Optimal Stabilization Control for Discrete-Time Mean-Field Stochastic Systems

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***Abstract*—This paper will investigate the stabilization and optimal linear quadratic (LQ) control problems for infinite horizon discrete-time mean-field systems. Unlike the previous works, for the first time, the necessary and sufficient stabilization conditions are explored under mild conditions, and the optimal LQ controller for infinite horizon is designed with a coupled algebraic Riccati equation (ARE). More specifically, we show that under the exact detectability (exact observability) assumption, the mean-field system is stabilizable in the mean square sense with the optimal controller if and only if a coupled ARE has a unique positive semidefinite (positive definite) solution. The presented results are parallel to the classical results for the standard LQ control.**

***Index Terms*—Algebraic Riccati equation (ARE), mean-field LQ (linear quadratic) control, optimal controller, stabilizing controller.**

I. Introduction

IN THIS paper, the optimal control and stabilization problems for infinite horizon discrete-time mean-field systems are considered. Different to the classical stochastic control problem, the system state is described by a controlled mean-field stochastic difference/differential equation (MF-SDE), which was first studied in [1] and [2]. Since then, significant contributions have been made in studying MF-SDEs and related topics by many researchers. See, for example, [3]–[7] and the references therein. Inspired by the progress made on MF-SDEs, the study of mean-field stochastic control has been a hot research topic since 1950s, which combines the mean-field theory with stochastic control problems. The recent development in mean-field control problems can be found in [8]–[15], [17], [18] and references therein.

In particular, Yong[20] first studied the finite horizon mean-field linear quadratic (LQ) control problem, a necessary and sufficient solvability condition of the problem was presented in terms of operator type criteria. Furthermore, the continuous-time mean-field LQ control and stabilization problem for the infinite horizon was investigated in [21], the equivalence of several notions of stability for mean-field systems was established. It was shown that the optimal mean-field LQ controller for the infinite horizon case can be presented via the maximal solution to the AREs.

For the discrete-time mean-field LQ control problem, a solvability condition for the finite horizon discrete-time mean-field LQ control problem was presented in terms of operator type conditions in [5]. Furthermore, under some conditions, the explicit optimal controller was derived by using the matrix dynamical optimization method. Besides, for the infinite time case, the equivalence of some stabilizability notions for mean-field SDEs was studied and the solution to the associated ARE was investigated in [27]. Moreover, Ni *et al.* [17], [18] studied the indefinite mean-field LQ control problems.

However, it should be stressed that the stabilization problem for mean-field systems remains to be further investigated, although a lot of progress has been made as mentioned in [5], [17], [18], [27], and references therein. The main reasons are as follows: 1) Stabilization properties of the optimal controllers have not been studied, or the necessary and sufficient stabilization conditions for mean-field systems have not been provided in the literature; 2) Previous works on optimal control design rely on strong assumptions and it is crucial to relax the assumptions. The following fundamental question has not been answered: Under what conditions can the closed-loop mean-field system with the optimal controller be mean-square stabilizable.

In this paper, we aim to provide a thorough solution to the problems of optimal LQ control and stabilization for infinite horizon discrete time mean-field systems. As the preliminaries, the results of finite horizon mean-field LQ control are presented, and the necessary and sufficient solvability condition of finite horizon case is given in an explicit expression. By doing the convergence analysis on the coupled Riccati equation for the finite horizon case, the infinite horizon LQ controller and the stabilization conditions (necessary and sufficient) are derived. In addition, the Lyapunov function for stabilization is expressed with the optimal cost function. Two stabilization results are obtained under two different assumptions. One is the assumption of exact detectability, under which it is shown that the mean-field system is mean square stabilizable if and only if the

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coupled algebraic Riccati equation (ARE) admits a unique positive semidefinite solution. The other one is the assumption of exact observability, under which it is shown that the mean-field system is mean square stabilizable if and only if the coupled ARE has a unique positive definite solution.

It should be pointed out the stabilization conditions will be explored under the assumption of $R \geq 0$ and $R + \bar{R} \geq 0$. This weakens the assumption of $R > 0$, even for the classical LQ case, see [24], [29], and [30]. On one hand, the control weighting matrices R are not necessary positive definite, especially for finance applications, like mean-variance portfolio selection problem [16]; on the other hand, the relaxation is significant and more general for mathematical reasons, which includes $R > 0$ as a special case.

The remainder of this paper is organized as follows. Section II provides the problem formulation and the preliminary results of finite horizon mean-field LQ control. In Section III, main results of the infinite horizon optimal control and stabilization problems are presented. Numerical examples are given in Section IV to illustrate main results of this paper. Some concluding remarks are given in Section V. Finally, relevant proofs are detailed in Appendices.

Notations: I_n means the unit matrix with rank n; superscript $'$ denotes the transpose of a matrix. Real symmetric matrix $A > 0$ (or ≥ 0) implies that A is strictly positive definite (or positive semidefinite). $\mathcal{R}^n$ signifies the n-dimensional Euclidean space. B^{-1} is used to indicate the inverse of real matrix B, and $C^\dagger$ means the Moore–Penrose inverse of C. $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k\geq 0}\}$ represents a complete probability space, with $\mathcal{F}_k$ the σ-algebra generated by $\{x_0, w_0, \ldots, w_k\}$. $E[\cdot|\mathcal{F}_k]$ means the conditional expectation with respect to $\mathcal{F}_k$ and $\mathcal{F}_{-1}$ is understood as $\{\emptyset, \Omega\}$. a.s. denotes in the "almost surely" sense.

Definition 1: For random vector x, if $E(x'x) = 0$, we call it zero random vector, i.e., $x = 0$, a.s.

II. Problem Formulation and Preliminaries

A. Problem Formulation

We consider the following discrete-time mean-field system:

$$\begin{cases} x_{k+1} = (Ax_k + \bar{A}Ex_k + Bu_k + \bar{B}Eu_k) \\ \qquad\quad +(Cx_k + \bar{C}Ex_k + Du_k + \bar{D}Eu_k)w_k, \\ x_0 = \xi \end{cases} \tag{1}$$

where $A, \bar{A}, C, \bar{C} \in \mathcal{R}^{n\times n}$, and $B, \bar{B}, D, \bar{D} \in \mathcal{R}^{n\times m}$, all the coefficient matrices are given deterministic. $x_k \in \mathcal{R}^n$ is the state process and $u_k \in \mathcal{R}^m$ is the control process. The system noise $\{w_k\}_{k=0}^N$ is scalar valued random white noise with zero mean and variance σ^2, defined on a complete probability space $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k\geq 0}\}$. E is the expectation taken over the noise $\{w_k\}_{k=0}^N$ and initial state ξ. $\{\mathcal{F}_k\}_{k\geq 0}$ is the natural filtration generated by $\{\xi, w_0, \ldots, w_k\}$ augmented by all the $\mathcal{P}$-null sets.

By taking expectations on both sides of (1), we obtain

$$Ex_{k+1} = (A + \bar{A})Ex_k + (B + \bar{B})Eu_k, Ex_0 = E\xi. \tag{2}$$

In this paper, the infinite horizon mean-field stochastic LQ control problem is solved. Besides, the necessary and sufficient stabilization conditions for mean-field systems are investigated. Unlike the finite horizon case, the infinite horizon solution also needs to guarantee the closed-loop stability.

The associated cost function is given by

$$J = \sum_{k=0}^{\infty} E[x_k'Qx_k + Ex_k'\bar{Q}Ex_k + u_k'Ru_k + Eu_k'\bar{R}Eu_k] \tag{3}$$

where $Q, \bar{Q}, R, \bar{R}$ are deterministic symmetric weighting matrices with appropriate dimensions.

The admissible control set of the infinite horizon case is presented as follows:

$$\mathcal{U}_\infty = \{u_0, u_1 \cdots | u_k \in \mathcal{R}^m, u_k \text{ is } \mathcal{F}_{k-1}\text{-measurable}, \text{ and } \sum_{k=0}^{\infty} E(u_k'u_k) < +\infty\}. \tag{4}$$

Throughout this paper, the weighting matrices in the cost function are required to satisfy:

Assumption 1: $Q \geq 0, Q + \bar{Q} \geq 0$ and $R \geq 0, R + \bar{R} \geq 0$.

Remark 1: In previous works like [24], [29], and [30], $R > 0$ was required to solve the stabilization problems. While in this paper, R is only required to be a positive semidefinite.

The following notions of stability and stabilization are introduced.

Definition 2: System (1) with $u_k = 0$ is called asymptotically mean square stable if for any initial values x_0, there holds

$$\lim_{k\to+\infty} E(x_k'x_k) = 0.$$

Definition 3: System (1) is called mean square stabilizable if there exists a $\mathcal{F}_{k-1}$-measurable linear controller $u \in \mathcal{U}_\infty$, such that for any initial state x_0, the closed loop of system (1) is asymptotically mean square stable.

Following from [25], [29], and [30], the definitions of exact observability and exact detectability are respectively given below.

Definition 4: We consider the following mean-field system:

$$\begin{cases} x_{k+1} = (Ax_k + \bar{A}Ex_k) + (Cx_k + \bar{C}Ex_k)w_k, \\ Y_k = \mathcal{Q}^{1/2}\mathbb{X}_k \end{cases} \tag{5}$$

where $\mathcal{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q+\bar{Q} \end{bmatrix}$ and $\mathbb{X}_k = \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix}$. System (5) is said to be exact observable, if for any $N \geq 0$

$$Y_k = 0 \quad \forall 0 \leq k \leq N \Rightarrow x_0 = 0$$

where the meaning of $Y_k = 0$ and $x_0 = 0$ are given in Definition 1.

For simplicity, we rewrite system (5) as $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$.

Definition 5: The system $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ in (5) is said to be exact detectable, if for any $N \geq 0$

$$Y_k = 0 \quad \forall 0 \leq k \leq N, \Rightarrow \lim_{k\to+\infty} E(x_k'x_k) = 0.$$

Remark 2: Definition 5 gives a different definition of "exact detectability" from the one given in previous work [27]. In fact, Ni *et al.* [27] considered the system with different observation y_k

$$\begin{cases} x_{k+1} = (Ax_k + \bar{A}Ex_k) + (Cx_k + \bar{C}Ex_k)w_k, \\ y_k = Qx_k + \bar{Q}Ex_k. \end{cases} \tag{6}$$

Obviously, it is different to the definition given in this paper. In [27], system (6) was called "exact detectable," if for any $N \geq 0$

$$y_k = 0 \quad \forall 0 \leq k \leq N \Rightarrow \lim_{k\to+\infty} E(x_k' x_k) = 0.$$

Remark 3: The exact detectability made in Definition 5 is weaker than the exact detectability made in [27]. In fact, noting that $Y_k = \mathcal{Q}^{1/2}\mathbb{X}_k = 0$ implies

$$\begin{bmatrix} Q & 0 \\ 0 & Q+\bar{Q} \end{bmatrix}^{1/2} \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix} = 0. \tag{7}$$

Equation (7) indicates that

$$Q(x_k - Ex_k) = 0, \text{ and } (Q+\bar{Q})Ex_k = 0 \tag{8}$$

and thus, $Qx_k + \bar{Q}Ex_k = 0$.

