A Sequential Cluster-Based Approach to Node Localizability of Sensor Networks

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***Abstract*—This paper addresses the node localizability problem in 2-D, aiming at determining which nodes are localizable in a sensor network with internode distance measurements. Toward this objective, a sequential cluster-based approach is proposed. The idea is to expand the set of localizable nodes starting from the set of anchor nodes cluster by cluster rather than node by node. Taking computation efficiency in practice into account, we consider the size of a cluster up to four nodes each time. Then, we develop a necessary and sufficient condition to determine whether a cluster up to four nodes is jointly localizable given the distance measurements among them and the distance measurements between them and their neighbors with known coordinates. In terms of this necessary and sufficient condition, both centralized and distributed algorithms are developed for detecting localizable nodes in a given sensor network. It is demonstrated that our approach outperforms well-known techniques, such as trilateration, bilatration, and wheel extension in finding as many localizable nodes as possible.**

***Index Terms*—Graph rigidity, localizability, sensor networks.**

I. Introduction

LOCALIZABILITY is a fundamental issue in localization of sensor networks. The latter focuses on how to compute the coordinates of sensor nodes by using measurements among nodes, including distance [1], [2]; angle-of-arrival [3]; and hop counts [4]. In contrast, the former aims to determine whether all available measurements are enough to uniquely determine the locations of sensor nodes. In this paper, rather than providing algorithms for locating sensor nodes, we concentrate on the problem of identifying localizable nodes in a given sensor network with internode distance measurements, called the *node localizability problem*. This problem is important because without excluding nonlocalizable nodes in a sensor network, the localization accuracy of the entire network will be greatly degraded, especially for concurrent localization algorithms. The sensor networks considered in this paper are assumed to be in the plane.

It is well known that the distance-based localizability of a sensor network is closely related to the topology of the distance graph of the network. Graph rigidity theory has been widely adopted recently to characterize the topological conditions for localizability of the entire network [5]–[10] or a single sensor node [10]–[12]. More precisely, a wireless-sensor network is modeled as a distance graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of sensor nodes and $\mathcal{E}$ denotes the set of edges between nodes to indicate the availability of distance measurements. In other words, for every pair of nodes $i, j \in \mathcal{V}$, there exists an edge $(i, j) \in \mathcal{E}$ if nodes i and j have a distance measurement between them. The network is called localizable if every node in the network is localizable [13]. Eren *et al.* [8] give a necessary and sufficient condition for *network localizability*, which states that a network is localizable if and only if it contains at least three anchors and its corresponding distance graph is globally rigid. However, many sensor networks in practice (e.g., randomly deployed networks) are not entirely localizable [11]. In such a case, the necessary and sufficient condition in [8] does not tell how to discover localizable or unlocalizable nodes in the network.

In contrast to network localizability, the *node localizability problem* [11] is concerned with determining whether a specific node in a network is localizable, how many nodes in a sensor network are localizable, and which ones they are. Unfortunately, there are no easily computable results for node localizability. On the one hand, two necessary conditions are developed in [10] and [11] to determine whether a node is localizable or not. In [10], Goldenberg *et al.* show that if a node is localizable, then it has three disjoint paths to three anchors. This is called the *3-path necessary condition*. Yang *et al.* derive a tighter but more complicated necessary condition in [11], yet not a necessary and sufficient one. On the other hand, the most commonly used approach for node localization is the so-called *trilateration* approach, which requires each node to have distance measurements with at least three nodes, whose coordinates are already known, in order to sequentially compute the coordinates node by node [6]. Moreover, the Sweeps algorithm is proposed in [7] and the extended analysis of the algorithm is given in [14]. It relies on a bilateration ordering for sequential localization. However, a localizable node may not be in a bilateration ordering in a sensor network. In particular,

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the four-bar linkage case is noticed in [7] as unsolved by any existing bilateration-based localization technique. In addition to the trilateration and biletaration techniques, Yang *et al.* [12] give a different sequential method called WHEEL extension, which detects localizable nodes by looking at a set of nodes with a wheel structure. But the approach is still not able to find all localizable nodes in a sensor network.

Motivated by the fact that a cluster of nodes, which cannot be localized one by one, may be jointly localizable by considering all of the distance measurements related to them at the same time, we propose a sequential cluster-based approach in this paper to address the node localizability problem and aim at providing principles, according to which as many localizable nodes as possible in a 2-D network can be detected. The idea is to check a cluster than a node each time to verify whether it is localizable. In theory, when the size of a cluster to be checked can vary from one to the total number of networks, then the sequential cluster-based approach is able to successfully find all localizable nodes in the network. But for computation efficiency and feasibility of distributed verification in practice, the size of each cluster cannot be large. Therefore, in this paper, we consider each cluster of four nodes at the most. Simulation results show that the number of localizable nodes detected by our approach is very close to the total number of all localizable nodes in the network.

In order to apply the sequential cluster-based approach for the node localizability problem, another fundamental question has to be answered. Namely, under what conditions, is a cluster up to four nodes jointly localizable given their distance measurements about other nodes with known coordinates? To the best of our knowledge, there is no complete solution for this problem. This paper develops a necessary and sufficient condition for the localizability of a cluster up to four nodes by mainly using the Henneberg 1-extension operation. The condition is characterized by the number of edges and distinct neighbors as well as the topological pattern of the cluster to be checked. Thus, it is more suitable for distributed verification by the nodes themselves.

Compared with the trilateration [6], bilateration [7], [14], and wheel extension techniques [12], our approach shows advantages in the capability of finding more localizable nodes in a sensor network. The reason behind this is that all of these techniques are special cases of our sequential cluster-based result with each cluster up to four nodes every step, and the networks satisfying our sequential cluster-based localizability condition represent a broader class of networks. A very similar result to ours is obtained recently in [15], but it addresses the problem from a different perspective by considering a split-stitch paradigm. That is, a network is split into pieces of clusters that may overlap each other and are then stitched together by choosing appropriate nodes. The stitch paradigm of [15] checks all numerical solutions for nodes locations by exploring both bilateration ordering and four-bar linkage structure. The idea of checking clusters sequentially for localizability developed in this paper and [15] has its roots in an earlier paper [16] that explores the principles to control autonomous merging formation. Compared with [15] and [16], the contribution of this paper (and our preliminary work in [17]) lies in the following aspects. First, the four-bar linkage structure considered in [15] assumes that the cluster to be checked is globally rigid, while we show in this paper that it is not necessary. Second, we provide a necessary and sufficient condition regarding the edge numbers and connectivity patterns for joint localizability of a cluster up to four nodes, which is more suitable for the sequential cluster-based localizabillity test by varying the cluster size from one to four iteratively. Third, for our approach, no split operation is required and a cluster of size from one to four can be verified for localizability by our proposed distributed algorithm in a sequential order. But how to split the network in the approach of [15] certainly affects the localizability result.

