the DARE (35) given by Lemma 9 is written as follows:

$$Y_\Gamma = B_1\sigma_0^2\lambda_0^2 B_1^T + B_2\Gamma^{-1}\Sigma\Lambda B_2^T. \quad (41)$$

On the other hand, compared with the DARE (35), the DARE associated with optimal $\mathcal{H}_2$ design would be more complicated when the measurement y includes the feed-forward term from the input signal v. For left-invertible plants, the solution to this DARE is a zero matrix (see [22]). The result in the remainder part holds in these cases.

Using Lemma 9, the minimum performance cost for the closed-loop system studied in Lemma 8 and the parameters $F_\Gamma, L_\Gamma, L_{0\Gamma}$ of the optimal controller are given in the following result.

Lemma 10: Suppose the plant (34) satisfies Assumptions 3 and 4. For the given Γ and Λ_e, the control parameters F_Γ, L_Γ and $L_{\Gamma 0}$ of the optimal $\mathcal{H}_2$ output feedback are given by (37) and

$$L_\Gamma = -A\Psi(C_2\Psi)^\dagger, \quad L_{\Gamma 0} = -F_\Gamma \Psi(C_2\Psi)^\dagger. \tag{42}$$

The minimum performance cost for the resulting closed-loop system is given by

$$\inf_K \bar{J}_{H_2} = \sum_{i=0}^{m} \sigma_i^2 \lambda_i^2 \gamma_i^{-2} \bar{\phi}_i(X_\Gamma) \tag{43}$$

where $\gamma_0 = 1$ and

$$\bar{\phi}_0(X_\Gamma) = \mathrm{tr}\left\{B_1^T A^{T^{r_0-1}} X_\Gamma A^{r_0-1} B_1\right\} + \sum_{j=0}^{r_0-2} \mathrm{tr}\left\{B_1^T A^{T^j} C_1^T C_1 A^j B_1\right\} \tag{44}$$

$$\bar{\phi}_i(X_\Gamma) = B_{2i}^T A^{T^{r_i-1}} X_\Gamma A^{r_i-1} B_{2i} + \sum_{j=0}^{r_i-2} B_{2i}^T A^{T^j} C_1^T C_1 A^j B_{2i},\ i = 1, \ldots, m. \tag{45}$$

Proof: See Appendix F. ■

It is shown in Lemma 10 that for the given Γ and Λ_e, the optimal parameters L_Γ and L_{Γ_0} in the observer-like part are determined by the plant model and the optimal state feedback F_Γ. Moreover, these optimal parameters are not related to the parameter Λ_e. Thus, the optimal output feedback controller given by Lemma 10 minimizes all components $\|\Gamma_e^{1/2} G_{i+1}\|_2^2 (\sigma_i^2/\gamma_i^2)$, $i = 0, 1, \ldots, m$ in the performance cost $\bar{J}_{H_2}$. Now, the key issue in the mean-square optimal output feedback design is to find the optimal state feedback F_Γ and associated parameter matrix Γ subject to the design constraint (15).

To this end, we study the mean-square optimal $\mathcal{H}_2$ state feedback problem for a new auxiliary plant with stochastic multiplicative uncertainties below

$$\tilde{x}(k+1) = A\tilde{x}(k) + A^{r_0-1}B_1 v(k) + [A^{r_1-1}B_{21} \cdots A^{r_m-1}B_{2m}]d(k) + B_2 u(k) \tag{46}$$

$$\tilde{z}(k) = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}\tilde{x}(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} v(k) + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} d(k)$$

$$d(k) = \Delta(k) u(k) \tag{47}$$

where $\Delta(k) = \mathrm{diag}\{\Delta_1, \ldots, \Delta_m\}$, Δ_i, $i = 1, \ldots, m$, are stochastic multiplicative uncertainties satisfying Assumption 1, $v(k)$ is an i.i.d input noise as studied in Section II

$$D_1^T D_1 = \sum_{j=0}^{r_0-2} B_1^T A^{T^j} C_1^T C_1 A^j B_1$$

$$D_2 = \mathrm{diag}\left\{\left(\sum_{j=0}^{r_1-2} B_{21}^T A^{T^j} C_1^T C_1 A^j B_{21}\right)^{\frac{1}{2}}, \ldots, \left(\sum_{j=0}^{r_m-2} B_{2m}^T A^{T^j} C_1^T C_1 A^j B_{2m}\right)^{\frac{1}{2}}\right\}.$$

Lemma 11: Suppose the plant (46), (47) satisfies Assumption 1. The following MARE:

$$A^T X A - X - A^T X B_2 \left[\bar{\Phi}(X)\Sigma + B_2^T X B_2\right]^{-1} B_2^T X A + C_1^T C_1 = 0 \tag{48}$$

with $\bar{\Phi}(X) = \mathrm{diag}\{\bar{\phi}_1(X), \ldots, \bar{\phi}_m(X)\}$ and $\bar{\phi}_i(\cdot)$ in (45), has a mean-square stabilizing solution if and only if this plant is mean-square stabilizable and (A, C_1) has no unobservable poles on the unit circle. In addition, this solution is the unique mean-square stabilizing solution and the largest positive semi-definite solution to the MARE (48).

Proof: See Appendix G. ■

Lemma 12: Suppose the plant (46), (47) satisfies Assumption 1 and (A, C_1) has no unobservable poles on the unit circle. Then, the plant is mean-square stabilizable if and only if the MARE (48) has a positive semidefinite solution. When the above holds, the mean-square optimal $\mathcal{H}_2$ state feedback gain F_{opt} is given by

$$F_{\mathrm{opt}} = -\left[\bar{\Phi}(X)\Sigma + B_2^T X B_2\right]^{-1} B_2^T X A \tag{49}$$

and the minimum performance cost for the resulting closed-loop system is given by

$$\min_F J_{H_2} = \mathrm{tr}\left\{B_1^T A^{T^{r_0-1}} X A^{r_0-1} B_1\right\} + \sum_{j=0}^{r_0-2} \mathrm{tr}\left\{B_1^T A^{T^j} C_1^T C_1 A^j B_1\right\}. \tag{50}$$

The proof follows that of Theorem 2 and is omitted.

Now we are ready to present the main result for the plant (1), (2) with input delays.