Hence, if $(A, \bar{A}, C, \bar{C}, Q, \bar{Q})$ is "exact detectable" as defined in [27], then $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ will be exact detectable as defined in Definition 5.

Remark 4: Definitions 4 and 5 can be reduced to the standard exact observability and exact detectability for standard stochastic systems, respectively. Actually, with $\bar{A} = 0, \bar{C} = 0, \bar{Q} = 0$ in system (5), Definition 4 becomes $Q^{1/2}x_k = 0 \Rightarrow x_0 = 0$, which is the standard exact observability definition for standard stochastic linear systems. Similarly, the exact detectability given in Definition 5 can be reduced to the standard exact detectability definition for standard stochastic system. One can refer to [22], [25], and [26].

The problems of infinite horizon LQ control and stabilization for discrete-time mean-field systems are stated as the following.

Problem 1: We find the $\mathcal{F}_{k-1}$-measurable linear controller $u \in \mathcal{U}_\infty$ to minimize the cost function (3) and stabilize system (1) in the mean square sense.

B. Preliminaries

In this section, we recall the finite horizon mean-field LQ control problem, which serves as the preliminary results.

The finite horizon cost function associated with system (1) is given by

$$J_N = \sum_{k=0}^{N} E\Big[x_k' Q x_k + (Ex_k)'\bar{Q}Ex_k + u_k' R u_k + (Eu_k)'\bar{R}Eu_k\Big] + E(x_{N+1}' P_{N+1} x_{N+1}) + (Ex_{N+1})'\bar{P}_{N+1}Ex_{N+1} \tag{9}$$

where $Q, \bar{Q}, R, \bar{R}, P_{N+1}, \bar{P}_{N+1}$ are deterministic symmetric matrices with compatible dimensions and $P_{N+1} \geq 0$, $P_{N+1} + \bar{P}_{N+1} \geq 0$.

The admissible control set is defined as follows:

$$\mathcal{U}_N = \{u_0, \ldots, u_N | u_k \in \mathcal{R}^m, u_k \text{ is } \mathcal{F}_{k-1}\text{-measurable}, \text{ and } \sum_{k=0}^{N} E(u_k' u_k) < +\infty\}. \tag{10}$$

Any $u_k \in \mathcal{U}_N$ is called an admissible control, and it is clear that $\mathcal{U}_N$ is a nonempty, closed, and convex subset of $\mathcal{R}^m$.

The results for the finite horizon mean-field LQ control problem are stated as below.

The following result is the maximum principle for system (1) associated with the finite horizon cost function (9).

Theorem 1: A necessary condition for minimizing (9) for system (1) is as follows:

$$0 = E\Bigg\{Ru_k + \bar{R}Eu_k + \begin{bmatrix} B + w_k D \\ 0 \end{bmatrix}' \lambda_k + E\Bigg\{ \begin{bmatrix} \bar{B} + w_k \bar{D} \\ B + \bar{B} \end{bmatrix}' \lambda_k \Bigg\} \Big| \mathcal{F}_{k-1} \Bigg\} \tag{11}$$

where the costate λ_k satisfies the following backward iteration:

$$\lambda_{k-1} = E\Bigg\{ \begin{bmatrix} Qx_k + \bar{Q}Ex_k \\ 0 \end{bmatrix} + \begin{bmatrix} A + w_k C & \bar{A} + w_k \bar{C} \\ 0 & A + \bar{A} \end{bmatrix}' \lambda_k \Big| \mathcal{F}_{k-1} \Bigg\} \tag{12}$$

with the final condition

$$\lambda_N = \begin{bmatrix} P_{N+1}x_{N+1} + \bar{P}_{N+1}Ex_{N+1} \\ 0 \end{bmatrix} \tag{13}$$

where P_{N+1} and $\bar{P}_{N+1}$ are as in the cost function (9).

Proof: See the detailed proof in [28]. ■

Remark 5: It is noted that previous results on maximum principle for mean-field LQ were based on the mean-field BSDE, see [18]–[20]. We develop a new maximum principle for mean-field LQ control problem in Theorem 1, which can be calculated more easily, and can be reduced to the standard LQ case.

Theorem 2: Under Assumption 1, the optimal controller $\{u_k\}_{k=0}^N$ is given as

$$u_k = K_k x_k + \bar{K}_k E x_k \tag{14}$$

and the controller is unique if and only if $\Upsilon_k^{(1)}$ and $\Upsilon_k^{(2)}$ for $k = 0, \ldots, N$, as given below, are all positive definite, where

$$K_k = -[\Upsilon_k^{(1)}]^{-1} M_k^{(1)}, \tag{15}$$

$$\bar{K}_k = -\Big\{[\Upsilon_k^{(2)}]^{-1} M_k^{(2)} - [\Upsilon_k^{(1)}]^{-1} M_k^{(1)}\Big\} \tag{16}$$

and $\Upsilon_k^{(1)}$, $M_k^{(1)}$, $\Upsilon_k^{(2)}$, $M_k^{(2)}$ are given as

$$\Upsilon_k^{(1)} = R + B' P_{k+1} B + \sigma^2 D' P_{k+1} D, \tag{17}$$

$$M_k^{(1)} = B' P_{k+1} A + \sigma^2 D' P_{k+1} C, \tag{18}$$

$$\Upsilon_k^{(2)} = R + \bar{R} + (B+\bar{B})'(P_{k+1} + \bar{P}_{k+1})(B+\bar{B}) + \sigma^2 (D+\bar{D})' P_{k+1} (D+\bar{D}), \tag{19}$$

$$M_k^{(2)} = (B+\bar{B})'(P_{k+1} + \bar{P}_{k+1})(A+\bar{A}) + \sigma^2 (D+\bar{D})' P_{k+1} (C+\bar{C}) \tag{20}$$

while P_k and $\bar{P}_k$ in the above obey the following coupled Riccati equation for $k = 0, \ldots, N$.

$$P_k = Q + A' P_{k+1} A + \sigma^2 C' P_{k+1} C - [M_k^{(1)}]'[\Upsilon_k^{(1)}]^{-1} M_k^{(1)}, \tag{21}$$

$$\begin{aligned}\bar{P}_k =& \bar{Q} + A' P_{k+1}\bar{A} + \sigma^2 C' P_{k+1}\bar{C} + \bar{A}' P_{k+1} A + \sigma^2 \bar{C}' P_{k+1} C\\ &+ \bar{A}' P_{k+1}\bar{A} + \sigma^2 \bar{C}' P_{k+1}\bar{C} + (A+\bar{A})'\bar{P}_{k+1}(A+\bar{A})\\ &+ [M_k^{(1)}]'[\Upsilon_k^{(1)}]^{-1} M_k^{(1)} - [M_k^{(2)}]'[\Upsilon_k^{(2)}]^{-1} M_k^{(2)}\end{aligned} \tag{22}$$

with final condition P_{N+1} and $\bar{P}_{N+1}$ given in (9).

The associated optimal cost function is given by

$$J_N^* = E(x_0' P_0 x_0) + (Ex_0)'\bar{P}_0(Ex_0). \tag{23}$$

Proof: The detailed proof can be found in [28]. ■

III. Main Results

In this section, the main results of this paper will be presented, and the necessary and sufficient stabilization conditions for mean-field systems will be developed.

Before presenting the solution to *Problem 1*, the following lemmas will be given at first.

Lemma 1: Under Assumption 1, for the following coupled Riccati equation

$$\begin{aligned}P_k(N) =& Q + A' P_{k+1}(N) A + \sigma^2 C' P_{k+1}(N) C\\ &- [M_k^{(1)}(N)]'[\Upsilon_k^{(1)}(N)]^{\dagger} M_k^{(1)}(N),\end{aligned} \tag{24}$$

$$\begin{aligned}\bar{P}_k(N) =& \bar{Q} + A' P_{k+1}(N)\bar{A} + \sigma^2 C' P_{k+1}(N)\bar{C} + \bar{A}' P_{k+1}(N) A\\ &+ \sigma^2 \bar{C}' P_{k+1}(N) C + \bar{A}' P_{k+1}(N)\bar{A} + \sigma^2 \bar{C}' P_{k+1}(N)\bar{C}\\ &+ (A+\bar{A})'\bar{P}_{k+1}(N)(A+\bar{A}) + [M_k^{(1)}(N)]'[\Upsilon_k^{(1)}(N)]^{\dagger} M_k^{(1)}(N)\\ &- [M_k^{(2)}(N)]'[\Upsilon_k^{(2)}(N)]^{\dagger} M_k^{(2)}(N)\end{aligned} \tag{25}$$

with the final condition $P_{N+1} = \bar{P}_{N+1} = 0$, where $\dagger$ means that the Moore–Penrose inverse. If the regular condition below holds:

$$\Upsilon_k^{(i)}(N)[\Upsilon_k^{(i)}(N)]^{\dagger} M_k^{(i)}(N) = M_k^{(i)}(N), i = 1,2 \tag{26}$$

where

$$\Upsilon_k^{(1)}(N) = R + B' P_{k+1}(N) B + \sigma^2 D' P_{k+1}(N) D, \tag{27}$$

$$M_k^{(1)}(N) = B' P_{k+1}(N) A + \sigma^2 D' P_{k+1}(N) C, \tag{28}$$

$$\begin{aligned}\Upsilon_k^{(2)}(N) =& (B+\bar{B})'[P_{k+1}(N) + \bar{P}_{k+1}(N)](B+\bar{B})\\ &+ \sigma^2 (D+\bar{D})' P_{k+1}(N)(D+\bar{D}) + R + \bar{R},\end{aligned} \tag{29}$$

$$\begin{aligned}M_k^{(2)}(N) =& (B+\bar{B})'[P_{k+1}(N) + \bar{P}_{k+1}(N)](A+\bar{A})\\ &+ \sigma^2 (D+\bar{D})' P_{k+1}(N)(C+\bar{C})\end{aligned} \tag{30}$$

with $P_k(N), \bar{P}_k(N)$ satisfying (24) and (25). Then, the cost function (9) with $P_{N+1} = \bar{P}_{N+1} = 0$ can be minimized by

$$u_k = \mathcal{K}_k(N) x_k + \bar{\mathcal{K}}_k(N) E x_k \tag{31}$$

where

$$\begin{aligned}\mathcal{K}_k(N) &= -[\Upsilon_k^{(1)}(N)]^{\dagger} M_k^{(1)}(N),\\ \bar{\mathcal{K}}_k(N) &= -\left\{[\Upsilon_k^{(2)}(N)]^{\dagger} M_k^{(2)}(N) - [\Upsilon_k^{(1)}(N)]^{\dagger} M_k^{(1)}(N)\right\}.\end{aligned} \tag{32}$$

Moreover, the optimal cost function is

$$J_N^* = E[x_0' P_0(N) x_0] + (Ex_0)'\bar{P}_0(N)(Ex_0). \tag{33}$$

Proof: See Appendix A. ■

Remark 6: It should be noted that the presented results in Lemma 1 are different to that in Theorem 2. First, the matrix $\Upsilon_k^{(1)}, \Upsilon_k^{(2)}$ are no longer positive definite but regular condition (26) is required; Second, the optimal controller (31) is not necessarily unique.