The rest of this paper is organized as follows. In Section II, we present some basic notions from graph theory and introduce the problem we study. In Section III, we derive a necessary and sufficient localizability condition for joint localizability of a cluster up to four nodes. In Section IV, we compare our proposed localizability condition with several known conditions. In Section V, localizability test algorithms for finding localizable nodes are given. In Section VI, we evaluate the proposed localizability conditions through several simulations. Section VII concludes our work.

II. Preliminary and Problem Statement

A. Graph Rigidity and Network Localizability

The localizability problem of a sensor network is often characterized through analyzing rigidity properties of the associated graph, which describes the internode distance constraints of the sensor network. In the following text, we first introduce the concepts of framework, rigidity, and localizability.

A framework is a pair $(\mathcal{G}, \mathbf{p})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes a graph with each edge specifying a distance constraint and $\mathbf{p} = [p_1, \cdots, p_N]^T$ associates each node $i \in \mathcal{V}$ with an Euclidean coordinate $p_i \in \mathbb{R}^2$. A framework $(\mathcal{G}, \mathbf{p})$ is said to be *generic* if the coordinates $p_1, \cdots, p_N$ do not satisfy any nontrivial algebraic equation with rational coefficients [14], [18], [19]. Intuitively speaking, a generic configuration has no degeneracy, that is, no three points staying on the same line, no three lines go through the same point, etc.

A framework $(\mathcal{G}, \mathbf{p})$ is called *rigid* if it cannot be continuously deformed without changing the distances for pairs of nodes connected by the edges in $\mathcal{G}$ and is called *redundantly rigid* if it is still rigid after removing any one edge. Moreover, a framework $(\mathcal{G}, \mathbf{p})$ is called *globally rigid* if there is a single realization with the given distance constraints for pairs of nodes connected by the edges in $\mathcal{G}$ and is called *minimally globally rigid* if it is globally rigid but no longer globally rigid after removing any one edge. A graph $\mathcal{G}$ is then said to be (generically) rigid (redundantly rigid, globally rigid, minimally globally rigid) if, for any generic configuration $\mathbf{p}$, the framework $(\mathcal{G}, \mathbf{p})$ is rigid (redundantly rigid, globally rigid, minimally globally rigid). These concepts are illustrated by examples in Fig. 1, where the top-left graph is not rigid because it can be continuously deformed; the top-right graph is rigid but not redundantly rigid; the bottom-left graph (solid line) is also rigid but not globally rigid, and a "flip" of the bottom node is

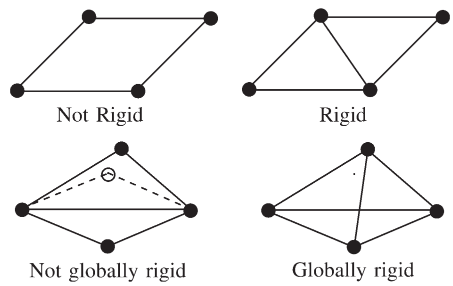


Fig. 1. Examples of rigidity notions.

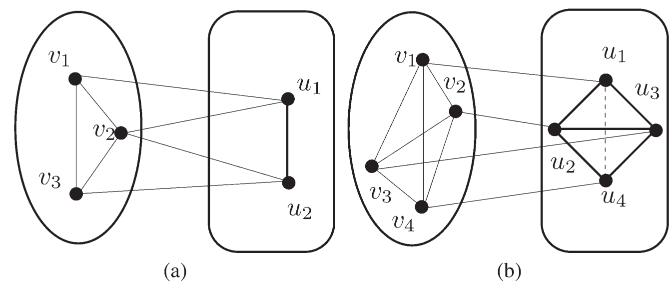


Fig. 2. (a) Set of jointly localizable nodes with bilateration ordering. (b) A set of jointly localizable nodes without bilateration ordering.

shown (dashed line); the bottom-right graph is globally rigid. For more details on graph rigidity, see [13] for an example.

Consider a sensor network consisting of *anchor nodes*, whose positions are known, and normal sensor nodes, whose positions are unknown. We use a graph $\mathcal{G}$, called the *distance graph*, to model the sensor network, where each sensor node including anchor nodes corresponds to a node of $\mathcal{G}$ and for any pair of anchor nodes, there is always an edge in $\mathcal{G}$ between them, while for any other pair, there is an edge if and only if the distance between the pair of nodes is available. A sensor network is said to be *localizable* if the positions of all the sensor nodes in the network can be uniquely determined given the known positions of anchor nodes and the internode distance measurements as described in $\mathcal{G}$. This is called a *network localizability problem*. The solution to the network localizability problem is closely related to global rigidity. A necessary and sufficient condition of network localizability in 2-D is presented in [8] and is stated as follows.

Lemma 1: A sensor network in 2-D is localizable if and only if it contains at least three anchor nodes and the distance graph $\mathcal{G}$ is globally rigid.

B. Problem Statement

Although the theory for network localizability is complete, the following two fundamental questions still remain open, which ARE raised in [11].

P1. Given a sensor network modeled by the distance graph $\mathcal{G}$, whether or not a specific node in the network is localizable?

P2. How many nodes in a sensor network can be located and which are them?

The two questions P1 and P2 are closely related, called the *node localizability problem*. A necessary condition for node localizability in 2-D is developed in [7], which is stated as follows.

Lemma 2: If a node of a sensor network in 2-D is localizable, then it has at least three disjoint paths to three different anchor nodes.

However, for the time being, there is still no complete solution for the node localizability problem. This paper aims to address the node localizability problem from two aspects. First, develop an approach to find as many localizable nodes as possible with a very small gap to find all localizable nodes in a sensor network. Second, develop a distributed scheme for node localizability verification.

A sequential cluster-based approach is proposed in this paper to deal with the node localizability problem. The basic idea is to expand the set of localizable nodes *cluster by cluster* starting from the set of anchor nodes. This is mainly motivated by the fact that a node alone may not be localizable given its distance measurements about its neighbors whose positions are already determined, but a cluster may be localizable jointly by using all of the distance measurements among them and the distance measurements about their neighbors whose positions are already determined. To demonstrate this, two examples are given in Fig. 2. In Fig. 2(a), the cluster in the left corresponds to the set of nodes with their positions known, while the cluster in the right corresponds to the set of nodes to be localized. For this case, each node to be localized has two edges connecting to the nodes with known positions (in a bilateration order [14]). So neither u_1 nor u_2 alone can be localized using its two distance measurements about the cluster nodes in the left. But nodes u_1 and u_2 together are localizable as we will show later. In Fig. 2(b), each node u_i to be localized has only one edge connecting to nodes $v_1, \ldots, v_4$ with known positions. So it is even not in a bilateraton order and cannot be localized alone by using its distance measurement about the nodes in the left. However, as we will show in this paper, the cluster of the four nodes $u_1, \ldots, u_4$ is localizable given the distance measurements shown in the graph.