Theorem 5: Suppose the plant (1), (2) satisfies Assumptions 1, 3, and 4. If the MARE (48) has a mean-square stabilizing solution $X \geq 0$ (i.e., the plant (46), (47) is mean-square stabilizable), then the minimum performance cost J_{H_2} of the plant with an output feedback (3) is given by $\min_F J_{H_2}$ in (50) and the optimal parameters of the output feedback controller are given by (49) and

$$L = -A\Psi(C_2\Psi)^\dagger, \quad L_0 = -F_{\mathrm{opt}}\Psi(C_2\Psi)^\dagger. \tag{51}$$

Proof: In view of Lemmas 3, 8, and (13), the optimal output feedback design problem is to find F, L and L_0 in minimizing $1/\sigma_0^2$ subject to the constraints

$$\left\|\Gamma_e^{\frac{1}{2}} G_i\right\|_2^2 \frac{\sigma_i^2}{\gamma_i^2} < 1, \; i = 0, 1, \ldots, m, \text{ with } \gamma_0 = 1. \quad (52)$$

Under Assumptions 3 and 4, it is shown by Lemma 10 that for any given $\Gamma = \text{diag}\{\gamma_1^2, \ldots, \gamma_m^2\}$, the output feedback controller with the parameters given by (37) and (42) minimizes $\|\Gamma_e^{1/2} G_i\|_2^2$, $i = 0, 1 \ldots, m$, simultaneously. Moreover, it holds that

$$\begin{aligned} \left\|\Gamma_e^{\frac{1}{2}} G_0\right\|_2^2 \frac{\sigma_0^2}{\gamma_0^2} &= \sigma_0^2 \bar{\phi}_0(X_\Gamma) \\ \left\|\Gamma_e^{\frac{1}{2}} G_i\right\|_2^2 \frac{\sigma_i^2}{\gamma_i^2} &= \frac{\sigma_i^2}{\gamma_i^2} \bar{\phi}_i(X_\Gamma), \; i = 1, \ldots, m \end{aligned} \quad (53)$$

where X_Γ is the stabilizing solution to the DARE (19). Substituting (53) into (52) leads to the optimal output feedback design constraint below

$$\sigma_0^2 \bar{\phi}_0(X_\Gamma) < 1 \quad \text{and} \quad \frac{\sigma_i^2}{\gamma_i^2} \bar{\phi}_i(X_\Gamma) < 1, \; i = 1, \ldots, m.$$

Hence the optimal output feedback design is to find the minimum $1/\sigma_0^2$ under the constraint that the inequalities above hold. This problem is equivalent to the optimal state feedback design studied in Lemmas 11 and 12. Applying these lemmas, we obtain the optimal parameter F_{opt} for the output feedback and the minimum performance cost given by (49) and (50), respectively. The controller parameters L and L_0 are obtained by applying Lemma 10. ■

Remark 4: This theorem shows that, for a minimum phase plant (1), (2) with input delays, a *separation principle* holds in certain sense for mean-square optimal $\mathcal{H}_2$ output feedback design. More precisely, the optimal state feedback gain is obtained from the mean-square stabilizing solution to the MARE (48). The remaining parameters L_0 and L, which are related to an observer-like part in the feedback law, are obtained from the plant model and the optimal state feedback gain matrix. It must be noticed that, in this case, the optimal state feedback design is based on the plant (46), (47) which is generated from the original plant (1), (2) but involves the input delay information in its model. This is conceptually different to the separation principle for the conventional optimal $\mathcal{H}_2$ output feedback design.

Remark 5: In addition, it is shown in the preceding section that the optimal state feedback design is an EVP problem and a numerical algorithm for solving it is presented in Theorem 3. This means that the mean-square optimal $\mathcal{H}_2$ control via output feedback is numerically solvable for a minimum phase plant (1), (2) satisfying Assumptions 3 and 4. It is also explicitly explained by (50) that the delay size degrades the performance exponentially.

Now, we consider the case when the plant (1), (2) is minimum phase with relative degree one and without input delays. In this case, the auxiliary plant (46), (47) which is used to design the optimal state feedback gain F_{opt} in this optimal output feedback reduces to the plant (1), (2). This leads to the corollary below, straightforwardly.

Corollary 1: Suppose that the plant (1), (2) satisfies Assumptions 1, 3, and 4 with $r_0 = r_1 = \cdots = r_m = 1$. Suppose the plant is mean-square stabilizable (i.e. the MARE (22) has the mean-square stabilizing solution). Then, the minimum performance cost via output feedback (3) is given by

$$\inf_K J_{H_2} = \text{tr}\left\{B_1^T X B_1\right\}$$

and the optimal parameters of the output feedback controller K_{opt} are given by

$$\begin{aligned} F &= -\left[\Phi(X)\Sigma + B_2^T X B_2\right]^{-1} B^T X A \quad (54) \\ L &= -A[B_1 \quad B_2]\left(C_2[B_1 \quad B_2]\right)^{\dagger} \\ L_0 &= -F[B_1 \quad B_2]\left(C_2[B_1 \quad B_2]\right)^{\dagger} \quad (55) \end{aligned}$$

where $X \geq 0$ is the mean-square stabilizing solution to the MARE (22).

V. Conclusion

In this paper, we have solved the mean-square optimal $\mathcal{H}_2$ control design problem for linear systems subject to stochastic multiplicative uncertainties. The optimal mean-square state feedback is designed using the mean-square stabilizing solution to a modified algebraic Riccati equation (MARE) or, alternatively, a set of linear matrix inequalities. A necessary and sufficient condition is presented for the existence of the mean-square stabilizing solution to the MARE. In the output feedback design, a separation principle holds for minimum-phase plants with/without fixed input delays in the sense that the optimal controller is designed using the optimal mean-square state feedback, and two observer gains determined by this optimal state feedback and the parameters of the plant model. Given the close connection between stochastic multiplicative uncertainties and modeling of network transmission errors (such as packet dropouts), the results in this paper are expected to have important applications in networked control. Future research can focus on the general output feedback control design where the minimum-phase assumption is removed.

Appendix A
Proof of Lemma 2

Let $\{G_e(0), G_e(1), G_e(2), \ldots\}$ be the impulse response associated with the transfer function G_e in (5). Following the partition of G_e in (5), we write $G_e(k)$, $k = 0, 1, 2, \ldots$ as:

$$G_e(k) = \begin{bmatrix} g_{z0}(k) & g_{z1}(k) & \cdots & g_{zm}(k) \\ g_{10}(k) & g_{11}(k) & \cdots & g_{1m}(k) \\ \vdots & \vdots & \ddots & \vdots \\ g_{m0}(k) & g_{m1}(k) & \cdots & g_{mm}(k) \end{bmatrix}.$$

Note the fact that the nominal plant (4) has a strict proper transfer function from input $\{v, d, u\}$ to output $\{z, y\}$. The transfer function G_e of the plant with a proper feedback control law is also strict proper. Hence, it holds that $g_{ij}(0) = 0$, $i = 0, 1, \ldots, m$, $j = 0, \ldots, m$. When the initial state of the system in Fig. 2 is at rest, the output of the system is given by

$$\begin{bmatrix} z(k) \\ u(k) \end{bmatrix} = \sum_{i=1}^{k} G_e(i) \begin{bmatrix} v(k-i) \\ d(k-i) \end{bmatrix} \quad (56)$$

where $d(k-i)=\Delta(k-i)u(k-i)$. Write the output $z(k)$ is

$$z(k) = z_v(k) + \sum_{j=1}^{m} z_{d_j}(k) \tag{57}$$

where

$$z_v(k) = \sum_{i=1}^{k} G_{z0}(i)v(k-i)$$
$$z_{d_j}(k) = \sum_{i=1}^{k} G_{zj}(i)d_j(k-i),\ j=1,\ldots,m. \tag{58}$$