Lemma 2: Under Assumption 1, for any $N \geq 0$, $P_k(N)$ and $\bar{P}_k(N)$ in (21) and (22) satisfy $P_k(N) \geq 0$ and $P_k(N) + \bar{P}_k(N) \geq 0$.

Proof: See Appendix B. ■

Lemma 3: Under Assumption 1, if the mean-field system (1) is mean square stabilizable, then the following coupled ARE has a solution satisfying $P \geq 0$ and $P + \bar{P} \geq 0$:

$$\begin{cases} P = Q + A'PA + \sigma^2 C'PC - [M^{(1)}]'[\Upsilon^{(1)}]^{\dagger} M^{(1)},\\ \bar{P} = \bar{Q} + A'P\bar{A} + \sigma^2 C'P\bar{C} + \bar{A}'PA + \sigma^2 \bar{C}'PC\\ \quad + \bar{A}'P\bar{A} + \sigma^2 \bar{C}'P\bar{C} + (A+\bar{A})'\bar{P}(A+\bar{A})\\ \quad + [M^{(1)}]'[\Upsilon^{(1)}]^{\dagger} M^{(1)} - [M^{(2)}]'[\Upsilon^{(2)}]^{\dagger} M^{(2)},\\ \Upsilon^{(i)}[\Upsilon^{(i)}]^{\dagger} M^{(i)} = M^{(i)}, i = 1,2 \end{cases} \tag{34}$$

while

$$\Upsilon^{(1)} = R + B'PB + \sigma^2 D'PD \geq R \geq 0, \tag{35}$$

$$M^{(1)} = B'PA + \sigma^2 D'PC, \tag{36}$$

$$\begin{aligned}\Upsilon^{(2)} =& R + \bar{R} + (B+\bar{B})'(P+\bar{P})(B+\bar{B})\\ &+ \sigma^2 (D+\bar{D})'P(D+\bar{D}) \geq R + \bar{R} \geq 0,\end{aligned} \tag{37}$$

$$\begin{aligned}M^{(2)} =& (B+\bar{B})'(P+\bar{P})(A+\bar{A})\\ &+ \sigma^2 (D+\bar{D})'P(C+\bar{C}).\end{aligned} \tag{38}$$

Proof: See Appendix C. ■

It is noted from (35)–(38) that

$$\left[M^{(1)}\right]'\left[\Upsilon^{(1)}\right]^{\dagger} M^{(1)} = -\left[M^{(1)}\right]'\mathcal{K} - \mathcal{K}'M^{(1)} - \mathcal{K}'\Upsilon^{(1)}\mathcal{K}, \tag{39}$$

$$\begin{aligned}\left[M^{(2)}\right]'[\Upsilon^{(2)}]^{\dagger} M^{(2)} =& -[M^{(2)}]'(\mathcal{K}+\bar{\mathcal{K}}) - (\mathcal{K}+\bar{\mathcal{K}})'M^{(2)}\\ &- (\mathcal{K}+\bar{\mathcal{K}})'\Upsilon^{(2)}(\mathcal{K}+\bar{\mathcal{K}})\end{aligned} \tag{40}$$

where $\mathcal{K}$ and $\bar{\mathcal{K}}$ satisfy

$$\mathcal{K} = -[\Upsilon^{(1)}]^{\dagger} M^{(1)}, \tag{41}$$

$$\bar{\mathcal{K}} = -\{[\Upsilon^{(2)}]^{\dagger} M^{(2)} - [\Upsilon^{(1)}]^{\dagger} M^{(1)}\}. \tag{42}$$

By using (39)–(40), we can rewrite (34) as follows:

$$P = \mathbf{Q} + \mathbf{A}'P\mathbf{A} + \sigma^2 \mathbf{C}'P\mathbf{C}, \tag{43}$$

$$P + \bar{P} = \bar{\mathbf{Q}} + \bar{\mathbf{A}}'(P+\bar{P})\bar{\mathbf{A}} + \sigma^2 \bar{\mathbf{C}}'P\bar{\mathbf{C}} \tag{44}$$

where

$$\begin{aligned}&\mathbf{Q} = Q + \mathcal{K}'R\mathcal{K} \geq 0, \mathbf{A} = A + B\mathcal{K}, \mathbf{C} = C + D\mathcal{K},\\&\bar{\mathbf{Q}} = Q + \bar{Q} + (\mathcal{K} + \bar{\mathcal{K}})'(R + \bar{R})(\mathcal{K} + \bar{\mathcal{K}}) \geq 0,\\&\bar{\mathbf{A}} = A + \bar{A} + (B + \bar{B})(\mathcal{K} + \bar{\mathcal{K}}),\\&\bar{\mathbf{C}} = C + \bar{C} + (D + \bar{D})(\mathcal{K} + \bar{\mathcal{K}}). \qquad (45)\end{aligned}$$

Definition 6: The Riccati equation (34) is said to have a positive definite (resp. positive semidefinite) solution, if there exist $P > 0$ and $P + \bar{P} > 0$ (resp. $P \geq 0$ and $P + \bar{P} \geq 0$) satisfying (34).

Lemma 4: Under Assumption 1, if system $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ is exact detectable, then:

1) The following system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$ is exact detectable:

$$\begin{cases}\mathbb{X}_{k+1} = \tilde{\mathbb{A}}\mathbb{X}_k + \tilde{\mathbb{C}}\mathbb{X}_k w_k, \ \mathbb{X}_0,\\ \tilde{Y}_k = \tilde{\mathcal{Q}}^{1/2}\mathbb{X}_k\end{cases} \qquad (46)$$

where $\tilde{\mathbb{A}} = \begin{bmatrix}\mathbf{A} & 0\\ 0 & \bar{\mathbf{A}}\end{bmatrix}$, $\tilde{\mathbb{C}} = \begin{bmatrix}\mathbf{C} & \bar{\mathbf{C}}\\ 0 & 0\end{bmatrix}$, $\tilde{\mathcal{Q}} = \begin{bmatrix}\mathbf{Q} & 0\\ 0 & \bar{\mathbf{Q}}\end{bmatrix} \geq 0$, and $\mathbb{X}_k = \begin{bmatrix}x_k - Ex_k\\ Ex_k\end{bmatrix}$, i.e., for any N, if $\tilde{Y}_k = 0$, $a.s.$, $0 \leq k \leq N$, then $\lim_{k\to+\infty} E(\mathbb{X}_k'\mathbb{X}_k) = 0$.

2) If $\mathbb{P} \geq 0$, then $E(\mathbb{X}_0'\mathbb{P}\mathbb{X}_0) = 0$ if and only if $\mathbb{X}_0$ is an unobservable state of the system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$, where $\mathbb{P} = \begin{bmatrix}P & 0\\ 0 & P + \bar{P}\end{bmatrix}$ and $P, \bar{P}$ satisfy (34).

Proof: See Appendix D. ■

Remark 7: Similar to Lemma 4 and its proof, under Assumption 1, it is easy to verify if the system $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ is exact observable, then the system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$ is exact observable.

We are now in the position to present the main results of this section. Two results are to be given, one is based on the assumption of exact detectability, and the other is based on the assumption of exact observability.

Theorem 3: Suppose Assumption 1 holds and system (5) $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ is exact detectable, then the mean-field system (1) is stabilizable in the mean square sense if and only if there exists a unique positive semidefinite solution to the coupled ARE (34).

In this case, a stabilizing controller is given by

$$u_k = \mathcal{K}x_k + \bar{\mathcal{K}}Ex_k \qquad (47)$$

where $\mathcal{K}$ and $\bar{\mathcal{K}}$ are given by (41) and (42). In addition, the stabilizing controller (47) minimizes the cost function (3), and the optimal cost function is given as

$$J^* = E(x_0'Px_0) + Ex_0'\bar{P}Ex_0. \qquad (48)$$

Proof: See Appendix E. ■

Theorem 4: Under the conditions of Assumption 1 and the exact observability of $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$, the mean-field system (1) is mean square stabilizable if and only if the coupled ARE (34) has a unique positive definite solution, and a stabilizing controller is given by (47) and that the cost function (3) is minimized by (47).

Proof: See Appendix F. ■

Remark 8: Theorems 3 and 4 propose a new approach to stabilization problems for mean-field systems. The necessary and sufficient stabilization conditions are provided under the assumption of $R \geq 0$ and $R + \bar{R} \geq 0$, in comparison with the results in the literature (e.g., [24], [29], and [30]) where $R > 0$ is required, even for the standard LQ problems.

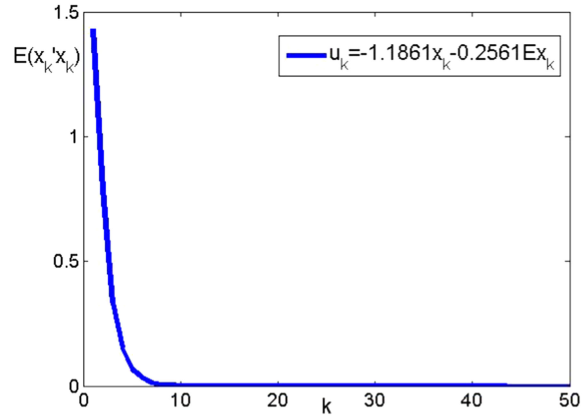

Fig. 1. Mean square stabilization of the mean-field system.

IV. NUMERICAL EXAMPLE

We consider system (1) and the cost function (3) with:

$$A = 1.1, \bar{A} = 0.2, B = 0.4, \bar{B} = 0.1, C = 0.9, \bar{C} = 0.5,$$

$$D = 0.8, \bar{D} = 0.2, Q = 2, \bar{Q} = 1, R = 1, \bar{R} = 1, \sigma^2 = 1$$

the initial state $x_0 \sim N(1, 2)$, i.e., x_0 obeys the normal distribution with mean 1 and covariance 2.

Note that $Q = 2$, $Q + \bar{Q} = 3$, $R = 1$, $R + \bar{R} = 2$ are all positive, then Assumption 1 and the exact observability of $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ are satisfied. By using coupled ARE (34), we have $P = 5.6191$ and $\bar{P} = 5.1652$. From (35)–(38), we can obtain $\Upsilon^{(1)} = 5.4953, M^{(1)} = 6.5182, \Upsilon^{(2)} = 10.3152$, and $M^{(2)} = 14.8765$.