In theory, when the size of a cluster to be localized can vary from one to the total number of sensor nodes in the network, then it is certain that all localizable nodes in the network can be determined by our proposed sequential cluster-based approach (i.e., the node localizability problem can be solved). However, due to the requirements of computation efficiency and distributed verification in practice, the size of a cluster to be checked cannot be very large. Therefore, to make a tradeoff, we consider a cluster of size up to four nodes in this paper. As we will show later in the simulation, the gap between the set of all localizable nodes in the network and the set of localizable nodes verified by our proposed sequential cluster-based approach with each cluster up to four nodes is very minor.

In order to address the node localizability problem using the sequential cluster-based approach with each cluster up to four nodes, we need to provide the answer to the following subproblem, namely, what is the necessary and sufficient condition for a cluster up to four nodes that is localizable. A recent work [15] has tackled a similar problem by looking into two cases (the bilateration case and the four-bar linkage case). However, no necessary and sufficient condition is characterized. Moreover, it is assumed in [15] that both clusters are globally rigid. This, however, may not be the case for the sequential cluster-based localizability test. Fig. 2(b) is an example where the cluster in the right is not globally rigid but is localizable.

III. Joint Localizability for a Cluster up to Four Nodes

This section deals with the subproblem of whether a cluster up to four nodes is localizable and will provide a necessary and sufficient condition. To formally state the problem, we let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ denote the graph corresponding to the cluster of nodes whose positions are already known, let $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ denote the graph corresponding to the cluster of nodes to be localized, and let $\mathcal{E}_3$ be the set of edges connecting $\mathcal{G}_1$ and $\mathcal{G}_2$. Notice that all the nodes in $\mathcal{G}_1$ know their positions, so we assume that every pair of nodes in $\mathcal{G}_1$ is adjacent. Moreover, we assume that $\mathcal{G}_1$ has at least three nodes. On the other hand, if $\mathcal{G}_2$ is not connected, then it is either not localizable or can be decomposed into multiple smaller clusters that are localizable independently. Therefore, we assume that $\mathcal{G}_2$ is connected. Now the problem becomes to characterize the connectivity patterns of $\mathcal{E}_3$ such that the concatenated graph $\mathcal{G} = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$ is globally rigid, which is equivalent to the localizability of the cluster $\mathcal{G}_2$ by Lemma 1.

The analysis of our main result on the localizability of a cluster up to four nodes will heavily rely on Henneberg extension operations on a graph. One type of Henneberg extension operation, called *1-extension*, will be used. The concept of 1-extension is introduced here first. A *1-extension* on a graph is the operation where two adjacent nodes of the graph are first selected, say nodes u and v, and a new node w is added to the graph by making w adjacent to node u, v, and x, where x is distinct from u and v, and removing the edge between u and v [20]. In [21], it is shown that the graph resulting from a 1-extension operation on any globally rigid graph of four or more nodes is again globally rigid.

Remark 1: The 1-extension operation can also be understood as follows. For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we remove a node $u \in \mathcal{V}$ and edges $(u, v), (u, w), (u, t) \in \mathcal{E}$ and then add one edge between a pair of distinct and nonadjacent nodes x, y taken from $\{v, w, t\}$ to obtain a new graph $\mathcal{G}^-$. If $\mathcal{G}^-$ is globally rigid, then $\mathcal{G}$ is also globally rigid.

Next, we introduce several notations and concepts from graph theory to clearly present our main result. We use $\pi(u_i), u_i \in \mathcal{G}_2$, to indicate the neighbor set of u_i in $\mathcal{G}_1$, that is, $\pi(u_i) = \{v_i \in \mathcal{V}_1 : (v_i, u_i) \in \mathcal{E}_3\}$. Moreover, we use $\pi(\mathcal{G}_2)$ to represent the set of nodes in $\mathcal{G}_1$, which have edges connecting to the nodes in $\mathcal{G}_2$, that is, $\pi(\mathcal{G}_2) = \bigcup_{u_i \in \mathcal{G}_2} \pi(u_i)$. A graph is called a *cycle graph* if it consists of a single cycle. A graph is called *complete* if every pair of vertices are adjacent, and the complete graph on n vertices is denoted by $\mathcal{K}_n$ [22]. A graph is called a *goblet graph* if it is formed by adding a new node and a new edge connecting the new node to a node in the complete graph K_3.

Now we are ready to state the necessary and sufficient condition for localizability of $\mathcal{G}_2$, which only requires checking the number of edges and the number of distinct neighbors in $\mathcal{G}_1$ as well as certain connectivity patterns for two special cases. Thus, the condition is more suitable for distributed verification by the nodes themselves in the cluster.

Theorem 1: Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ be a complete graph having arbitrary number of nodes satisfying $|\mathcal{V}_1| \geq 3$ and let $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be a connected graph satisfying $|\mathcal{V}_2| \leq 4$. Suppose the positions of the nodes in $\mathcal{G}_1$ are known and the positions of the nodes in $\mathcal{G}_2$ are unknown. Then, $\mathcal{G}_2$ is localizable via a set of connecting edges $\mathcal{E}_3$ to $\mathcal{G}_1$ if and only if the following three conditions are satisfied:

1) *Three-Edge Condition*: Each node in $\mathcal{G}_2$ has at least three edges;
2) *Connecting-Edge Condition*: $|\mathcal{E}_3| + |\mathcal{E}_2| \geq 2|\mathcal{V}_2| + 1$;
3) *Connectivity Pattern Condition*:
   a) $|\pi(\mathcal{G}_2)| \geq 3$ if $|\mathcal{V}_2| = 1$;
   b) $|\pi(\mathcal{G}_2)| \geq 3$ if $|\mathcal{V}_2| = 2$;
   c) $|\pi(\mathcal{G}_2)| \geq 3$ and $|\pi(u_i) \cup \pi(u_j)| \geq 2$ for any $u_i, u_j \in \mathcal{G}_2$ if $|\mathcal{V}_2| = 3$;
   d) $|\pi(\mathcal{G}_2)| \geq 3$ if $|\mathcal{V}_2| = 4$ and $|\mathcal{E}_2| = 3$ or 5 or 6;
   e) $|\pi(\mathcal{G}_2)| \geq 3$ and $|\pi(u_i) \cup \pi(u_j) \cup \pi(u_k)| \geq 2$ for any $u_i, u_j, u_k \in \mathcal{G}_2$ if $\mathcal{G}_2$ is a cycle graph with $|\mathcal{V}_2| = 4$ and $|\mathcal{E}_2| = 4$;
   f) $|\pi(\mathcal{G}_2)| \geq 3$ and $|\pi(u_i^*) \cup \pi(u_j^*)| \geq 2$ if $\mathcal{G}_2$ is a goblet graph with $|\mathcal{V}_2| = 4$ and $|\mathcal{E}_2| = 4$, where u_i^* and u_j^* are the two corner nodes of the goblet graph [Fig. 3(f.1)-(f.2)].