Under Assumption 1, we obtain that

$$E\left\{z^T(k)z(k)\right\} = E\left\{z_v^T(k)z_v(k)\right\} + \sum_{j=1}^{m} E\left\{z_{d_j}^T(k)z_{d_j}(k)\right\}. \tag{59}$$

Substituting (58) into $E\{z_{d_j}^T(k)z_{d_j}(k)\}$ leads to

$$\begin{aligned}E&\left\{z_{d_j}^T(k)z_{d_j}(k)\right\}\\ &= E\left\{\mathrm{tr}\left\{\sum_{i=1}^{k} G_{zj}(i)d_j(k-i)\left[\sum_{i=1}^{k} G_{zj}(i)d_j(k-i)\right]^T\right\}\right\}\\ &= \sum_{i=1}^{k} \mathrm{tr}\left[G_{zj}(i)G_{zj}^T(i)\right]\sigma_j^2 E\left\{u_j^2(k-i)\right\}.\end{aligned} \tag{60}$$

Taking summation on both sides of (60), we obtain that

$$\begin{aligned}\sum_{k=1}^{\bar{k}} &E\left\{z_{d_j}^T(k)z_{d_j}(k)\right\}\\ &= \sum_{k=1}^{\bar{k}}\sum_{i=1}^{k} \mathrm{tr}\left[G_{zj}(i)G_{zj}^T(i)\right]\sigma_j^2 E\left\{u_j^2(k-i)\right\}\\ &= \sigma_j^2 \sum_{i=1}^{\bar{k}}\sum_{k=i}^{\bar{k}} \mathrm{tr}\left[G_{zj}(i)G_{zj}^T(i)\right] E\left\{u_j^2(k-i)\right\}.\end{aligned}$$

Then, the averaged power of the signal z_{d_j} is given by

$$\begin{aligned}E&\left\{\|z_{d_j}\|_p^2\right\}\\ &= \lim_{\bar{k}\to\infty} \frac{1}{\bar{k}+1}\sum_{k=1}^{\bar{k}} E\left\{z_{d_j}^T(k)z_{d_j}(k)\right\}\\ &= \sigma_j^2 \lim_{\bar{k}\to\infty} \frac{1}{\bar{k}+1}\sum_{i=1}^{\bar{k}} \mathrm{tr}\left[G_{zj}(i)G_{zj}^T(i)\right]\sum_{k=i}^{\bar{k}} E\left\{u_j^2(k-i)\right\}\\ &= \sigma_j^2 \sum_{i=1}^{\infty} \mathrm{tr}\left[G_{zj}(i)G_{zj}^T(i)\right] E\left\{\|u_j\|_p^2\right\}.\end{aligned} \tag{61}$$

Noting the fact that $\|G_{zj}\|_2^2 = \sum_{i=1}^{\infty} \mathrm{tr}[G_{zj}(i)G_{zj}^T(i)]$, we rewrite (61) as below

$$E\left\{\|z_{d_j}\|_p^2\right\} = \sigma_j^2\|G_{zj}\|_2^2 E\left\{\|u_j\|_p^2\right\}. \tag{62}$$

Similarly, noting the stochastic properties of the signal v, we have that

$$E\left\{\|z_v\|_p^2\right\} = \|G_{z0}\|_2^2. \tag{63}$$

With (59), (62) and (63), we have that

$$E\left\{\|z\|_p^2\right\} = \|G_{z0}\|_2^2 + \sum_{i=1}^{m}\sigma_i^2\|G_{zi}\|_2^2 E\left\{\|u_i\|_p^2\right\}. \tag{64}$$

In the light of the discussion above, we can also obtain that

$$E\left\{\|u_j\|_p^2\right\} = \|G_{j0}\|_2^2 + \sum_{i=1}^{m}\sigma_i^2\|G_{ji}\|_2^2 E\left\{\|u_i\|_p^2\right\},\ j=1,\ldots,m$$

or

$$\begin{bmatrix} E\left\{\|u_1\|_p^2\right\} \\ \vdots \\ E\left\{\|u_m\|_p^2\right\}\end{bmatrix} = \begin{bmatrix} E\left\{\|G_{10}\|_p^2\right\} \\ \vdots \\ E\left\{\|G_{m0}\|_p^2\right\}\end{bmatrix} + \hat{G}\Sigma \begin{bmatrix} E\left\{\|u_1\|_p^2\right\} \\ \vdots \\ E\left\{\|u_m\|_p^2\right\}\end{bmatrix}. \tag{65}$$

From (64) and (65), we obtain (10).

Appendix B
Proof of Lemma 7

Following Lemma 5, we can see that the plant (1), (2) is mean-square stabilizable if and only if there exist F, $\Gamma > 0$ and $\sigma_0 > 0$ satisfying (17). Hence, it holds for some $\varepsilon > 0$ such that

$$\left\|\begin{bmatrix}\varepsilon I \\ C_1 \\ \Gamma^{\frac{1}{2}}F\end{bmatrix}(zI-A_F)^{-1}B_1\right\|_2^2 < \frac{1}{\sigma_0^2}$$
$$\left\|\begin{bmatrix}\varepsilon I \\ C_1 \\ \Gamma^{\frac{1}{2}}F\end{bmatrix}(zI-A_F)^{-1}B_{2i}\right\|_2^2 < \frac{\gamma_i^2}{\sigma_i^2},\ i=1,\ldots,m.$$

According to Lemma 6, the terms on the left-hand sides of the inequalities above are minimized simultaneously by

$$F_\varepsilon = -\left(\Gamma + B_2^T X_{\Gamma,\varepsilon}B_2\right)^{-1} B_2^T X_{\Gamma,\varepsilon}A$$

where $X_{\Gamma,\varepsilon} > 0$ is the stabilizing solution to the DARE

$$\begin{aligned}X_{\Gamma,\varepsilon} &= A^T X_{\Gamma,\varepsilon}A + C_1^T C_1 + \varepsilon^2 I\\ &\quad - A^T X_{\Gamma,\varepsilon}B_2\left(\Gamma + B_2^T X_{\Gamma,\varepsilon}B_2\right)^{-1} B_2^T X_{\Gamma,\varepsilon}A.\end{aligned}$$