Notice that $P > 0$ and $P + \bar{P} > 0$, according to Theorem 4, there exists a unique optimal controller to stabilize the mean-field system (1) in the mean square sense as well as to minimize the cost function (3), the controller in (47) is presented as

$$u_k = \mathcal{K}x_k + \bar{\mathcal{K}}Ex_k = -1.1861x_k - 0.2561Ex_k, \ k \geq 0.$$

Using the designed controller, the simulation of the system state is shown in Fig. 1. With the optimal controller, the regulated system state is stabilized in the mean square sense as shown in Fig. 1.

To explore the improvement of the main results presented in this paper, we consider the mean-field system (1) and cost function (3) with

$$A = 2, \bar{A} = 0.8, B = 0.5, \bar{B} = 1, C = 1, \bar{C} = 1,$$

$$D = -0.8, \bar{D} = 0.6, Q = 1, \bar{Q} = 1, R = 1, \bar{R} = 1, \sigma^2 = 1.$$

The initial state $x_0 \sim N(1, 2)$.

By solving the coupled ARE (34), it is found that P has two negative roots as $P = -1.1400$ and $P = -0.2492$. Thus,

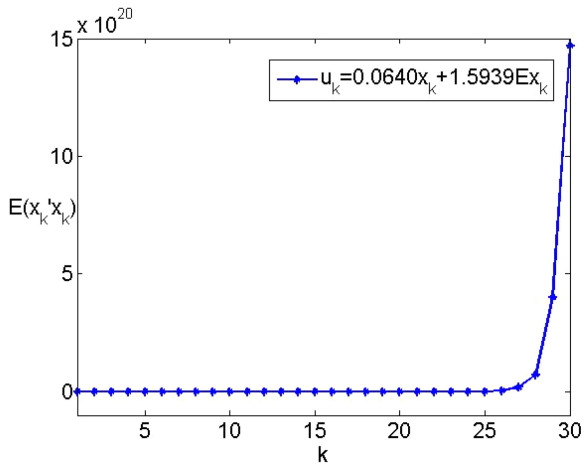

Fig. 2. Simulation for the state trajectory $E(x_k' x_k)$.

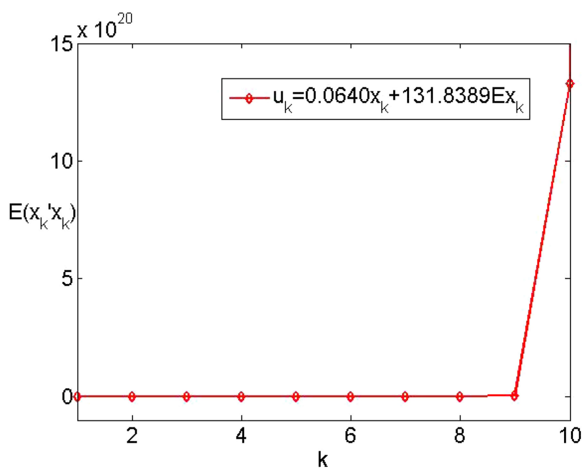

Fig. 3. Simulation for the state trajectory $E(x_k' x_k)$.

according to Theorems 3 and 4, we know that system (1) is not stabilizable in the mean square sense.

Actually, when $P = -1.1400$, it can be checked that (34) has no real roots for $\bar{P}$. While in the case of $P = -0.2492$, $\bar{P}$ has two real roots which can be solved from (34) as $\bar{P} = 7.0597$ and $\bar{P} = -0.6476$, respectively.

In the latter case, with $P = -0.2492$ and $\bar{P} = 7.0597$, we can calculate $\mathcal{K}$ and $\bar{\mathcal{K}}$ from (41) and (42) as $\mathcal{K} = 0.0640$, $\bar{\mathcal{K}} = 1.5939$. Similarly, with $P = -0.2492$ and $\bar{P} = -0.6476$, $\mathcal{K}$ and $\bar{\mathcal{K}}$ can be computed as $\mathcal{K} = 0.0640$, $\bar{\mathcal{K}} = 131.8389$. Accordingly, the controllers are designed as $u_k = 0.0640x_k + 1.5939Ex_k$, $u_k = 0.0640x_k + 131.8389Ex_k$, respectively.

Simulation results of the corresponding state trajectories with the designed controllers are shown as in Figs. 2 and 3, respectively. As expected, the state trajectories are not convergent.

V. Conclusion

In this paper, the necessary and sufficient stabilization conditions for mean-field systems have been investigated. It is shown that, under the exact detectability assumption, the mean-field system (1) is mean square stabilizable if and only if a coupled ARE has a unique positive semidefinite solution. Furthermore, under the exact observability assumption, we show that the mean-field system is mean square stabilizable if and only if the coupled ARE admits a unique positive definite solution.

Appendix A
Proof of Lemma 1

Proof: The positive semidefiniteness of $\Upsilon_N^{(1)}(N)$, $\Upsilon_N^{(2)}(N)$ can be easily obtained under Assumption 1. By using the induction method as in Theorem 2, we can conclude that $\Upsilon_k^{(1)}(N) \geq 0$ and $\Upsilon_k^{(1)}(N) \geq 0$ for any $k \geq 0$.

On the other hand, using the Riccati equation (24)–(25) and the regular condition (26), following the proof of [28, Th. 2], then J_N in (9) with $P_{N+1} = \bar{P}_{N+1} = 0$ can be calculated as

$$\begin{aligned} J_N = & \sum_{k=0}^{N} E\Big\{\Big[u_k - Eu_k - \mathcal{K}_k(N)(x_k - Ex_k)\Big]'\Upsilon_k^{(1)}(N) \\ & \times \Big[u_k - Eu_k - \mathcal{K}_k(N)(x_k - Ex_k)\Big]\Big\} \\ & + \sum_{k=0}^{N}\{Eu_k - [\mathcal{K}_k(N) + \bar{\mathcal{K}}_k(N)]Ex_k\}'\Upsilon_k^{(2)}(N) \\ & \times \{Eu_k - [\mathcal{K}_k(N) + \bar{\mathcal{K}}_k(N)]Ex_k\} \\ & + E[x_0' P_0(N) x_0] + Ex_0' \bar{P}_0(N) Ex_0 \end{aligned} \tag{49}$$

where $\mathcal{K}_k(N), \bar{\mathcal{K}}_k(N)$ are given by (32).

Since $\Upsilon_k^{(1)}(N) \geq 0$ and $\Upsilon_k^{(2)}(N) \geq 0$, then $J_N \geq E[x_0' P_0(N) x_0] + Ex_0' \bar{P}_0(N) Ex_0$. Thus, J_N in (9) with $P_{N+1} = \bar{P}_{N+1} = 0$ is minimized by (31) and the optimal cost function is (33). ■

Appendix B
Proof of Lemma 2

Proof: Using Lemma 1, under Assumption 1, if the solution to the Riccati equation (24)–(25) satisfies the regular condition (26), then (9) with $P_{N+1}(N) = \bar{P}_{N+1}(N) = 0$ can be minimized by (31), and the optimal cost function is as (33):

$$J_N^* = E[x_0' P_0(N) x_0] + Ex_0' \bar{P}_0(N) Ex_0. \tag{50}$$

Moreover, Assumption 1 indicates $J_N \geq 0$ for any controller u_k, thus the optimal $J_N^* \geq 0$ can be derived.

We choose x_0 to be any random variable with $Ex_0 = 0$, (50) reduces to $E[x_0' P_0(N) x_0] \geq 0$, then we have $P_0(N) \geq 0$. On the other hand, if $x_0 = Ex_0$, i.e., x_0 is deterministic, from (50) we know $x_0'[P_0(N) + \bar{P}_0(N)]x_0 \geq 0$, then $P_0(N) + \bar{P}_0(N) \geq 0$ can be derived.

Noting the time-variance of the coefficient matrices in (24)–(32), there holds $P_k(N) = P_0(N-k)$, $\bar{P}_k(N) = \bar{P}_0(N-k)$.

In conclusion, under Assumption 1, $P_k(N) \geq 0$, $P_k(N) + \bar{P}_k(N) \geq 0$ for any $0 \leq k \leq N$. ■

APPENDIX C
PROOF OF LEMMA 3

Proof: Under Assumption 1, suppose the mean-field system (1) is stabilizable in the mean square sense, we will show that the coupled ARE (34) has a unique solution P and $\bar{P}$ with $P \geq 0$ and $P + \bar{P} \geq 0$.

First, we shall show $P_0(N)$ and $P_0(N) + \bar{P}_0(N)$ are monotonically increasing with N.

Actually, since $J_N \leq J_{N+1}$, then for any initial state x_0, we have $J_N^* \leq J_{N+1}^*$, it holds from (33) that

$$\begin{aligned} &E[(x_0 - Ex_0)'P_0(N)(x_0 - Ex_0)] \\ &+ (Ex_0)'[P_0(N) + \bar{P}_0(N)](Ex_0) \\ &\leq E[(x_0 - Ex_0)'P_0(N+1)(x_0 - Ex_0)] \\ &+ (Ex_0)'[P_0(N+1) + \bar{P}_0(N+1)](Ex_0). \end{aligned} \tag{51}$$

For any initial state $x_0 \neq 0$ with $Ex_0 = 0$, (51) can be reduced to

$$E[x_0'P_0(N)x_0] \leq E[x_0'P_0(N+1)x_0]$$

i.e., $E\{x_0'[P_0(N) - P_0(N+1)]x_0\} \leq 0$. Therefore, we can obtain

$$P_0(N) \leq P_0(N+1) \tag{52}$$

which implies that $P_0(N)$ increases with respect to N.

On the other hand, for arbitrary initial state $x_0 \neq 0$ with $x_0 = Ex_0$, i.e., $x_0 \in \mathcal{R}^n$ is arbitrary deterministic, (51) indicates that

$$x_0'[P_0(N) + \bar{P}_0(N)]x_0 \leq x_0'[P_0(N+1) + \bar{P}_0(N+1)]x_0.$$

Note that x_0 is arbitrary, then we have

$$P_0(N) + \bar{P}_0(N) \leq P_0(N+1) + \bar{P}_0(N+1) \tag{53}$$

which implies that $P_0(N) + \bar{P}_0(N)$ increases with respect to N, too.

Next we will show that $P_0(N)$ and $P_0(N) + \bar{P}_0(N)$ are bounded. Since system (1) is stabilizable in the mean square sense, there exists u_k has the form

$$u_k = Lx_k + \bar{L}Ex_k \tag{54}$$

with constant matrices L and $\bar{L}$ such that the closed-loop system (1) satisfies

$$\lim_{k\to+\infty} E(x_k'x_k) = 0. \tag{55}$$

As $(Ex_k)'Ex_k + E(x_k - Ex_k)'(x_k - Ex_k) = E(x_k'x_k)$, thus, (55) implies $\lim_{k\to+\infty}(Ex_k)'Ex_k = 0$.