Proof: Since the nodes in $\mathcal{G}_1$ all have known positions and they can be considered as anchor nodes, then it follows from Lemma 1 that $\mathcal{G}_2$ is localizable via $\mathcal{E}_3$ if and only if the concatenated graph $\mathcal{G} = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$ is globally rigid. Hence, it equivalent to show that the Three-Edge condition on $\mathcal{G}_2$ and the Connecting-Edge condition on $|\mathcal{E}_3|$ and the Connectivity Pattern condition are necessary and sufficient for $\mathcal{G}$ to be globally rigid.

(Necessity): First, the Three-Edge condition is necessary because by Lemma 2, every node in $\mathcal{G}_2$ should have at least three disjoint paths to three different anchor nodes, which implies that each node has at least three edges.

Second, it is known from [19] that if a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is globally rigid, then $|\mathcal{E}| \geq 2|\mathcal{V}| - 2$. For our case, we then have

$$|\mathcal{E}_1| + |\mathcal{E}_2| + |\mathcal{E}_3| \geq 2(|\mathcal{V}_1| + |\mathcal{V}_2|) - 2. \quad (1)$$

Since $\mathcal{G}_2$ is localizable via a set $\mathcal{E}_3$ of connecting edges to a complete graph $\mathcal{G}_1$ with an arbitrary number of nodes satisfying $|\mathcal{V}_1| \geq 3$, it must hold when $\mathcal{G}_1$ is a complete graph with three nodes, for which $|\mathcal{V}_1| = 3$ and $|\mathcal{E}_1| = 3$. Thus, the Connecting-Edge condition follows from (1).

Third, we prove that $|\pi(\mathcal{G}_2)| \geq 3$ is necessary for all cases listed in the Connectivity Pattern condition. Suppose by contradiction that there exists a globally rigid concatenated graph $\mathcal{G}$ satisfying $|\pi(\mathcal{G}_2)| \leq 2$. Then for any node in $\mathcal{G}_2$, say u_i, it holds $|\pi(u_i)| \leq 2$ according to the definition of $\pi(\mathcal{G}_2)$. This leads each node in $\mathcal{G}_2$ to have no more than two neighbors in $\mathcal{G}_2$ and, thus, no more than two disjoint paths to anchor nodes, which contradicts with Lemma 2. So $|\pi(\mathcal{G}_2)| \geq 3$ is necessary.

Finally, it remains to prove the necessity of the Connectivity Pattern condition for cases c), e), and f).

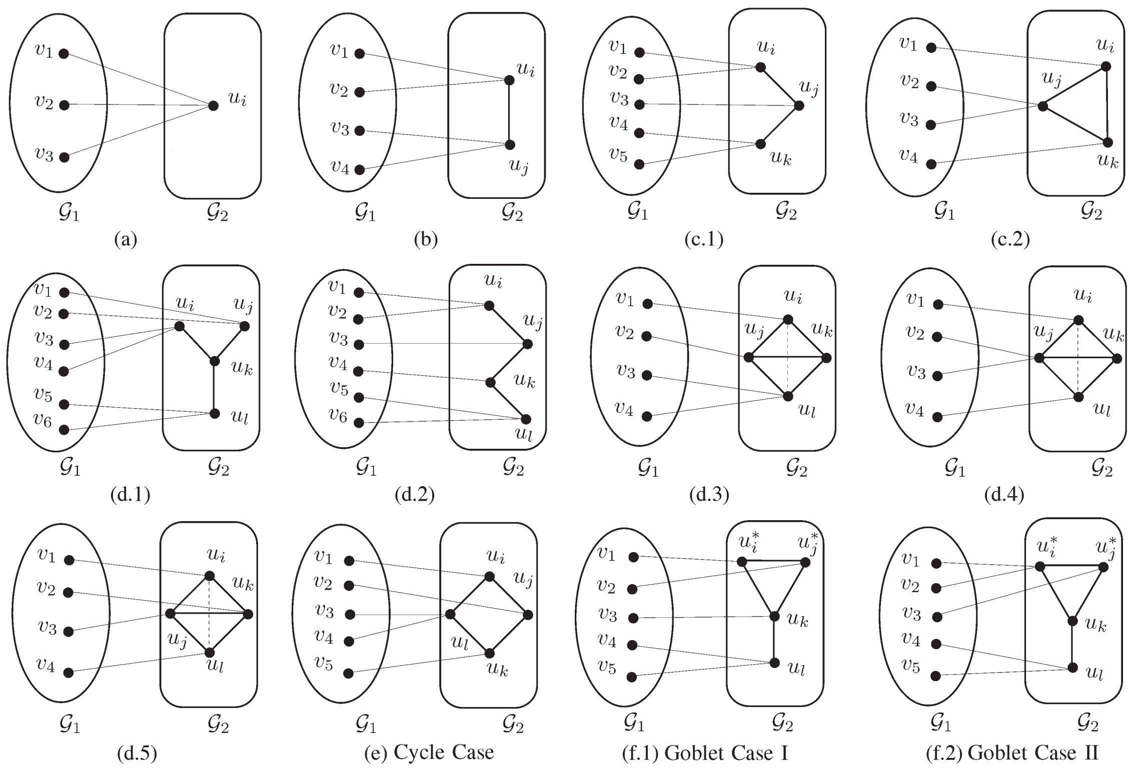


Fig. 3. Feasible connectivity patterns for localizability. Note that these shown in this figure are graphs rather than frameworks.

For case (c), suppose by contradiction that there exists a pair of u_i and u_j satisfying $|\pi(u_i) \cup \pi(u_j)| < 2$. This means that $\pi(u_i)$ is the same node with $\pi(u_j)$. We name this node v_x. Since $|\mathcal{V}_2| = 3$, there exists another node u_k in $\mathcal{G}_2$. Then, removing v_x and u_k will lead the graph $\mathcal{G}$ to be not connected and, thus, not globally rigid. So, $|\pi(u_i) \cup \pi(u_j)| \geq 2$ is necessary in this case.

For case (e), suppose by contradiction that there exists a group of u_i, u_j, and u_k satisfying $|\pi(u_i) \cup \pi(u_j) \cup \pi(u_k)| < 2$ and u_l has two connecting edges to $\mathcal{G}_1$. We name the intersected node of $\pi(u_i) \cup \pi(u_j) \cup \pi(u_k)$ as v_x. Then, removing v_x and u_l will lead the graph $\mathcal{G}$ to not be connected and, thus, not globally rigid. So $|\pi(u_i) \cup \pi(u_j) \cup \pi(u_k)| \geq 2$ is necessary.