Moreover, it holds that

$$\mathrm{tr}\left\{B_1^T X_{\Gamma,\varepsilon}B_1\right\} = \left\|\begin{bmatrix}\varepsilon I \\ C_1 \\ \Gamma^{\frac{1}{2}}F_\varepsilon\end{bmatrix}(zI-A_{F_\varepsilon})^{-1}B_1\right\|_2^2 < \frac{1}{\sigma_0^2}$$
$$B_{2i}^T X_{\Gamma,\varepsilon}B_{2i} = \left\|\begin{bmatrix}\varepsilon I \\ C_1 \\ \Gamma^{\frac{1}{2}}F_\varepsilon\end{bmatrix}(zI-A_{F_\varepsilon})^{-1}B_{2i}\right\|_2^2 < \frac{\gamma_i^2}{\sigma_i^2}$$
$$i=1,\ldots,m.$$

From F_ε and the DARE above, we obtain the following inequality:

$$X_{\Gamma,\varepsilon} > (A+B_2F_\varepsilon)^T X_{\Gamma,\varepsilon}(A+B_2F_\varepsilon) + C_1^T C_1 + F_\varepsilon^T \Gamma F_\varepsilon.$$

The inequalities (24), (25) follow from the above by setting ignoring the subscript ε.

Appendix C
Proof of Theorem 1

Sufficiency: Suppose the plant (1), (2) is mean-square stabilizable and (A, C_1) has no unobservable poles on the unit circle. Then, there exists a state feedback $u(k) = F_0 x(k)$ such that the performance cost J_{H_2} of the resulting closed-loop system is bounded by a constant $1/\sigma_0^2$. Moreover, for some $\Gamma_0 = \mathrm{diag}\{\gamma_{01}^2, \ldots, \gamma_{0m}^2\}$ and $\Gamma_{e0} = \mathrm{diag}\{I, \Gamma_0\}$, the inequalities in (17) hold, i.e.,

$$J_0 = \left\| \begin{bmatrix} C_1 \\ \Gamma_0^{\frac{1}{2}} F_0 \end{bmatrix} (zI - A_{F_0})^{-1} B_1 \right\|_2^2 < \frac{1}{\sigma_0^2} \tag{66}$$

$$J_i = \left\| \begin{bmatrix} C_1 \\ \Gamma_0^{\frac{1}{2}} F_0 \end{bmatrix} (zI - A_{F_0})^{-1} B_{2i} \right\|_2^2 < \frac{\gamma_{0i}^2}{\sigma_i^2}, \; i = 1, \ldots, m. \tag{67}$$

The function J_0 on the left-hand side of the inequality (66) is the square of the $\mathcal{H}_2$ norm of the auxiliary plant (18) with the state feedback law $u(k) = F_0 x(k)$, $\Gamma = \Gamma_0$ and $D = 0$. Applying Lemma 6, we design the optimal state feedback gain F_{Γ_0} to minimize J_0 as follows:

$$F_{\Gamma_0} = -\left(\Gamma_0 + B_2^T X_{\Gamma_0} B_2\right)^{-1} B_2^T X_{\Gamma_0} A \tag{68}$$

where X_{Γ_0} is the stabilizing solution to the following DARE:

$$\begin{aligned} X_{\Gamma_0} &= A^T X_{\Gamma_0} A + C_1^T C_1 \\ &\quad - A^T X_{\Gamma_0} B_2 \left(\Gamma_0 + B_2^T X_{\Gamma_0} B_2\right)^{-1} B_2^T X_{\Gamma_0} A. \end{aligned} \tag{69}$$

For the closed-loop system, the corresponding minimum performance cost of J_0 is given by

$$J_{0,\min}(\Gamma_0) = \mathrm{tr}\left\{B_1^T X_{\Gamma_0} B_1\right\}. \tag{70}$$

It must also be noted that the state feedback gain F_{Γ_0} in (68) minimizes the cost functions $J_1, \ldots, J_m$, simultaneously and the minimum values of these functions are

$$J_{i,\min}(\Gamma_0) = B_{2i}^T X_{\Gamma_0} B_{2i}, \; i = 1, \ldots, m. \tag{71}$$

Substituting $J_{0,\min}(\Gamma_0)$ and $J_{i,\min}(\Gamma_0)$ into (66) and (67) leads to

$$\sigma_0^2 \mathrm{tr}\left\{B_1^T X_{\Gamma_0} B_1\right\} < 1, \; \sigma_i^2 B_{2i}^T X_{\Gamma_0} B_{2i} < \gamma_{0i}^2, \; i = 1, \ldots, m. \tag{72}$$

Now, let $\Gamma_{e1} = \mathrm{diag}\{I, \Gamma_1\}$ with $\Gamma_1 = \mathrm{diag}\{\gamma_{11}^2, \ldots, \gamma_{1m}^2\}$ and $\gamma_{1i}^2 = \sigma_i^2 B_{2i}^T X_{\Gamma_0} B_{2i}$, $i = 1, \ldots, m$. It follows from (72) that $\Gamma_1 < \Gamma_0$. Similarly, consider the auxiliary plant (18) with $\Gamma = \Gamma_1$. According to Lemma 6, we design the optimal state feedback F_{Γ_1} for the plant using the stabilizing solution to the new DARE below

$$\begin{aligned} X_{\Gamma_1} &= A^T X_{\Gamma_1} A + C_1^T C_1 \\ &\quad - A^T X_{\Gamma_1} B_2 \left(\Gamma_1 + B_2^T X_{\Gamma_1} B_2\right)^{-1} B_2^T X_{\Gamma_1} A. \end{aligned} \tag{73}$$

Since $\Gamma_1 < \Gamma_0$, it holds that $X_{\Gamma_1} \leq X_{\Gamma_0}$. This leads to

$$\begin{aligned} &\sigma_0^2 \mathrm{tr}\left\{B_1^T X_{\Gamma_1} B_1\right\} < 1 \\ &\max\left\{\frac{\sigma_i^2}{\gamma_{1i}^2} B_{2i}^T X_{\Gamma_1} B_{2i}, \; i = 1, \ldots, m\right\} < 1. \end{aligned} \tag{74}$$

Subsequently, we can construct $\Gamma_2 < \Gamma_1$ with the positive semidefinite solution X_{Γ_1}. Repeating the process above, we obtain a sequence $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$ satisfying $\Gamma_0 > \Gamma_1 > \Gamma_2 > \cdots > 0$. The sequence of positive semidefinite solutions $\{X_{\Gamma_0}, X_{\Gamma_1}, X_{\Gamma_2}, \ldots\}$ to the DAREs associated with $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$ satisfies $X_{\Gamma_0} \geq X_{\Gamma_1} \geq \cdots \geq 0$, thus converges to a positive semidefinite matrix $\bar{X}$, and satisfies the equation below

$$\begin{aligned} \bar{\Gamma}_e &= \mathrm{diag}\{1, \bar{\Gamma}\} \\ \bar{\Gamma} &= \mathrm{diag}\left\{\sigma_1^2 B_{21}^T \bar{X} B_{21}, \ldots, \sigma_m^2 B_{2m}^T \bar{X} B_{2m}\right\}. \end{aligned} \tag{75}$$