Substituting (54) into (1), we obtain

$$\begin{aligned} x_{k+1} &= [(A + w_kC) + (B + w_kD)L]x_k \\ &+ [(B + w_kD)\bar{L} + (\bar{A} + w_k\bar{C}) + (\bar{B} + w_k\bar{D})(L + \bar{L})]Ex_k, \end{aligned} \tag{56}$$

$$Ex_{k+1} = [(A + \bar{A}) + (B + \bar{B})(L + \bar{L})]Ex_k. \tag{57}$$

We denote $X_k \triangleq \begin{bmatrix} x_k \\ Ex_k \end{bmatrix}$, and $\mathcal{X}_k \triangleq E[X_kX_k']$.

Following from (56) and (57), it holds

$$X_{k+1} = \mathcal{A}X_k \tag{58}$$

where $\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $A_{11} = (A + w_kC) + (B + w_kD)L$, $A_{12} = (B + w_kD)\bar{L} + (\bar{A} + w_k\bar{C}) + (\bar{B} + w_k\bar{D})(L + \bar{L})$, and $A_{22} = (A + \bar{A}) + (B + \bar{B})(L + \bar{L})$.

The mean square stabilization of $\lim_{k\to+\infty} E(x_k'x_k) = 0$ implies $\lim_{k\to+\infty} \mathcal{X}_k = 0$, thus, it follows from [23] that

$$\sum_{k=0}^{\infty} E(x_k'x_k) < +\infty, \text{ and } \sum_{k=0}^{\infty}(Ex_k)'(Ex_k) < +\infty.$$

Therefore, there exists constant c such that

$$\sum_{k=0}^{\infty} E(x_k'x_k) \leq cE(x_0'x_0). \tag{59}$$

Since $Q \geq 0$, $Q + \bar{Q} \geq 0$, $R \geq 0$ and $R + \bar{R} \geq 0$, thus there exists constant λ such that $\begin{bmatrix} Q & 0 \\ 0 & Q + \bar{Q} \end{bmatrix} \leq \lambda I$ and $\begin{bmatrix} L'RL & 0 \\ 0 & (L + \bar{L})'(R + \bar{R})(L + \bar{L}) \end{bmatrix} \leq \lambda I$, using (54) and (59), we obtain that

$$\begin{aligned} J &= \sum_{k=0}^{\infty} E[x_k'Qx_k + u_k'Ru_k + Ex_k'\bar{Q}Ex_k + Eu_k'\bar{R}Eu_k] \\ &= \sum_{k=0}^{\infty} E\left\{ \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & Q + \bar{Q} \end{bmatrix} \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix} \right\} \\ &+ \sum_{k=0}^{\infty} E\left\{ \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix}' \begin{bmatrix} L'RL & 0 \\ 0 & (L + \bar{L})'(R + \bar{R})(L + \bar{L}) \end{bmatrix} \right. \\ &\left. \times \begin{bmatrix} x_k - Ex_k \\ Ex_k \end{bmatrix} \right\} \\ &\leq 2\lambda \sum_{k=0}^{\infty} E[(Ex_k)'Ex_k + (x_k - Ex_k)'(x_k - Ex_k)] \\ &= 2\lambda \sum_{k=0}^{\infty} E(x_k'x_k) \leq 2\lambda cE(x_0'x_0). \end{aligned} \tag{60}$$

On the other hand, by (23), notice the fact that

$$E[x_0'P_0(N)x_0] + (Ex_0)'\bar{P}_0(N)(Ex_0) = J_N^* \leq J$$

thus, (60) yields

$$E[x_0'P_0(N)x_0] + (Ex_0)'\bar{P}_0(N)(Ex_0) \leq 2\lambda cE(x_0'x_0). \tag{61}$$

Now we let the initial state value be a random vector with zero mean, i.e., $Ex_0 = 0$, it follows from (61) that

$$0 \leq E[x_0'P_0(N)x_0] \leq 2\lambda cE(x_0'x_0).$$

Since x_0 is arbitrary with $Ex_0 = 0$, there holds that

$$0 \leq P_0(N) \leq 2\lambda cI.$$

Similarly, let the initial state value be arbitrary deterministic, i.e., $x_0 = Ex_0$, (61) yields that

$$0 \leq x_0'[P_0(N) + \bar{P}_0(N)]x_0 = J_N^* \leq J \leq 2\lambda cx_0'x_0$$

which implies

$$0 \leq P_0(N) + \bar{P}_0(N) \leq 2\lambda cI.$$

Therefore, both $P_0(N)$ and $P_0(N)+\bar{P}_0(N)$ are bounded. Recall that $P_0(N)$ and $P_0(N)+\bar{P}_0(N)$ are monotonically increasing, we conclude that $P_0(N)$ and $P_0(N)+\bar{P}_0(N)$ are convergent, i.e., there exists P and $\bar{P}$ such that

$$\lim_{N\to+\infty} P_k(N) = \lim_{N\to+\infty} P_0(N-k) = P \geq 0,$$
$$\lim_{N\to+\infty} \bar{P}_k(N) = \lim_{N\to+\infty} \bar{P}_0(N-k) = \bar{P}, P+\bar{P} \geq 0.$$

Furthermore, in view of (27)–(30), we know that $\Upsilon_k^{(1)}(N)$, $M_k^{(1)}(N)$, $\Upsilon_k^{(2)}(N)$, and $M_k^{(2)}(N)$ are convergent, i.e.

$$\lim_{N\to+\infty} \Upsilon_k^{(i)}(N) = \Upsilon^{(i)}, \lim_{N\to+\infty} M_k^{(i)}(N) = M^{(i)}, i=1,2 \tag{62}$$

where $\Upsilon^{(1)}, M^{(1)}, \Upsilon^{(2)}, M^{(2)}$ are given by (35)–(38). Taking limitation on both sides of (21) and (22), we know that P and $\bar{P}$ satisfy the coupled ARE (34). ■

Appendix D
Proof of Lemma 4

Proof: 1) The mean-field system (1) and (2) with $u_k=0$ can be rewritten as follows:

$$\begin{cases} \mathbb{X}_{k+1} = \mathbb{A}\mathbb{X}_k + \mathbb{C}\mathbb{X}_k w_k, \ \mathbb{X}_0, \\ Y_k = \mathcal{Q}^{1/2}\mathbb{X}_k \end{cases} \tag{63}$$

where $\mathbb{A} = \begin{bmatrix} A & 0 \\ 0 & A+\bar{A}\end{bmatrix}$, $\mathbb{C} = \begin{bmatrix} C & C+\bar{C} \\ 0 & 0\end{bmatrix}$, and $\mathcal{Q}$ is as in (5). Thus, the exact detectability of system $(A,\bar{A},C,\bar{C},\mathcal{Q}^{1/2})$ is equivalent to the exact detectability of system $(\mathbb{A},\mathbb{C},\mathcal{Q}^{1/2})$ in (63).

The systems (1) and (2) with controller (47) can be presented as

$$\mathbb{X}_{k+1} = \tilde{\mathbb{A}}\mathbb{X}_k + \tilde{\mathbb{C}}\mathbb{X}_k w_k \tag{64}$$

where $\mathbb{X}_k$, $\tilde{\mathbb{A}}$, and $\tilde{\mathbb{C}}$ are given below (46).

Using the symbols in (45) and (46), there holds

$$\tilde{\mathbb{A}} = \mathbb{A}+\mathbb{B}\mathbb{K}, \tilde{\mathbb{C}} = \mathbb{C}+\mathbb{D}\mathbb{K}, \tilde{\mathcal{Q}} = \mathcal{Q}+\mathbb{K}'\mathbb{R}\mathbb{K}$$

where $\mathbb{B} = \begin{bmatrix} B & 0 \\ 0 & B+\bar{B}\end{bmatrix}$, $\mathbb{D} = \begin{bmatrix} D & D+\bar{D} \\ 0 & 0\end{bmatrix}$, $\mathbb{K} = \begin{bmatrix} \mathcal{K} & 0 \\ 0 & \mathcal{K}+\bar{\mathcal{K}}\end{bmatrix}$, and $\mathbb{R} = \begin{bmatrix} R & 0 \\ 0 & R+\bar{R}\end{bmatrix}$.

From [29, Th. 4 and Proposition 1], we conclude that if the system $(\mathbb{A},\mathbb{C},\mathcal{Q}^{1/2})$ (i.e., $(A,\bar{A},C,\bar{C},\mathcal{Q}^{1/2})$) is exact detectable, then for any feedback gain $\mathbb{K}$ ($\mathcal{K}$ and $\bar{\mathcal{K}}$), system $(\tilde{\mathbb{A}},\tilde{\mathbb{C}},\tilde{\mathcal{Q}}^{1/2})$ is exact detectable.

2) Similar to the derivation of [28, Th. 2] for the finite horizon case, we have

$$\begin{aligned} &E(x'_{N+1}Px_{N+1}) + (Ex_{N+1})'\bar{P}Ex_{N+1} \\ &\quad - [E(x'_0Px_0) + (Ex_0)'\bar{P}Ex_0] \\ &= E(\mathbb{X}'_{N+1}\mathbb{P}\mathbb{X}_{N+1}) - E(\mathbb{X}'_0\mathbb{P}\mathbb{X}_0) \\ &= -\sum_{k=0}^{N} E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k) \leq 0 \end{aligned} \tag{65}$$

where the controller (47) has been used, $\tilde{\mathcal{Q}}$ is as in (45).

Following from (65), since $\mathbb{P}\geq 0$, if $E(\mathbb{X}'_0\mathbb{P}\mathbb{X}_0)=0$, then

$$0 \leq \sum_{k=0}^{N} E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k) = -E(\mathbb{X}'_{N+1}\mathbb{P}\mathbb{X}_{N+1}) \leq 0. \tag{66}$$

This implies $\sum_{k=0}^{N} E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k)=0$. Then, $\sum_{k=0}^{N} E(\tilde{Y}'_k\tilde{Y}_k) = \sum_{k=0}^{N} E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k)=0$, which means that for any $k\geq 0$, $\tilde{Y}_k = \tilde{\mathcal{Q}}^{1/2}\mathbb{X}_k = 0$. Hence, $\mathbb{X}_0$ is an unobservable state of system $(\tilde{\mathbb{A}},\tilde{\mathbb{C}},\tilde{\mathcal{Q}}^{1/2})$.