For case (f), suppose by contradiction that there exists a pair of u_i^* and u_j^* satisfying $|\pi(u_i^*) \cup \pi(u_j^*)| < 2$. Assume that $v_x = \pi(u_i^*) \cup \pi(u_j^*)$ and the middle node in the goblet graph is u_k, as shown in Fig. 3(f.1)-(f.2). Then, removing node v_x and u_k will lead $\mathcal{G}$ to not be connected and, thus, not globally rigid. So, $|\pi(u_i^*) \cup \pi(u_j^*)| \geq 2$ is necessary for the goblet case.

(Sufficiency): We need to show that the graphs that satisfy each case of Connectivity Pattern condition, combining the Three-Edge condition and Connecting-Edge condition, are globally rigid. We will prove theses cases one by one. For easy understanding, we let $|\pi(\mathcal{G}_2)|$, that is, the cardinality of $\pi(\mathcal{G}_2)$, equal to a fixed constant for each situation. When $|\pi(\mathcal{G}_2)|$ is equal to other values, the proof is similar and, thus, omitted.

When $|\mathcal{V}_2| = 1$, the Three-Edge condition guarantees the only node in $\mathcal{G}_2$, say u_i as shown in Fig. 3(a), having no less than three edges to $\mathcal{G}_1$. Then, $|\pi(G_2)| \geq 3$ leads u_i to have at least three neighbors in $\mathcal{G}_1$. In this way, node u_i fits the trilateration condition and, thus, the concatenated graph $\mathcal{G}$ is globally rigid.

When $|\mathcal{V}_2| = 2$, we prove the global rigidity of $\mathcal{G}$ by using 1-extension. Take the graph in Fig. 3(b) for example. After removing node u_j and the edges between u_j with u_i, v_3 and v_4, and adding one edge between node u_i and v_3, we obtain a new graph $\mathcal{G}^-$, which has the same topology as the graph shown in Fig. 3(a), that is, the case of $|\mathcal{V}_2| = 1$. As we have proved, the graph in the case of $|\mathcal{V}_2| = 1$ is globally rigid. So, according to Remark 1, $\mathcal{G}$ is also globally rigid when $|\mathcal{V}_2| = 2$.

When $|\mathcal{V}_2| = 3$ and $|\mathcal{E}_2| = 2$, take the graph in Fig. 3(c.1) for example. Removing node u_k and edges $u_k u_j$, $u_k v_4$, and $u_k v_5$, and adding edge $u_j v_4$ leads to the same graph with the case in Fig. 3(b). Also, according to Remark 1, we know that $\mathcal{G}$ is globally rigid when $|\mathcal{V}_2| = 3$ and $|\mathcal{E}_2| = 2$.

When $|\mathcal{V}_2| = 3$ and $|\mathcal{E}_2| = 3$, see Fig. 3(c.2) for example. Removing node u_k and edges $u_k u_i$, $u_k u_j$ and $u_k v_4$ and adding edge $u_i v_4$ between u_i and v_4 derive a new graph $\mathcal{G}^-$, which has the same topology as the graph in Fig. 3(b). Thus, according to Remark 1, graph $\mathcal{G}$ when $|\mathcal{V}_2| = 3$ and $|\mathcal{E}_2| = 3$ is globally rigid.

When $|\mathcal{V}_2| = 4$ and $|\mathcal{E}_2| = 3$, two examples with $|\pi(\mathcal{G}_2)| = 6$ are shown in Fig. 3(d.1)-(d.2). In each case, removing node u_l and three edges $u_l u_k$, $u_l v_5$, $u_l v_6$, and adding one edge $u_k v_5$ produce the same globally rigid graph as the case in Fig. 3(c.1). So, according to Remark 1, $\mathcal{G}$, in this case, is also globally rigid.

When $|\mathcal{V}_2| = 4$, $|\mathcal{E}_2| = 5$, take Fig. 3(d.3)-(d.5) for example. In each case, removing node u_i and edges $u_i u_j$, $u_i u_k$, $u_i v_1$, and adding edge $u_k v_1$ lead to the same graph with that in Fig. 3(c.2). As a result, graph $\mathcal{G}$ proved to be globally rigid. Note that when $|\mathcal{E}_2| = 6$ (with the dashed line in $\mathcal{G}_2$), the global rigidity is identical to $|\mathcal{E}_2| = 5$ because the extra internal edge does not affect the allocation of the connecting edges in $\mathcal{E}_3$.

When $|\mathcal{V}_2| = 4$, $|\mathcal{E}_2| = 4$ and $\mathcal{G}_2$ is a cycle graph, take Fig. 3(e) for example, removing node u_k and edges u_ku_j, u_ku_l, u_kv_5, and adding one edge between u_j and any one of v_1, v_3, v_4 lead to a graph $\mathcal{G}^-$ with the same topology as the graph shown in Fig. 3(c.1). So, it follows from the case in Fig. 3(c.1) and Remark 1 that $\mathcal{G}$ in Fig. 3(e) is globally rigid.

When $|\mathcal{V}_2| = 4$, $|\mathcal{E}_2| = 4$, and $\mathcal{G}_2$ is a goblet graph, two examples with $|\pi(\mathcal{G}_2)| = 5$ are shown in Fig. 3(f.1)-(f.2). Removing node u_l and edges u_lu_k, u_lv_4, u_lv_5 and adding edge u_k to any one of v_1, v_2, v_3 derive a new graph $\mathcal{G}^-$ having the same topology as the graph shown in Fig. 3(c.2). Thus, the graphs in Fig. 3(f.1)-(f.2) are globally rigid. ■

Remark 2: The necessary and sufficient condition of a cluster $\mathcal{G}_2$ up to four nodes is characterized by the number of edges and the number of distinct neighbors in $\mathcal{G}_1$ for most cases (namely, the cluster has one node, two nodes, three nodes, or four nodes except for the case with four edges). The case corresponding to a cluster with four nodes and four edges is a little bit complicated, which is treated in case e) and f) depending on its pattern. The technical proof for Theorem 1 is straightforward by iteratively using 1-extension operation. However, the condition we develop in Theorem 1 is simple enough such that the nodes in the cluster can check it in a distributed way.

IV. Beyond Trilateration, Bilateration and WHEEL Extension

Theorems 1 in the preceding section shows a necessary and sufficient condition for localizability of a cluster up to four nodes. This result can then be used to sequentially find localizable nodes in a given sensor network. In other words, starting from the set of anchor nodes, localizable nodes can be detected cluster by cluster for the node localizability problem.