The DARE associated with $\bar{\Gamma}$ becomes the MARE (22) and it holds that

$$\begin{aligned} &\sigma_0^2 \mathrm{tr}\left\{B_1^T \bar{X} B_1\right\} < 1 \\ &\max\left\{\frac{\sigma_i^2}{\bar{\gamma}_i^2} B_{2i}^T \bar{X} B_{2i}, \; i = 1, \ldots, m\right\} = 1. \end{aligned}$$

This yields

$$\inf_F J_{H_2} = \inf_{F,\sigma_0}\left\{\frac{1}{\sigma_0^2} : \text{subject to } (17)\right\} = \mathrm{tr}\left\{B_1^T \bar{X} B_1\right\}.$$

Moreover, it holds that $\rho(\hat{G}\Sigma) < 1$. Indeed, it holds when $C_1(zI - A_{\bar{F}})^{-1} B_{2i} \neq 0, i = 1, \ldots, m$.

In the case when $C_1(zI - A_{\bar{F}})^{-1} B_{2i} = 0$ for some $1 \leq i \leq m$, and there exists j, $1 \leq j \leq m$ such that $C_1(zI - A_{\bar{F}})^{-1} B_{2j} \neq 0$, we have that

$$\left\|\bar{\Gamma}^{\frac{1}{2}} \bar{F}(zI - A_{\bar{F}})^{-1} B_{2j}\right\|_2^2 < \frac{\bar{\gamma}_j^2}{\sigma_j^2}.$$

It is clear that there exists a $\hat{\gamma}_j < \bar{\gamma}_j$ satisfying

$$\left\|\hat{\Gamma}^{\frac{1}{2}} \bar{F}(zI - A_{\bar{F}})^{-1} B_{2j}\right\|_2^2 < \left\|\bar{\Gamma}^{\frac{1}{2}} \bar{F}(zI - A_{\bar{F}})^{-1} B_{2j}\right\|_2^2 < \frac{\hat{\gamma}_j^2}{\sigma_j^2} \tag{76}$$

where $\hat{\Gamma}$ is the matrix constructed from $\bar{\Gamma}$ by replacing the jth diagonal element with $\hat{\gamma}_j$. Subsequently, it holds that

$$\left\|\hat{\Gamma}^{\frac{1}{2}} \bar{F}(zI - A_{\bar{F}})^{-1} B_{2k}\right\|_2^2 < \frac{\bar{\gamma}_k^2}{\sigma_k^2}, \; k = 1, \ldots, m \text{ and } k \neq j. \tag{77}$$

The inequalities (76) and (77) result in $\rho(\hat{G}\Sigma) < 1$.

Note the fact that

$$C_1(zI - A_{\bar{F}})^{-1} B_2 = C_1(zI - A)^{-1} B_2 \left[I - \bar{F}(zI - A)^{-1} B_2\right]^{-1}.$$

For the case when $C_1(zI - A_{\bar{F}})^{-1} B_{2i} = 0$ for $i = 1, \ldots, m$, it holds that $C_1(zI - A)^{-1} B_2 = 0$. In this case, each of observable modes for (A, C_1) is not controllable for (A, B_2). The optimal design problem reduces to mean-square stabilization problem.

Necessity: Suppose the MARE (22) has a mean-square stabilizing solution X^*. By the definition of mean-square stabilizing solutions, the plant (1), (2) is mean-square stabilizable. We claim that (A, C_1) has no unobservable poles on the unit circle, following Lemma 6. Indeed, if this were not true, we take $\Gamma^* = \mathrm{diag}\{\sigma_1^2 B_{21}^T X^* B_{21}, \ldots, \sigma_m^2 B_{21}^T X^* B_{21}\}$. Then the DARE (19) would have no stabilizing solution for this Γ^*, which, according to Lemma 6, would contradict to the assumption that the MARE has a mean-square stabilizing solution.

Up to now, we see that for any sequence $\{X_{\Gamma_0}, X_{\Gamma_1}, X_{\Gamma_2}, \ldots\}$ generated by an initial Γ_0, it is convergent to a minimum X which is a mean-square stabilizing solution to the MARE (22). Next, we show that this minimum X is the unique mean-square solution and the largest solution to the MARE.

It follows from Lemmas 5 and 7 that the mean-square optimal $\mathcal{H}_2$ problem amounts to finding the supremum σ_0 such that inequalities (24) and (25) hold. Applying Schur Complement, we write these inequalities as follows:

$$\begin{bmatrix} X_\Gamma & (A+B_2F)^T & F^T & C_1^T \\ (A+B_2F) & X_\Gamma^{-1} & 0 & 0 \\ F & 0 & \Gamma^{-1} & 0 \\ C_1 & 0 & 0 & I \end{bmatrix} > 0$$

$$\mathrm{tr}\{Z\} < \frac{1}{\sigma_0^2}, \quad \begin{bmatrix} Z & B_1^T \\ B_1 & X_\Gamma^{-1} \end{bmatrix} > 0$$

$$\begin{bmatrix} \frac{\gamma_i^2}{\sigma_i^2} & B_{2i}^T \\ B_{2i} & X_\Gamma^{-1} \end{bmatrix} > 0, \; i = 1, \ldots, m.$$

Letting $S = X_\Gamma^{-1}$, $V = FX_\Gamma^{-1}$, $\Pi = \Gamma^{-1}$ and pre- and post-multiplying the matrices

$$\begin{bmatrix} X_\Gamma^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}; \quad \begin{bmatrix} \gamma_i^{-2} & 0 \\ 0 & I \end{bmatrix}$$

on both sides of the first one and last two inequalities above, respectively, leads to the following inequalities:

$$\begin{bmatrix} S & SA^T + V^T B_2^T & V^T & SC_1^T \\ AS + B_2 V & S & 0 & 0 \\ V & 0 & \Pi & 0 \\ C_1 S & 0 & 0 & I \end{bmatrix} > 0$$

$$\begin{bmatrix} Z & B_1^T \\ B_1 & S \end{bmatrix} > 0, \; \begin{bmatrix} \frac{\pi_i}{\sigma_i^2} & \pi_i B_{2i}^T \\ \pi_i B_{2i} & S \end{bmatrix} > 0 \quad i = 1, \ldots, m$$

$$S > 0, \quad Z > 0, \quad \Pi = \mathrm{diag}\{\pi_1, \pi_2, \ldots, \pi_m\} > 0.$$

The optimal state feedback design is to minimize $\mathrm{tr}\{Z\}$ subject to the inequality constraints above. This is an EVP problem which has a global optimal solution (see [3]). Denote the optimal solution to this problem by S_{opt} and V_{opt}. Hence, all stabilizing solution sequences $\{X_{\Gamma_0}, X_{\Gamma_1}, X_{\Gamma_2}, \ldots\}$ to the DAREs converge to $S_{\mathrm{opt}}^\dagger$. Since these stabilizing solutions are the unique stabilizing solutions and largest positive semidefinite solutions to the DAREs, respectively, $S_{\mathrm{opt}}^\dagger$ is the unique mean-square stabilizing solution and the largest positive semidefinite solution to the MARE (22). Otherwise, there exists a stabilizing solution sequence to the DAREs which converges to another point, i.e., S_{opt} is not a global optimal solution, which is not possible.