On the contrary, if we choose $\mathbb{X}_0$ as an unobservable state of $(\tilde{\mathbb{A}},\tilde{\mathbb{C}},\tilde{\mathcal{Q}}^{1/2})$, i.e., $\tilde{Y}_k = \tilde{\mathcal{Q}}^{1/2}\mathbb{X}_k \equiv 0$, $k\geq 0$. Noting from Lemma 4 that $(\tilde{\mathbb{A}},\tilde{\mathbb{C}},\tilde{\mathcal{Q}}^{1/2})$ is exact detectable, it holds $\lim_{N\to+\infty} E(\mathbb{X}'_{N+1}\mathbb{P}\mathbb{X}_{N+1})=0$. Thus, from (65), we can obtain that

$$E(\mathbb{X}'_0\mathbb{P}\mathbb{X}_0) = \sum_{k=0}^{\infty} E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k) = \sum_{k=0}^{\infty} E(\tilde{Y}'_k\tilde{Y}_k) = 0. \tag{67}$$

This ends the proof. ■

Appendix E
Proof of Theorem 3

Proof: The sketch of the "sufficiency" proof is broken into two parts. First, we define the Lyapunov function candidate with the optimal cost function, if the solution to the ARE (34) is strictly positive definite, the stabilization of mean-field systems will be derived; second, if the solution to the ARE (34) is strictly positive semidefinite, we decompose the solution of the ARE (34) into two parts, and the corresponding system state of the mean-field systems is divided into two parts. By using the exact detectability assumption, we show the stabilization of the two parts, respectively. While for the "necessity" proof, we show that the ARE (34) admits a positive semidefinite solution, and the proof can be found in Lemma 3. In the end, we show the uniqueness of the solution to the ARE (34).

"Sufficiency:" If Assumption 1 holds and the system $(A,\bar{A},C,\bar{C},\mathcal{Q}^{1/2})$ is exact detectable, suppose P and $\bar{P}$ are the unique solution to (34) satisfying $P\geq 0$ and $P+\bar{P}\geq 0$, we will show that (47) stabilizes system (1) in the mean square sense.

From (33), we define the Lyapunov function candidate $V(k,x_k)$ with the optimal cost function as

$$V(k,x_k) \triangleq E(x'_kPx_k) + Ex'_k\bar{P}Ex_k. \tag{68}$$

Apparently we have

$$\begin{aligned} V(k,x_k) &= E[(x_k-Ex_k)'P(x_k-Ex_k) + Ex'_k(P+\bar{P})Ex_k] \\ &\geq 0. \end{aligned} \tag{69}$$

From (65), there holds

$$V(k,x_k) - V(k+1,x_{k+1}) = E(\mathbb{X}'_k\tilde{\mathcal{Q}}\mathbb{X}_k) \geq 0 \tag{70}$$

where $\tilde{\mathcal{Q}} = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \bar{\mathbf{Q}}\end{bmatrix} \geq 0$, and $\mathbb{X}_k = \begin{bmatrix} x_k - Ex_k \\ Ex_k\end{bmatrix}$. Thus, $V(k,x_k)$ is convergent.

Following from Lemma 4, we know the stabilization of system (1) with controller (47) is equivalent to the stability of system (64), or $(\tilde{\mathbb{A}},\tilde{\mathbb{C}})$.

In the following, we will consider $\mathbb{P} > 0$ and $\mathbb{P} \geq 0$, respectively. And we shall show that system (1) is mean square stabilizable in these two different cases.

1) $\mathbb{P} > 0$, i.e., $P > 0$ and $P + \bar{P} > 0$.

In this case, $E(\mathbb{X}_0'\mathbb{P}\mathbb{X}_0) = 0$ implies that $\mathbb{X}_0 = 0$, i.e., $x_0 = Ex_0 = 0$. Following from Lemma 4 and Remark 7, we know that system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$ is exact observable.

Taking summation on both sides of (70) from 0 to N for any $N > 0$, we have that

$$\begin{aligned}\sum_{k=0}^{N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k) &= V(0,x_0) - V(N+1, x_{N+1})\\ &= \sum_{k=0}^{N} E(x_k - Ex_k)'\mathbf{Q}(x_k - Ex_k) + Ex_k'\bar{\mathbf{Q}}Ex_k\\ &= E(x_0 - Ex_0)'H_0(N)(x_0 - Ex_0)\\ &\quad + Ex_0'[H_0(N) + \bar{H}_0(N)]Ex_0 \end{aligned} \tag{71}$$

where $H_0(N), \bar{H}_0(N)$ can be obtained by

$$H_k(N) = \mathbf{Q} + \mathbf{A}'H_{k+1}(N)\mathbf{A} + \sigma^2\mathbf{C}'H_{k+1}(N)\mathbf{C}, \tag{72}$$

$$\begin{aligned} H_k(N) + \bar{H}_k(N) &= \bar{\mathbf{Q}} + \bar{\mathbf{A}}'[H_{k+1}(N) + \bar{H}_{k+1}(N)]\bar{\mathbf{A}}\\ &\quad + \sigma^2\bar{\mathbf{C}}'H_k(N)\bar{\mathbf{C}} \end{aligned} \tag{73}$$

with final condition $H_{N+1}(N) = \bar{H}_{N+1}(N) = 0$.

Since $\mathbf{Q} \geq 0$ and $\bar{\mathbf{Q}} \geq 0$, we conclude that (72) and (73) admit a unique solution $H_0(N) \geq 0, H_0(N) + \bar{H}_0(N) \geq 0$ by backward iterations.

Then we will show $H_0(N) > 0$ and $H_0(N) + \bar{H}_0(N) > 0$. Otherwise, there exists nonzero y and $\bar{y}$ satisfying

$$y \neq 0,\ E[y'H_0(N)y] = 0,\ Ey = 0, \tag{74}$$

$$\bar{y} \neq 0,\ \bar{y}'[H_0(N) + \bar{H}_0(N)]\bar{y} = 0,\ \bar{y} = E\bar{y}. \tag{75}$$

If the initial state is chosen to be y, (71) implies

$$\sum_{k=0}^{N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k) = E[y'H_0(N)y] = 0. \tag{76}$$

Then, $\tilde{\mathcal{Q}}^{1/2}\mathbb{X}_k = 0$, a.s. Following from the exact observability of the system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$, we have $y = 0$, which contradicts with $y \neq 0$ defined in (74).

On the other hand, if we assume the initial state to be $\bar{y}$, (71) reduces to

$$\sum_{k=0}^{N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k) = \bar{y}'[H_0(N) + \bar{H}_0(N)]\bar{y} = 0. \tag{77}$$

Since system (46) is exact observable, then $\bar{y} = 0$, this contradicts with $\bar{y} \neq 0$ in (75).

In conclusion, we have proved $H_0(N) > 0$ and $H_0(N) + \bar{H}_0(N) > 0$.

Via a time shift of l, there holds from (71) that

$$\begin{aligned}&\sum_{k=l}^{l+N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k)\\ &= E[(x_l - Ex_l)'H_0(N)(x_l - Ex_l)]\\ &\quad + Ex_l'[H_0(T) + \bar{H}_0(N)]Ex_l\\ &= V(l, x_l) - V(l+N, x_{l+N}) \end{aligned} \tag{78}$$

where $H_l(l+N) = H_0(N), \bar{H}_l(l+N) = \bar{H}_0(N)$ has been inserted above.

By taking limitation $l \to +\infty$ on both sides of (78), using the convergence of $V(k, x_k)$, we have that

$$\lim_{l\to+\infty} E[(x_l - Ex_l)'(x_l - Ex_l)] = 0,\ \lim_{l\to+\infty} Ex_l'Ex_l = 0. \tag{79}$$

Therefore, $\lim_{k\to+\infty} E(x_k'x_k) = 0$, i.e., system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}})$ is stable, and system (1) is mean square stabilizable with controller (47).

2) $\mathbb{P} \geq 0$.

It is noticed from (43) and (44) that $\mathbb{P}$ satisfies the following Lyapunov equation:

$$\mathbb{P} = \tilde{\mathcal{Q}} + \tilde{\mathbb{A}}'\mathbb{P}\tilde{\mathbb{A}} + \sigma^2[\tilde{\mathbb{C}}^{(1)}]'\mathbb{P}\tilde{\mathbb{C}}^{(1)} + \sigma^2[\tilde{\mathbb{C}}^{(2)}]'\mathbb{P}\tilde{\mathbb{C}}^{(2)} \tag{80}$$

where $\tilde{\mathbb{C}}^{(1)} = \begin{bmatrix} \mathbf{C} & 0\\ 0 & 0\end{bmatrix}$, $\tilde{\mathbb{C}}^{(2)} = \begin{bmatrix} 0 & \bar{\mathbf{C}}\\ 0 & 0\end{bmatrix}$, and $\tilde{\mathbb{C}}^{(1)} + \tilde{\mathbb{C}}^{(2)} = \tilde{\mathbb{C}}$.

Since $\mathbb{P} \geq 0$, there exists an orthogonal matrix U with $U' = U^{-1}$ such that

$$U'\mathbb{P}U = \begin{bmatrix} 0 & 0\\ 0 & \mathbb{P}_2\end{bmatrix}, \mathbb{P}_2 > 0. \tag{81}$$

Obviously from (80), we can obtain that

$$\begin{aligned} U'\mathbb{P}U &= U'\tilde{\mathcal{Q}}U + U'\tilde{\mathbb{A}}'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{A}}U\\ &\quad + \sigma^2U'[\tilde{\mathbb{C}}^{(1)}]'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{C}}^{(1)}U\\ &\quad + \sigma^2U'[\tilde{\mathbb{C}}^{(2)}]'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{C}}^{(2)}U. \end{aligned} \tag{82}$$

Assuming $U'\tilde{\mathbb{A}}U = \begin{bmatrix} \tilde{\mathbb{A}}_{11} & \tilde{\mathbb{A}}_{12}\\ \tilde{\mathbb{A}}_{21} & \tilde{\mathbb{A}}_{22}\end{bmatrix}$, $U'\tilde{\mathcal{Q}}U = \begin{bmatrix} \tilde{\mathcal{Q}}_1 & \tilde{\mathcal{Q}}_{12}\\ \tilde{\mathcal{Q}}_{21} & \tilde{\mathcal{Q}}_2\end{bmatrix}$, $U'\tilde{\mathbb{C}}^{(1)}U = \begin{bmatrix} \tilde{\mathbb{C}}_{11}^{(1)} & \tilde{\mathbb{C}}_{12}^{(1)}\\ \tilde{\mathbb{C}}_{21}^{(1)} & \tilde{\mathbb{C}}_{22}^{(1)}\end{bmatrix}$, and $U'\tilde{\mathbb{C}}^{(2)}U = \begin{bmatrix} \tilde{\mathbb{C}}_{11}^{(2)} & \tilde{\mathbb{C}}_{12}^{(2)}\\ \tilde{\mathbb{C}}_{21}^{(2)} & \tilde{\mathbb{C}}_{22}^{(2)}\end{bmatrix}$, we have that

$$U'\tilde{\mathbb{A}}'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{A}}U = \begin{bmatrix} \tilde{\mathbb{A}}_{21}'\mathbb{P}_2\tilde{\mathbb{A}}_{21} & \tilde{\mathbb{A}}_{21}'\mathbb{P}_2\tilde{\mathbb{A}}_{22}\\ \tilde{\mathbb{A}}_{22}'\mathbb{P}_2\tilde{\mathbb{A}}_{21} & \tilde{\mathbb{A}}_{22}'\mathbb{P}_2\tilde{\mathbb{A}}_{22}\end{bmatrix},$$

$$\begin{aligned}&U'\{\tilde{\mathbb{C}}^{(1)}\}'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{C}}^{(1)}U\\ &= \begin{bmatrix} \{\tilde{\mathbb{C}}_{21}^{(1)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{21}^{(1)} & \{\tilde{\mathbb{C}}_{21}^{(1)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(1)}\\ \{\tilde{\mathbb{C}}_{22}^{(1)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{21}^{(1)} & \{\tilde{\mathbb{C}}_{22}^{(1)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(1)}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}&U'\{\tilde{\mathbb{C}}^{(2)}\}'U\cdot U'\mathbb{P}U\cdot U'\tilde{\mathbb{C}}^{(2)}U\\ &= \begin{bmatrix} \{\tilde{\mathbb{C}}_{21}^{(2)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{21}^{(2)} & \{\tilde{\mathbb{C}}_{21}^{(2)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(2)}\\ \{\tilde{\mathbb{C}}_{22}^{(2)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{21}^{(2)} & \{\tilde{\mathbb{C}}_{22}^{(2)}\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(2)}\end{bmatrix}.\end{aligned}$$