With the similar sequential idea for the node localizability problem, the trilateration approach detects localizable nodes node by node sequentially [7]. For a set of nodes $u_i, i = 1 \ldots n$, if there exists an ordering, say $u_1, u_2, \ldots, u_n$, such that each node u_j, for $j > 3$, can find at least three localizable neighbors in $u_1, u_2, \ldots, u_{j-1}$, then this set is *sequentially localizable* according to this ordering, when the first three nodes are anchor nodes. This ordering is called the *trilateration ordering*. In other words, the trilateration approach checks a cluster of only one node at a time, which is the case of $|\mathcal{V}_2| = 1$, as shown in Fig. 3(a).

Bilateration is one extended version of trilateration ordering. For a set of nodes $u_1, u_2, \ldots, u_n$, if for every u_j, $j > 2$, there exist at least two neighbors in $u_1, u_2, \ldots, u_{j-1}$, then we say this set of nodes has a bilateration ordering with the first two nodes being the anchor nodes. All nodes of this set can be finitely localized, which means that there are a finite number of possible positions for these nodes and, thus, it is possible to identify the correct positions of these nodes through traversing all candidate combinations of positions [7], [14].

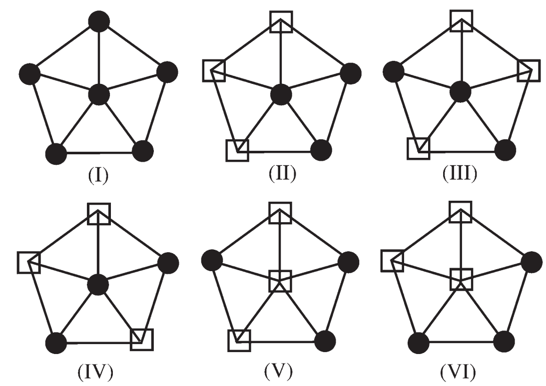


Fig. 4. (I) is a normal WHEEL structure. (II)–(VI) correspond to different anchor node deployments.

TABLE I
Relationship With WHEEL Extension

WHEEL Graph in Fig. 4	Graph Described in Fig. 3
Fig. 4 (II)	Fig. 3(c.2)
Fig. 4 (III)	Fig. 3(c.1)
Fig. 4 (IV)	Fig. 3(c.1)
Fig. 4 (V)	Fig. 3(a)+(b)
Fig. 4 (VI)	Fig. 3(c.1)

As pointed out in [7], the four-bar linkage case [similar to the one given in Fig. 3(d.5)] is not covered by any existing bilateration-based localization technique. Thus, our result can be used to find more localizable nodes than the bilateration approach does. As commented in [15], introducing the four-bar linkage case presents a broader class of networks than bilateration which can still be localized by efficient algorithms. [15] also addresses the same localizability problem by considering both the bilateration case and the four-bar linkage case. However, it does not provide a necessary and sufficient condition for localizability of a cluster. Indeed, the four-bar linkage case considered in [15] assumes a complete graph for the cluster of four nodes, which however is not necessary as shown in Theorem 1.

In addition to trilateration and bilateration ordering, there is another extension called WHEEL extension. In [12], Yang *et al.* explore the localizable nodes through detecting a WHEEL structure in the network. A node is claimed to be localizable if it is included in a WHEEL graph containing at least three anchor nodes. A common WHEEL graph is shown in Fig. 4(I). Since there are six nodes in a WHEEL graph, one can localize at most three nodes jointly using this method. The trilateration condition can be treated as a special case of a WHEEL graph. According to different positions of anchor nodes, there are at most five possible WHEEL structures as shown in Fig. 4(II)–(VI). In a WHEEL graph, each rectangle indicates one anchor node and the solid circle indicates a node to be localized. It turns out that every graph in Fig. 4(II)–(VI) is a special case in our conditions shown in Theorem. 1. The detailed relationship is shown in Table I.

In summary, Fig. 5 shows the inclusion relationship of the sets of localizable nodes detected by different techniques. Our sequential cluster-based approach with each cluster up to four

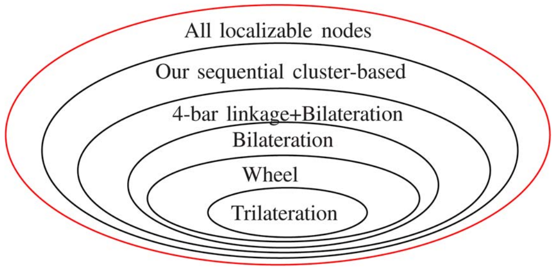


Fig. 5. Inclusion relationship of the set of localizable nodes detected by different techniques.

nodes can find more localizable nodes comparing to other known techniques. The percentage of localizable nodes not able to be verified by our approach is very small as we will see in the simulation.

V. Sequential Cluster-Based Algorithm for Localizable Nodes Detection

A. Algorithms for Localizability Test

Given a graph $\mathcal{G}$ with N nodes, which contains a set of anchor nodes $\mathcal{V}_0$, $|\mathcal{V}_0| \geq 3$, and a set of nodes to be tested, we consider how to apply Theorem 1 sequentially for localizability test. We offer two implementations: 1) a centralized algorithm and 2) a distributed algorithm.

1) Centralized Algorithm for Localizability Test: A centralized algorithm for localizability test is given in Algorithm 1, in which $\mathcal{V}_a$ denotes the set of nodes that have been checked to be localizable after each round including the anchor nodes and $\mathcal{U}_n$ represents the set of nodes to be tested. The pseudocode is developed by repeatedly applying the localizability conditions in Theorem 1. The first step is to detect Case (a) as described in Theorem 1 and we move the nodes in $\mathcal{U}_n$ having three direct connections with three anchor nodes to $\mathcal{V}_a$. The second step is to check Case (b), and so on. This process continues until no more new localizable nodes are detected.

Algorithm 1 A centralized description of the cluster localizability test.

Step 0: Initialize $\mathcal{V}_a = \mathcal{V}_0$ and $\mathcal{U}_n = \mathcal{V} \backslash \mathcal{V}_0$.
Step 1:
if $|\mathcal{U}_n| = 0$ **then**
    Go to Step 5.
end if
for each node $u \in \mathcal{U}_n$ **do**
    if $|\pi(u)| \geq 3$ **then**
        $\mathcal{V}_a \leftarrow \mathcal{V}_a \cup \{u\}$
        $\mathcal{U}_n \leftarrow \mathcal{U}_n \backslash \{u\}$
        Go to Step 1.
    end if
end for
Step 2:
for each connected pair $u_1, u_2 \in \mathcal{U}_n$ **do**
    if u_1 and u_2 fit Connectivity Pattern condition (b) and Connecting-Edge condition **then**
        $\mathcal{V}_a \leftarrow \mathcal{V}_a \cup \{u_1, u_2\}$
        $\mathcal{U}_n \leftarrow \mathcal{U}_n \backslash \{u_1, u_2\}$
        Go to Step 1.
    end if
end for
Step 3:
for each connected 3-tuple $u_1, u_2, u_3 \in \mathcal{U}_n$ **do**
    if u_1, u_2 and u_3 fit Connectivity Pattern condition (c) and Connecting-Edge condition **then**
        $\mathcal{V}_a \leftarrow \mathcal{V}_a \cup \{u_1, u_2, u_3\}$
        $\mathcal{U}_n \leftarrow \mathcal{U}_n \backslash \{u_1, u_2, u_3\}$
        Go to Step 1.
    end if
end for
Step 4:
for each connected 4-tuple $u_1, u_2, u_3, u_4 \in \mathcal{U}_n$ **do**
    if u_1, u_2, u_3 and u_4 fit Connectivity Pattern condition (d) or (e) or (f) and Connecting-Edge condition **then**
        $\mathcal{V}_a \leftarrow \mathcal{V}_a \cup \{u_1, u_2, u_3, u_4\}$
        $\mathcal{U}_n \leftarrow \mathcal{U}_n \backslash \{u_1, u_2, u_3, u_4\}$
        Go to Step 1.
    end if
end for
Step 5: Stop