Appendix D
Proof of Theorem 2

Sufficiency: Let X be the largest positive semidefinite solution to the MARE (22). Denote $\Gamma = \mathrm{diag}\{\sigma_1^2 B_{21}^T X B_{21}, \ldots, \sigma_m^2 B_{2m}^T X B_{2m}\}$. Then, X is a solution to the DARE

$$X = A^T X A - A^T X B_2 \left(\Gamma + B_2^T X B_2\right)^{-1} B_2^T X A + C_1^T C_1. \tag{78}$$

In fact, X is the largest solution to this DARE, otherwise we can find the other solution $\bar{X} > X$ to the MARE (22), which contradicts to the assumption on X. According to Lemma 6, X is the stabilizing solution to (78), the nominal closed-loop system G_e with this state feedback gain $F = -(\Gamma + B_2^T X B_2)^{-1} B_2^T X A$ is stable. Moreover, it holds for some $\sigma_0 > 0$ that

$$\left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F)^{-1} B_1 \right\|_2^2 < \frac{1}{\sigma_0^2}$$

$$\left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F)^{-1} B_{2i} \right\|_2^2 = \frac{\gamma_i^2}{\sigma_i^2}, \; i = 1, \ldots, m. \tag{79}$$

According to Lemma 4, it means that $\rho(\hat{G}_e \Sigma_e) \leq 1$. Subsequently, from Lemma 3, we obtain that the performance cost J_{H_2} of the plant (1), (2) with this state feedback law is bounded and $\rho(\hat{G}\Sigma) \leq 1$. Moreover, noting (10), one can see that $\rho(\hat{G}\Sigma) < 1$, i.e. the plant is man-square stabilizable, otherwise the performance cost J_{H_2} of the closed-loop system is unbounded.

Necessity: Notice the assumption that (A, C_1) has no unobservable poles on the unit circle. It follows from Theorem 1 that if the plant (1), (2) is mean-square stabilizable, the the MARE (22) has the unique mean-square stabilizing solution. This solution is the largest positive semidefinite to the MARE.

The proof for the optimal state feedback design and minimum performance cost follows the proof of Theorem 1.

Appendix E
Proof of Theorem 4

Consider the plant (29) and note that $A_i = B_{si} C_{si}$, $i = 1, \ldots, m_s$. Let $y_{si}(k) = C_{si} x(k)$, $d_{si}(k) = \Delta_{si}(k) y_{si}(k)$, $\tilde{y}(k) = [y_{s1}(k), \ldots, y_{sm_s}(k), u_1(k), \ldots, u_m(k)]^T$ and $\tilde{d}(k) = [d_{s1}(k), \ldots, d_{sm_s}(k), d_1(k), \ldots, d_m(k)]^T$. The closed-loop system of the plant with a state feedback $u(k) = Fx(k)$ is written as

$$x(k+1) = A_F x(k) + B_1 v(k) + \tilde{B}_2 \tilde{d}(k) \tag{80}$$

$$z(k) = C_1 x(k)$$

$$\tilde{y}(k) = \begin{bmatrix} C_s \\ F \end{bmatrix} x(k)$$

$$\tilde{d}(k) = \tilde{\Delta}(k) \tilde{y}(k) \tag{81}$$

where $A_F = A + B_2 F$, $\tilde{B}_2 = [B_{s1} \; \cdots \; B_{sm_s} \; B_{21} \; \cdots \; B_{2m}]$, $C_s = [C_{s1}^T \; \cdots \; C_{sm_s}^T]^T$ and

$$\tilde{\Delta} = \mathrm{diag}\{\Delta_{s1}, \ldots, \Delta_{sm_s}, \Delta_1, \ldots, \Delta_m\}.$$

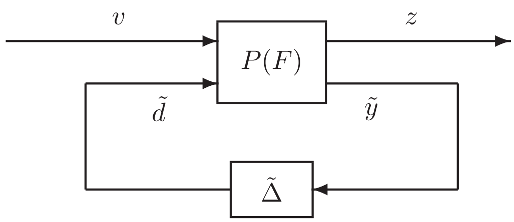

Fig. 5. The closed-loop system with multiplicative noises via a state feedback.

The closed-loop system (80) and (81) is depicted in Fig. 5 where $P(F)$ is the nominal closed-loop system (80). Denote the transfer function of this nominal closed-loop system by $\tilde{G}_e$ which is given below

$$\tilde{G}_e = \begin{bmatrix} C_1 \\ C_{s1} \\ \vdots \\ F_m \end{bmatrix} (zI - A_F)^{-1} \begin{bmatrix} B_1 & B_{s1} & \cdots & B_{2m} \end{bmatrix}.$$

Then define $\hat{\tilde{G}}_e$ by following (11). Let $\tilde{\Sigma}_e = \text{diag}\{\sigma_0^2, \sigma_{s1}^2, \ldots, \sigma_m^2\}$. According to Lemma 3, the problem of mean-square optimal $\mathcal{H}_2$ control via state feedback is to find the state feedback gain F in minimizing $1/\sigma_0^2$ with the constraint $\rho(\hat{\tilde{G}}_e \tilde{\Sigma}_e) < 1$. Let $\tilde{\Gamma}_e = \text{diag}\{I, \Gamma_s, \Gamma\}$, $\Gamma_s = \text{diag}\{\gamma_{s1}^2, \ldots, \gamma_{sm_s}^2\}$ and $\Gamma = \text{diag}\{\gamma_1^2, \ldots, \gamma_m^2\}$. It follows from Lemma 4 that the inequality $\rho(\hat{\tilde{G}}_e \tilde{\Sigma}_e) < 1$ holds if and only if it holds for some $\tilde{\Gamma}_e$ that

$$\left\| \begin{bmatrix} C_1 \\ \Gamma_s^{\frac{1}{2}} C_s \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F)^{-1} B_1 \right\|_2^2 < \frac{1}{\sigma_0^2}$$
$$\left\| \begin{bmatrix} C_1 \\ \Gamma_s^{\frac{1}{2}} C_s \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F)^{-1} B_{si} \right\|_2^2 < \frac{\gamma_{si}^2}{\sigma_{si}^2}, \quad i = 1, \ldots, m_s \quad (82)$$
$$\left\| \begin{bmatrix} C_1 \\ \Gamma_s^{\frac{1}{2}} C_s \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A_F)^{-1} B_{2i} \right\|_2^2 < \frac{\gamma_i^2}{\sigma_i^2}, \quad i = 1, \ldots, m. \quad (83)$$