Thus, by comparing each block element on both sides of (82) and noting $\mathbb{P}_2 > 0$, we have that $\tilde{\mathbb{A}}_{21} = 0$, $\tilde{\mathbb{C}}_{21}^{(1)} = \tilde{\mathbb{C}}_{21}^{(2)} = 0$, and $\tilde{\mathcal{Q}}_1 = \tilde{\mathcal{Q}}_{12} = \tilde{\mathcal{Q}}_{21} = 0$, i.e.

$$U'\tilde{\mathbb{A}}U = \begin{bmatrix}\tilde{\mathbb{A}}_{11} & \tilde{\mathbb{A}}_{12}\\ 0 & \tilde{\mathbb{A}}_{22}\end{bmatrix}, U'\tilde{\mathbb{C}}U = \begin{bmatrix}\tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12}\\ 0 & \tilde{\mathbb{C}}_{22}\end{bmatrix}, U'\tilde{\mathcal{Q}}U = \begin{bmatrix}0 & 0\\ 0 & \tilde{\mathcal{Q}}_2\end{bmatrix} \tag{83}$$

where $\tilde{\mathcal{Q}}_2 \geq 0$, $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{11}^{(1)} + \tilde{\mathbb{C}}_{11}^{(2)}$, $\tilde{\mathbb{C}}_{12} = \tilde{\mathbb{C}}_{12}^{(1)} + \tilde{\mathbb{C}}_{12}^{(2)}$ and $\tilde{\mathbb{C}}_{22} = \tilde{\mathbb{C}}_{22}^{(1)} + \tilde{\mathbb{C}}_{22}^{(2)}$.

Substituting (81) and (83) into (82) yields that

$$\mathbb{P}_2 = \tilde{\mathcal{Q}}_2 + \tilde{\mathbb{A}}_{22}'\mathbb{P}_2\tilde{\mathbb{A}}_{22} + \sigma^2\left\{\tilde{\mathbb{C}}_{22}^{(1)}\right\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(1)} + \sigma^2\left\{\tilde{\mathbb{C}}_{22}^{(2)}\right\}'\mathbb{P}_2\tilde{\mathbb{C}}_{22}^{(2)}. \tag{84}$$

Defining $U'\mathbb{X}_k = \bar{\mathbb{X}}_k = \begin{bmatrix}\bar{\mathbb{X}}_k^{(1)}\\ \bar{\mathbb{X}}_k^{(2)}\end{bmatrix}$, where the dimension of $\bar{\mathbb{X}}_k^{(2)}$ is the same as the rank of $\mathbb{P}_2$. Thus, from (64), we have

$$U'\mathbb{X}_{k+1} = U'\tilde{\mathbb{A}}UU'\mathbb{X}_k + U'\tilde{\mathbb{C}}UU'\mathbb{X}_k w_k$$

i.e.

$$\bar{\mathbb{X}}_{k+1}^{(1)} = \tilde{\mathbb{A}}_{11}\bar{\mathbb{X}}_k^{(1)} + \tilde{\mathbb{A}}_{12}\bar{\mathbb{X}}_k^{(2)} + (\tilde{\mathbb{C}}_{11}\bar{\mathbb{X}}_k^{(1)} + \tilde{\mathbb{C}}_{12}\bar{\mathbb{X}}_k^{(2)})w_k, \tag{85}$$

$$\bar{\mathbb{X}}_{k+1}^{(2)} = \tilde{\mathbb{A}}_{22}\bar{\mathbb{X}}_k^{(2)} + \tilde{\mathbb{C}}_{22}\bar{\mathbb{X}}_k^{(2)}w_k. \tag{86}$$

First, we will show the stability of $(\tilde{\mathbb{A}}_{22}, \tilde{\mathbb{C}}_{22})$.

Actually, recall from (71) and (83), we have that

$$\begin{aligned}\sum_{k=0}^{N} E[(\bar{\mathbb{X}}_k^{(2)})'\tilde{\mathcal{Q}}_2\bar{\mathbb{X}}_k^{(2)}] &= \sum_{k=0}^{N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k)\\ &= E(\mathbb{X}_0'\mathbb{P}\mathbb{X}_0) - E(\mathbb{X}_{N+1}'\mathbb{P}\mathbb{X}_{N+1})\\ &= E[(\bar{\mathbb{X}}_0^{(2)})'\mathbb{P}_2\bar{\mathbb{X}}_0^{(2)}] - E[(\bar{\mathbb{X}}_{N+1}^{(2)})'\mathbb{P}_2\bar{\mathbb{X}}_{N+1}^{(2)}].\end{aligned} \tag{87}$$

Similar to Lemma 4, we conclude $\bar{\mathbb{X}}_0^{(2)}$ is an unobservable state of $(\tilde{\mathbb{A}}_{22}, \tilde{\mathbb{C}}_{22}, \tilde{\mathcal{Q}}_2^{1/2})$ if and only if $E[(\bar{\mathbb{X}}_0^{(2)})'\mathbb{P}_2\bar{\mathbb{X}}_0^{(2)}] = 0$. Since $\mathbb{P}_2 > 0$, thus $(\tilde{\mathbb{A}}_{22}, \tilde{\mathbb{C}}_{22}, \tilde{\mathcal{Q}}_2^{1/2})$ is exact observable. Therefore, following from the derivation of (71)–(79), we know that

$$\lim_{k\to+\infty} E(\bar{\mathbb{X}}_k^{(2)})'\bar{\mathbb{X}}_k^{(2)} = 0 \tag{88}$$

i.e., $(\tilde{\mathbb{A}}_{22}, \tilde{\mathbb{C}}_{22})$ is stable in the mean square sense.

Next, the stability of $(\tilde{\mathbb{A}}_{11}, \tilde{\mathbb{C}}_{11})$ will be shown as below.

We choose $\bar{\mathbb{X}}_0^{(2)} = 0$, then from (86), we have $\bar{\mathbb{X}}_k^{(2)} = 0$ for any $k \geq 0$. In this case, (85) becomes

$$\mathbb{Z}_{k+1} = \tilde{\mathbb{A}}_{11}\mathbb{Z}_k + \tilde{\mathbb{C}}_{11}\mathbb{Z}_k w_k \tag{89}$$

where $\mathbb{Z}_k$ is the value of $\bar{\mathbb{X}}_k^{(1)}$ with $\bar{\mathbb{X}}_k^{(2)} = 0$. Thus, for an arbitrary initial state $\mathbb{Z}_0 = \bar{\mathbb{X}}_0^{(1)}$, we have

$$E[\tilde{Y}_k'\tilde{Y}_k] = E[\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k] = E[(\bar{\mathbb{X}}_k^{(2)})'\tilde{\mathcal{Q}}_2\bar{\mathbb{X}}_k^{(2)}] \equiv 0. \tag{90}$$

From the exact detectability of $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$, it holds

$$\begin{aligned}\lim_{k\to+\infty} E(\bar{\mathbb{X}}_k'\bar{\mathbb{X}}_k) &= \lim_{k\to+\infty} E(\bar{\mathbb{X}}_k'U'U\bar{\mathbb{X}}_k)\\ &= \lim_{k\to+\infty} E(\mathbb{X}_k'\mathbb{X}_k) = 0.\end{aligned} \tag{91}$$

Therefore, in the case of $\bar{\mathbb{X}}_0^{(2)} = 0$, (91) indicates that

$$\begin{aligned}\lim_{k\to+\infty} E(\mathbb{Z}_k'\mathbb{Z}_k) &= \lim_{k\to+\infty} E[(\bar{\mathbb{X}}_k^{(1)})'\bar{\mathbb{X}}_k^{(1)}]\\ &= \lim_{k\to+\infty}\{E[(\bar{\mathbb{X}}_k^{(1)})'\bar{\mathbb{X}}_k^{(1)}] + E[(\bar{\mathbb{X}}_k^{(2)})'\bar{\mathbb{X}}_k^{(2)}]\}\\ &= \lim_{k\to+\infty} E(\bar{\mathbb{X}}_k'\bar{\mathbb{X}}_k) = 0.\end{aligned} \tag{92}$$

i.e., $(\tilde{\mathbb{A}}_{11}, \tilde{\mathbb{C}}_{11})$ is mean square stable.

Third, we will show that system (1) is stabilizable in the mean square sense.