2) Distributed Algorithm for Localizability Test: Algorithm 2 is a distributed implementation of Theorem 1. That is, we implement the localizability test on each sensor node allowing each node to communicate with its 1-hop neighbors. For every node i, it holds a binary value $flag(i)$ to indicate that it is verified localizable when $flag(i) = 1$. Moreover, node i updates the set of neighbor nodes of node i, denoted by $\pi(i)$, which are verified localizable. In the localizability test as shown in Algorithm 2, each node broadcasts its status $flag(i)$ and $\pi(i)$ to its neighbors. Then each node is able to check whether itself alone or itself together with its neighbors are jointly localizable according to Theorem 1. For most patterns in Fig. 3 except (d.2) and (e), the localizable pattern can be verified by one node in the cluster by using only received information $flag(j)$ and $\pi(j)$ from its neighbors. For example, u_j is such a node in Fig. 3(c.1), which can verify the localizability of the cluster. Though node u_i and u_k are not able to verify the localizability of the cluster, it will be informed by node u_j. Such verification using only 1-hop information is described in Algorithm 2 (from Step 1 to Step 4). For the two special cases in Fig. 3(d.2) and (e), the verification cannot be done using only 1-hop information, so a node will request a cooperation from its neighbors. In other words, when a node notices a pattern like the one of Fig. 3(d.2) and (e) with node u_i and its incident edge removed, it asks its neighbor u_j to check if node u_j has a neighbor u_i with the pattern shown in Fig. 3(d.2) and (e). In this way, node i will be able to determine the localizability of the cluster. Finally, it is worth to point out that Algorithm 2 can be implemented asynchronously. In other words, whether or not every node runs a localizability test is event-triggered, namely, when it notices a change of its neighbors' status.

Algorithm 2 A distributed realization of the cluster localizability test on node i.

Initialization: every node i broadcasts $flag(i)$ and $\pi(i)$ to its neighbors $j \in \mathcal{N}_i$.
while $flag(i) = 0$ and any one $flag(j)$ or $\pi(j)$, $j \in \mathcal{N}_i$, changes its value **do**
  Step 1:
  if it has three neighbors with $flag(j) = 1$ [pattern of Fig. 3(a)] **then**
    Changes $flag(i) = 1$ and broadcasts $flag(i)$ to its neighbors
  end if
  Step 2:
  if it has one neighbor j, together with which it forms the pattern of Fig. 3(b) **then**
    Changes $flag(i) = flag(j) = 1$ and broadcasts them to its neighbors
  end if
  Step 3:
  if it has two neighbors j and k, together with which it forms the patterns of Fig. 3(c.1)-(c.2) **then**
    Changes $flag(i) = flag(j) = flag(k) = 1$ and broadcasts them to its neighbors
  end if
  Step 4:
  if it has three neighbor j, k, l, together with which it forms the patterns of Fig. 3(d.1), (d.3)-(d.5), and (f.1)-(f.2) **then**
    Changes $flag(i)=flag(j)=flag(k)=flag(l)=1$ and broadcasts them to its neighbors
  end if
  Step 5:
  if it has two neighbor j and l, together with which it forms the pattern of Fig. 3(d.2) and (e) with node u_i and its incident edges removed **then**
    Sends to nodes j and l a request of checking whether it has a neighbor k, together which they form the patterns of Fig. 3(d.2) and (e)
    if Yes **then**
      Changes $flag(i) = flag(j) = flag(k) = flag(l) = 1$ and broadcasts them to its neighbors
    end if
  end if
  if $flag(i)$ or $\pi(i)$ changes its value **then**
    Broadcasts $flag(i)$ and $\pi(i)$ to its neighbors
  end if
end while

B. Computational Complexity Analysis

Given a network with n nodes, assume the connectivity degree, that is, the average number of neighbor nodes, for each node is equal to D.

According to Algorithm 1, to find a localizable cluster of one node, every node in the network needs to check all its neighbors and, thus, requires D iterations in average. To find a localizable cluster of two nodes, every node in the network needs to check all its neighbors and all its 2-hop neighbors. Thus, it requires D^2 iterations in average. Similarly, we can infer that every node requires D^3 and D^4 iterations in average to find a cluster of three nodes and four nodes, respectively. Thus, the computational complexity for the whole network is $\mathcal{O}(n(D + D^2 + D^3 + D^4))$.

For the distributed algorithm (Algorithm 2), every node only needs to know the status of its 1-hop neighbors and 2-hop neighbors in order to determine its own localizability. Moreover, each node remains silent unless its neighbors broadcast a message to indicate a change of their status or their neighbors status. Thus, the computational complexity is $\mathcal{O}(D + D^2)$ for each node and $\mathcal{O}(n(D + D^2))$ for the entire network.

It is worth pointing out that, as shown in [8], the computational complexity of the trilateration method is $\mathcal{O}(nD)$ if the anchor nodes are known in the network. In addition, since the WHEEL extension method considers at most the connectivity information of 2-hop neighbors of every node, it takes $\mathcal{O}(nD^2)$ for localizability test. From the above analysis we can see that although the computational complexity of our algorithms is higher than the trilateration and WHEEL extension methods, it is still linear with respect to the size of the whole network.

VI. Performance Evaluation

In order to show the effectiveness of our localizability test scheme, we use the Monte Carlo method to compare the number of nodes detected to be localizable in a given network by using Trilateration, WHEEL extension and our proposed localizability conditions. Moreover, the effect of communication radius (namely, connectivity degree) to localizability and the effect of anchor nodes distribution as well as total anchor numbers will also be explored.