Following the argument used in the proof of Theorem 1, we obtain that, for any given Γ_s and Γ, the terms on the left-hand sides of the inequalities in (82) and (83) have a common minimizer F_Γ which is designed by the stabilizing solution to the DARE

$$X_\Gamma = A^T X_\Gamma A + C_1^T C_1 + C_s^T \Gamma_s C_s - A^T X_\Gamma B_2 \left(\Gamma + B_2^T X_\Gamma B_2\right)^{-1} B_2^T X_\Gamma A. \quad (84)$$

With this common minimizer F_Γ, the inequalities in (82) and (83) are rewritten as

$$\text{tr}\left\{B_1^T X_\Gamma B_1\right\} < \frac{1}{\sigma_0^2}$$
$$B_{si}^T X_\Gamma B_{si} < \frac{\gamma_{si}^2}{\sigma_{si}^2}, \; i = 1, \ldots, m_s$$
$$B_{2i}^T X_\Gamma B_{2i} < \frac{\gamma_i^2}{\sigma_i^2}, \; i = 1, \ldots, m.$$

In the light of the proof for Theorem 1, we construct a sequence of $\tilde{\Gamma}_e$ and denote it by $\{\tilde{\Gamma}_{e0}, \tilde{\Gamma}_{e1}, \tilde{\Gamma}_{e2}, \ldots\}$. It is clear that the DARE (84) associated with $\tilde{\Gamma}_{ek}$, $k = 0, 1, 2, \ldots$, has a stabilizing solution if and only if (A, B_2) is stabilizable and $(A, [C_1^T, C_{s1}^T, \ldots, C_{sm_s}^T]^T)$ has no unobservable poles on the unit circle. The sequence of $\tilde{\Gamma}_e$ is convergent if the plant is mean-square stabilizable. The sequence of the DARE associated with $\tilde{\Gamma}_e$ converges to the MARE (30). By the arguments used in the proofs of Theorems 1–3, we can see that this theorem is true.

Appendix F
Proof of Lemma 10

It is shown in Lemma 8 that the square of the minimum $\mathcal{H}_2$ norm of the auxiliary plant (34) with the output feedback controller (36) is given by (39). Since the plant (34) satisfies Assumptions 3 and 4, so does the plant (1). By the stabilizing solution Y_Γ to the DARE (35) given in Lemma 9, we have

$$C_2 Y_\Gamma C_2^T = C_2 \Psi \text{diag}\left\{\sigma_0^2 \lambda_0^2 \gamma_0^{-2}, \ldots, \sigma_m^2 \lambda_m^2 \gamma_m^{-2}\right\} \Psi^T C_2^T$$
$$Y_\Gamma C_2^T = \Psi \text{diag}\left\{\sigma_0^2 \lambda_0^2 \gamma_0^{-2}, \ldots, \sigma_m^2 \lambda_m^2 \gamma_m^{-2}\right\} \Psi^T C_2^T \quad (85)$$

where Ψ is given in (32). Therefore, it holds that

$$\left(C_2 \Psi \text{diag}\left\{\sigma_0 \lambda_0 \gamma_0^{-1}, \ldots, \sigma_m \lambda_m \gamma_m^{-1}\right\}\right)^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger = \left(C_2 \Psi \text{diag}\left\{\sigma_0 \lambda_0 \gamma_0^{-1}, \ldots, \sigma_m \lambda_m \gamma_m^{-1}\right\}\right)^\dagger.$$

This leads to

$$\text{tr}\left\{Z F_\Gamma Y_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger C_2 Y_\Gamma F_\Gamma^T Z\right\} = \text{tr}\left\{Z F_\Gamma \Psi \text{diag}\left\{\sigma_0^2 \lambda_0^2 \gamma_0^{-2}, \ldots, \sigma_m^2 \lambda_m^2 \gamma_m^{-2}\right\} \Psi^T F_\Gamma^T Z\right\}.$$

Subsequently, we obtain that

$$\begin{aligned} &\text{tr}\{Z F_\Gamma Y_\Gamma F_\Gamma^T Z\} - \text{tr}\left\{Z F_\Gamma Y_\Gamma C_2^T \left(C_2 Y_\Gamma C_2^T\right)^\dagger C_2 Y_\Gamma F_\Gamma^T Z\right\} \\ &= \text{tr}\left\{Z F_\Gamma \left(\sum_{j=0}^{r_0-2} A^j B_1 \sigma_0^2 \lambda_0^2 \gamma_0^{-2} B_1^T A^{Tj} + \sum_{i=1}^{m} \sum_{j=0}^{r_i-2} A^j B_{2i} \sigma_i^2 \lambda_i^2 \gamma_i^{-2} B_{2i}^T A^{Tj}\right) F_\Gamma^T Z\right\} \\ &= \sum_{j=0}^{r_0-2} \sigma_0^2 \lambda_0^2 \gamma_0^{-2} \text{tr}\left\{B_1^T A^{Tj} F_\Gamma^T Z Z^T F_\Gamma A^j B_1\right\} \\ &\quad + \sum_{i=1}^{m} \sum_{j=0}^{r_i-2} \sigma_i^2 \lambda_i^2 \gamma_i^{-2} \text{tr}\left\{B_{2i}^T A^{Tj} F_\Gamma^T Z Z^T F_\Gamma A^j B_{2i}\right\}. \end{aligned} \quad (86)$$

It follows from (19) and (37) that:

$$F_\Gamma^T Z Z^T F_\Gamma = A^T X_\Gamma A - X_\Gamma + C_1^T C_1. \quad (87)$$

Substituting (86) and (87) into yields (39) leads to (43).

The controller parameters L_Γ and $L_{\Gamma 0}$ in (42) are obtained by applying (40) and (85) to (38).

Appendix G
Proof of Lemma 11

The proof is similar to that of Theorem 1. Here, a sketch of the proof is presented due to the space limit. Following

Lemma 5, we see that the problem of mean-square optimal $\mathcal{H}_2$ control via state feedback for the plant (46), (47) is to find $F_{\rm opt} = \arg\inf_F \inf_{\sigma_0}(1/\sigma_0^2)$ subject to

$$\left\|\Gamma_e^{\frac{1}{2}} G_0\right\|_2^2 = \left\|\Gamma_e^{\frac{1}{2}} \begin{bmatrix} C_1 \\ F \end{bmatrix} (zI - A_F)^{-1} A^{r_0} B_1 + \begin{bmatrix} D_1 \\ 0 \end{bmatrix}\right\|_2^2 < \frac{1}{\sigma_0^2}$$
$$\left\|\Gamma_e^{\frac{1}{2}} G_i\right\|_2^2 = \left\|\Gamma_e^{\frac{1}{2}} \begin{bmatrix} C_1 \\ F \end{bmatrix} (zI - A_F)^{-1} A^{r_i} B_{2i} + \begin{bmatrix} 0 \\ D_{2i} \end{bmatrix}\right\|_2^2 < \frac{\gamma_i^2}{\sigma_i^2}$$
$$i = 1, \ldots, m$$

where $A_F = A + B_2 F$ and D_{2i} is the i-th column of D_2.