In fact, we denote $\tilde{\mathcal{A}} = \begin{bmatrix}\tilde{\mathbb{A}}_{11} & 0\\ 0 & \tilde{\mathbb{A}}_{22}\end{bmatrix}$, $\tilde{\mathcal{C}} = \begin{bmatrix}\tilde{\mathbb{C}}_{11} & 0\\ 0 & \tilde{\mathbb{C}}_{22}\end{bmatrix}$. Hence, (85) and (86) can be reformulated as

$$\bar{\mathbb{X}}_{k+1} = \left\{\tilde{\mathcal{A}}\bar{\mathbb{X}}_k + \begin{bmatrix}\tilde{\mathbb{A}}_{12}\\ 0\end{bmatrix}\mathbb{U}_k\right\} + \left\{\tilde{\mathcal{C}}\bar{\mathbb{X}}_k + \begin{bmatrix}\tilde{\mathbb{C}}_{12}\\ 0\end{bmatrix}\mathbb{U}_k\right\}w_k \tag{93}$$

where $\mathbb{U}_k$ is as the solution to (86) with the initial condition $\mathbb{U}_0 = \mathbb{X}_0^{(2)}$. The stability of $(\tilde{\mathbb{A}}_{11}, \tilde{\mathbb{C}}_{11})$ and $(\tilde{\mathbb{A}}_{22}, \tilde{\mathbb{C}}_{22})$ as proved above indicates that $(\tilde{\mathcal{A}}, \tilde{\mathcal{C}})$ is stable in the mean square sense. Obviously from (88) it holds $\lim_{k\to+\infty} E(\mathbb{U}_k'\mathbb{U}_k) = 0$ and $\sum_{k=0}^{\infty} E(\mathbb{U}_k'\mathbb{U}_k) < +\infty$. Using [6, Proposition 2.8 and Remark 2.9], we know that there exists a constant c_0 such that

$$\sum_{k=0}^{\infty} E(\bar{\mathbb{X}}_k'\bar{\mathbb{X}}_k) < c_0\sum_{k=0}^{\infty} E(\mathbb{U}_k'\mathbb{U}_k) < +\infty. \tag{94}$$

Hence, $\lim_{k\to+\infty} E(\bar{\mathbb{X}}_k'\bar{\mathbb{X}}_k) = 0$ can be obtained from (94). Furthermore, it is noted from (91) that

$$\begin{aligned}\lim_{k\to+\infty} E(x_k'x_k) &= \lim_{k\to+\infty}[(x_k - Ex_k)'(x_k - Ex_k) + Ex_k'Ex_k]\\ &= \lim_{k\to+\infty} E(\mathbb{X}_k'\mathbb{X}_k) = \lim_{k\to+\infty} E(\bar{\mathbb{X}}_k'\bar{\mathbb{X}}_k) = 0.\end{aligned}$$

Note that the system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}})$ given in (64) is just the mean-field system (1) with the controller (47). In conclusion, the mean-field system (1) is stabilizable in the mean square sense.

In the following, we will show that the stabilizing controller (47) minimizes cost function (3).

In fact, from (65), there holds that

$$\begin{aligned}&E(x_{N+1}'Px_{N+1}) + Ex_{N+1}'\bar{P}Ex_{N+1}\\ &\quad - [E(x_0'Px_0) + Ex_0'\bar{P}Ex_0]\\ &= -E\sum_{k=0}^{N}[x_k'Qx_k + Ex_k'\bar{Q}Ex_k + u_k'Ru_k + Eu_k'\bar{R}Eu_k]\\ &\quad + E\sum_{k=0}^{N}[u_k - Eu_k - \mathcal{K}(x_k - Ex_k)]'\Upsilon^{(1)}\\ &\quad \times [u_k - Eu_k - \mathcal{K}(x_k - Ex_k)]\\ &\quad + \sum_{k=0}^{N}[Eu_k - (\mathcal{K} + \bar{\mathcal{K}})Ex_k]'\Upsilon^{(2)}[Eu_k - (\mathcal{K} + \bar{\mathcal{K}})Ex_k]\end{aligned} \tag{95}$$

where $\Upsilon^{(1)}$, $M^{(1)}$, $\Upsilon^{(2)}$, $M^{(2)}$ are given in (35)–(38), and $\mathcal{K}, \bar{\mathcal{K}}$ satisfy (41) and (42).

From $\lim_{k\to+\infty} Ex_k'x_k = 0$, obviously we can obtain that $\lim_{N\to+\infty}[E(x_{N+1}'Px_{N+1}) + Ex_{N+1}'\bar{P}Ex_{N+1}] = 0$. Thus, letting $N \to +\infty$, the cost function (3) can be rewritten from (95) as follows:

$$\begin{aligned} J &= E(x_0'P_0x_0) + Ex_0'\bar{P}_0Ex_0 \\ &+ E\sum_{k=0}^{\infty}[u_k - Eu_k - \mathcal{K}(x_k - Ex_k)]'\Upsilon^{(1)} \\ &\times [u_k - Eu_k - \mathcal{K}(x_k - Ex_k)] \\ &+ E\int_0^{\infty}[Eu_k - (\mathcal{K}+\bar{\mathcal{K}})Ex_k]'\Upsilon^{(2)}[Eu_k - (\mathcal{K}+\bar{\mathcal{K}})Ex_k]. \end{aligned} \tag{96}$$

It is noted from Assumption 1 that $\Upsilon^{(1)} \geq 0$ and $\Upsilon^{(2)} \geq 0$, then the cost function (3) is minimized by the controller (47), and the optimal cost function is given as (48).

Finally, $u_k \in \mathcal{U}_\infty$ can be shown as below.

In this case, for the stabilizing controller (47), i.e., $u_k = \mathcal{K}x_k + \bar{\mathcal{K}}Ex_k$, we have

$$E(u_k'u_k) = E[x_k'\mathcal{K}'\mathcal{K}x_k + Ex_k'(\bar{\mathcal{K}}'\mathcal{K} + \mathcal{K}'\bar{\mathcal{K}} + \bar{\mathcal{K}}'\bar{\mathcal{K}})Ex_k]. \tag{97}$$

From (59), we know that $\sum_{k=0}^{\infty} E(x_k'x_k) < +\infty$, therefore $\sum_{k=0}^{\infty} E(u_k'u_k) < +\infty$ can be induced from (97), i.e., $u_k \in \mathcal{U}_\infty$ can be verified.

"Necessity:" Under Assumption 1 and the exact detectability of system (5), if system (1) is mean square stabilizable, we will show the ARE (34) has a unique positive semidefinite solution. From Lemma 3, we know that ARE (34) admits a positive semidefinite solution.

In what follows, we will show the uniqueness of P and $\bar{P}$.

In fact, let S and $\bar{S}$ be another solution of (34) satisfying $S > 0$ and $S + \bar{S} > 0$, i.e.

$$S = Q + A'SA + \sigma^2C'SC - [T^{(1)}]'[\Delta^{(1)}]^{\dagger}T^{(1)}, \tag{98}$$

$$\begin{aligned} \bar{S} &= \bar{Q} + A'S\bar{A} + \sigma^2C'S\bar{C} + \bar{A}'SA + \sigma^2\bar{C}'SC \\ &+ \bar{A}'S\bar{A} + \sigma^2\bar{C}'S\bar{C} + (A+\bar{A})'\bar{S}(A+\bar{A}) \\ &+ [T^{(1)}]'[\Delta^{(1)}]^{\dagger}T^{(1)} - [T^{(2)}]'[\Delta^{(2)}]^{\dagger}T^{(2)} \end{aligned} \tag{99}$$

where

$$\begin{aligned} \Delta^{(1)} &= R + B'SB + \sigma^2D'SD, T^{(1)} = B'SA + \sigma^2D'SC, \\ \Delta^{(2)} &= R + \bar{R} + (B+\bar{B})'(S+\bar{S})(B+\bar{B}) \\ &+ \sigma^2(D+\bar{D})'S(D+\bar{D}), \\ T^{(2)} &= (B+\bar{B})'(S+\bar{S})(A+\bar{A}) \\ &+ \sigma^2(D+\bar{D})'S(C+\bar{C}) \end{aligned}$$

and the regular condition holds:

$$\Delta^{(i)}[\Delta^{(i)}]^{\dagger}T^{(i)} = T^{(i)}, i = 1, 2.$$

It is noted from the "sufficiency proof" that the optimal cost function has been proved to be (48), i.e.

$$J^* = E(x_0'Px_0) + Ex_0'\bar{P}Ex_0 = E(x_0'Sx_0) + Ex_0'\bar{S}Ex_0. \tag{100}$$

For any initial state x_0 satisfying $x_0 \neq 0$ and $Ex_0 = 0$, (100) implies that

$$E[x_0'(P - S)x_0] = 0$$

thus, we can conclude that $P = S$.

Moreover, if $x_0 = Ex_0$ is an arbitrary deterministic initial state, it follows from (100) that

$$x_0'(P + \bar{P} - S - \bar{S})x_0 = 0$$

which indicates $P + \bar{P} = S + \bar{S}$.

Hence, we have $S = P$ and $\bar{S} = \bar{P}$, i.e., the uniqueness has been proven. The proof is complete. ∎

APPENDIX F
PROOF OF THEOREM 4

Proof: "Sufficiency": Under Assumption 1 and the exact observability of $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$, if the coupled ARE (34) has a unique positive definite solution, $P > 0$ and $P + \bar{P} > 0$, we shall show that mean-field system (1) is stabilizable with controller (47) in the mean square sense.

Following from Remark 7, if $(A, \bar{A}, C, \bar{C}, \mathcal{Q}^{1/2})$ is exact observable, we know system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$ is exact observable.

By following the discussions of (71)–(79) in the proof of Theorem 3, then system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}})$ is mean square stable. Therefore, mean-field system (1) can be mean square stabilizable with the controller (47).

"Necessity": Suppose Assumption 1 and the exact observability of (5) hold, if the system (1) is mean square stabilizable, we will show that the coupled ARE (34) admits a unique positive definite solution.

First, under Assumption 1, from Lemma 3 and (98)–(100), we know that the coupled ARE (34) admits a unique solution satisfying $P \geq 0$ and $P + \bar{P} \geq 0$. In what follows, $P > 0$ and $P + \bar{P} > 0$ will be shown.

In fact, suppose this is not true, since $E(x_0'x_0) = E(\mathbb{X}_0'\mathbb{X}_0)$, then there exists $\mathbb{X}_0 \neq 0$ (i.e., $x_0 \neq 0$) satisfying $E(\mathbb{X}_0'\mathbb{P}\mathbb{X}_0) = 0$, the symbols $\mathbb{P}, \mathbb{X}_k$ are given in (5) and Lemma 4.

From Lemma 4, we know that the mean square stabilization of system (1) with controller (47) is equivalent to the mean square stable of system (46) $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$, and the solution to ARE $P, \bar{P}$ satisfies Lyapunov equation (43)–(44). Next, by following from the derivation of (65) and letting the initial state be $\mathbb{X}_0$ defined above, we obtain

$$0 \leq \sum_{k=0}^{N} E(\mathbb{X}_k'\tilde{\mathcal{Q}}\mathbb{X}_k) = -E(\mathbb{X}_N'\mathbb{P}\mathbb{X}_N) \leq 0 \tag{101}$$

which indicates $\tilde{\mathcal{Q}}^{1/2}\mathbb{X}_k \equiv 0$, a.s.

On the other hand, as stated in Remark 7, the exact observability of system $(\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathcal{Q}}^{1/2})$ can be obtained from the exact observability of system (5). Thus, we can conclude $\mathbb{X}_0 = 0$, which contradicts with $\mathbb{X}_0 \neq 0$. Therefore, we can conclude $P > 0$ and $P + \bar{P} > 0$.

Finally, by following (96), we know that the stabilizing controller (47) minimizes the cost function (3). Moreover, similar to (97), $u_k \in \mathcal{U}_\infty$ can be derived. The proof is complete. ∎

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