It is verified by Lemma 6 that F_Γ given in (37) minimizes $\|\Gamma_e G_i\|_2^2$, $i = 0, 1, \ldots, m$, simultaneously. Moreover, these minimum costs are given by

$$\|\Gamma_e G_0\|_2^2 = \bar{\phi}_0(X_\Gamma), \quad \|\Gamma_e G_i\|_2^2 = \bar{\phi}_i(X_\Gamma),\ i = 1, \ldots, m.$$

The design constraints for the mean-square optimal $\mathcal{H}_2$ control problem are written below

$$\sigma_0^2 \bar{\phi}_0(X_\Gamma) < 1 \quad \text{and} \quad \frac{\sigma_i^2}{\gamma_i^2} \bar{\phi}_i(X_\Gamma) < 1,\ i = 1, \ldots, m.$$

Then, similar to the proof of Theorem 1, we construct a sequence $\{\Gamma_k, k = 0, 1, 2, \ldots, \infty\}$ such that $\Gamma_k = \Phi(X_{\Gamma_{k-1}})\Sigma$ where $X_{\Gamma_{k-1}}$ is the stabilizing solution to the DARE (19) associated with Γ_{k-1}. According to the arguments used in the proof of Theorem 1, the sequences $\{\Gamma_k, k = 0, 1, 2, \ldots, \infty\}$ and $\{X_{\Gamma_k}, k = 0, 1, 2, \ldots, \infty\}$ are convergent. The MARE (48) has a mean-square stabilizing solution X. By the same argument used in the proof of of Theorem 1, X is the unique mean-square stabilizing solution to the MARE (48) and the largest positive semidefinite solution to this MARE.

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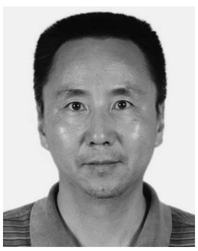

Weizhou Su (S'96–M'00–SM'05) received the B.Eng. and M.Eng. degrees in automatic control engineering from the Southeast University, Nanjing, Jiangsu, China, in 1983 and 1986, respectively, the M.Eng. degree in electrical and electronic engineering from Nanyang Technological University, in 1996, and the PhD. degree in electrical engineering from the University of Newcastle, Newcastle, NSW, Australia, in 2000.

From 2000 to 2004, he held research positions in the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Hong Kong, China; the School of QMMS, University of Western Sydney, Sydney, Australia, respectively. He joined the School of Automation Science and Engineering, South China University of Technology, Guangzhou, China, in 2004 where he is currently a full Professor. His research interests include networked control system, robust control, fundamental performance limitation of feedback control, and signal processing.

Jie Chen (S'87–M'89–SM'98–F'07) is a Chair Professor in the Department of Electronic Engineering, City University of Hong Kong, Hong Kong, China. He received the B.S. degree in aerospace engineering from Northwestern Polytechnic University, Xian, China, in 1982, the M.S.E. degree in electrical engineering, the M.A. degree in mathematics, and the Ph.D. degree in electrical engineering, all from The University of Michigan, Ann Arbor, MI, USA, in 1985, 1987, and 1990, respectively.

Prior to joining City University, he was with School of Aerospace Engineering and School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, Georgia from 1990 to 1993, and with University of California, Riverside, CA, USA, from 1994 to 2014, where he was a Professor and served as Professor and Chair for the Department of Electrical Engineering from 2001 to 2006. His main research interests are in the areas of linear multivariable systems theory, system identification, robust control, optimization, time-delay systems, networked control, and multi-agent systems. He served as an Associate Editor for *Automatica*. He was also the founding Editor-in-Chief for *Journal of Control Science and Engineering*. He serves as an Associate Editor for SIAM *Journal on Control and Optimization*.

An elected Fellow of AAAS, Fellow of IFAC and a Yangtze Scholar/Chair Professor of China, Dr. Chen was a recipient of 1996 US National Science Foundation CAREER Award, 2004 SICE International Award, and 2006 Natural Science Foundation of China Outstanding Overseas Young Scholar Award. He served on a number of journal editorial boards, as an Associate Editor, a Guest Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and a Guest Editor for IEEE Control Systems Magazine. He is currently an IEEE Control Systems Society Distinguished Lecturer.

Minyue Fu (F'03) received the B.Sc. degree in electrical engineering from the University of Science and Technology of China, Hefei, China, in 1982, and the M.S. and Ph.D. degrees in electrical engineering from the University of Wisconsin-Madison, Madison, WI, USA, in 1983 and 1987, respectively.

From 1987 to 1989, he was an Assistant Professor in the Department of Electrical and Computer Engineering, Wayne State University, Detroit, MI, USA. He joined the Department of Electrical and Computer Engineering at the University of Newcastle, Australia, in 1989. Currently, he is a Chair Professor of Electrical Engineering. He has been a Visiting Associate Professor at the University of Iowa, USA, Nanyang Technological University, Singapore and Tokyo University, Tokyo, Japan. He has held a Chang Jiang Visiting Professorship at Shandong University, Jinan, China, a Qian-ren Professorship at Zhejiang University, China, and a Qian-ren Professorship at Guangdong University of Technology, China. His current research projects include networked control systems, smart electricity networks, and super-precision positioning control systems. His main research interests include control systems, signal processing, and communications. Has been an Associate Editor for *Automatica* and the *Journal of Optimization and Engineering*.

Dr. Fu has been an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and the IEEE TRANSACTIONS ON SIGNAL PROCESSING.

Tian Qi (S'08–M'09) was born in Chaoyang, Liaoning Province, China, in 1981. She received the B.S. degree in automation from Hangzhou Dianzi University, Hangzhou, China, in 2003, the M.S. degree and the Ph.D. degree in control theory and control engineering from South China University of Technology, Guangzhou, China, in 2006 and 2009, respectively.

She joined the South China University of Technology in 2009, where she is currently a Lecturer. From May 2011 to August 2011, she was with the Department of Electronic Engineering, City University of Hong Kong, as a postdoctoral researcher. From January 2012 to December 2012, she was a Research Associate in the Department of Electrical and Computer Engineering, Hong Kong University of Science and Technology. Her main research interests are in the areas of networked control, multi-agent systems, time-delay systems, robust control, and optimization.