A. Evaluation Criteria

The metric we use to evaluate the performance of localizability test is the *localizable ratio* defined by

$$r_{\text{loc}} = \frac{N_d}{N} \tag{2}$$

where N is the total number of nodes in the network and N_d is the number of nodes that can be localized. We use $r_{\text{loc}}^{(T)}$, $r_{\text{loc}}^{(W)}$, and $r_{\text{loc}}^{(O)}$ to indicate the localizable ratios detected by trilateration, WHEEL Extension and the proposed scheme, respectively. Furthermore, we use $r_{\text{loc}}^{(P)}$ to indicate the ratio of the number of nodes that satisfy the 3-path necessary condition, to the total number. Notice that the 3-path is a necessary condition for node localizability tests. Thus, $r_{\text{loc}}^{(P)}$ is an upper-bound of the true localizable ratio.

TABLE II
THE LOCALIZABLE RATIO BY TRILATERATION, WHEEL EXTENSION AND OUR PROPOSED METHODS, AND THE RATIO OF NODES SATISFYING THE 3-PATH CONDITION (A NECESSARY CONDITION)

Radius	14	17	20	23	26
$r_{loc}^{(T)}$	8.19%	21.10%	52.57%	86.23%	97.69%
$r_{loc}^{(W)}$	11.79%	26.40%	63.57%	90.05%	97.69%
$r_{loc}^{(O)}$	11.80%	30.32%	67.57%	92.15%	98.58%
$r_{loc}^{(P)}$	12.9%	32.93%	69.74%	92.71%	98.63%

B. Comparisons in Terms of Localizable Ratio

We run a 200-round Monte Carlo simulation to compare the localizable ratio r_{loc} detected by the proposed scheme with trilateration and WHEEL extension. In each round, a 80-node network $\mathcal{G}$ is randomly deployed in a 100×100 unit area. The anchor nodes are randomly selected. All nodes are assumed to have the same communication radius. The result is summarized in Table II with the rows indicating the localizable ratios detected by trilateration $r_{\text{loc}}^{(T)}$, WHEEL Extension $r_{\text{loc}}^{(W)}$, our proposed scheme $r_{\text{loc}}^{(O)}$, and the ratio of the number of nodes that satisfy the 3-path necessary condition $r_{\text{loc}}^{(P)}$. The columns in the table indicate the localization rations under different communication radius 14, 17, 20, 23, and 26.

From Table II, we can see that for all the cases with different communication radius, the localizable ratio of our proposed scheme is higher than the localization ratio achieved by trilateration and WHEEL extension, and closer to the ratio $r_{\text{loc}}^{(P)}$ of nodes that satisfy only a necessary condition. Though some unlocalizable nodes are counted in computing $r_{\text{loc}}^{(P)}$ (because the 3-path condition is not sufficient), we see from Table II that the difference between the localizable ratio $r_{\text{loc}}^{(O)}$ resulted from our proposed condition and $r_{\text{loc}}^{(P)}$ is rather small, which means that our condition is very close to the necessary and sufficient one.

Next, we present a specific network in order to demonstrate why our proposed approach is advantageous over the WHEEL extension method. For a specific communication radius, the network is drawn in Fig. 6 with small circles representing the sensor nodes and lines indicating that the distance between two nodes joined by each line can be measured. In the figures, the three nodes linked by three solid lines are the anchor nodes. Localizable nodes verified by the WHEEL extension method are marked by triangles in Fig. 6(a) and localizable nodes verified by our proposed method are marked by triangles in Fig. 6(b). For this particular network, notice that at the lower right corner of Fig. 6(a) [at the coordinate around (60,15)], several nodes are not localizable by the WHEEL extension. However, they are localizable by our method due to four connections with localized nodes to the right. Moreover, once they become localizable, most nodes in the left become localizable as well, as shown in Fig. 6(b). This is why our proposed method can verify more localizable nodes (the higher localizable ratio) than the WHEEL extension method.

Moreover, we use another specific network to illustrate that our proposed method can detect more localizable nodes than Sweeps [7], [14], which detects all localizable nodes when localizability test is operated node by node. As before, the network is drawn in Fig. 7. Notice that at the lower right corner of Fig. 7(a) [at the coordinate from (50, 40) to (95, 30)], several nodes cannot be recognized as localizable ones by Sweeps. However, they are localizable according to our method. This is due to our method is checking localizability cluster by cluster, rather than node by node as done by Sweeps. Specifically, those "un-localizable" nodes around (50, 40) in Fig. 7 hold a structure as the graph shown in Fig. 3(d.5). The graph shown in Fig. 3(d.5) is not localizable by Sweeps because neither of nodes in $\mathcal{G}_2$ has two direct connections with localizable nodes.

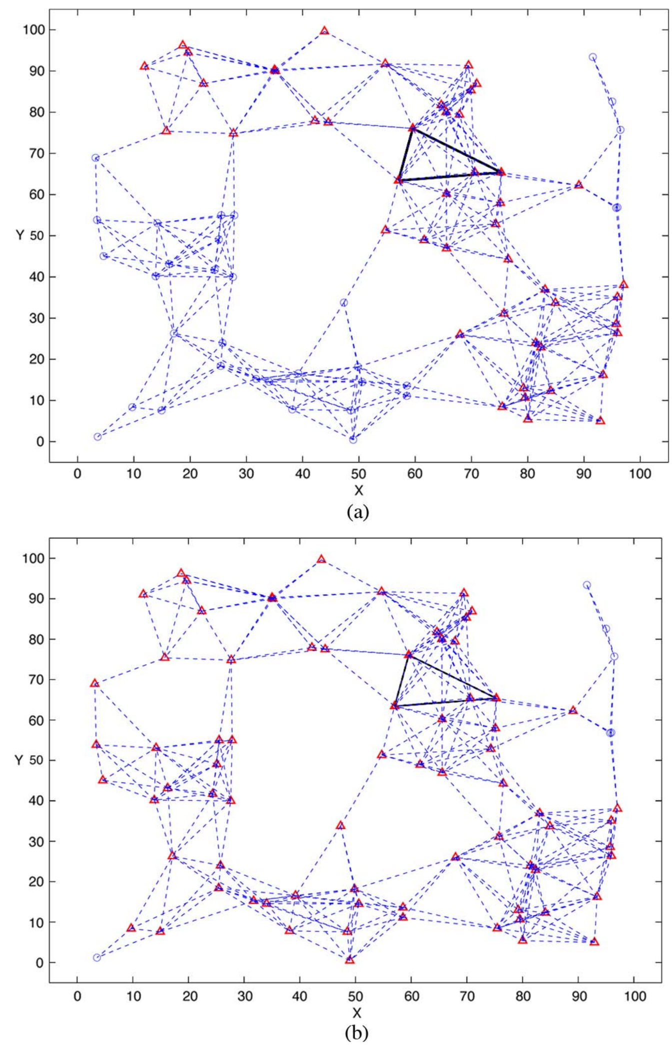
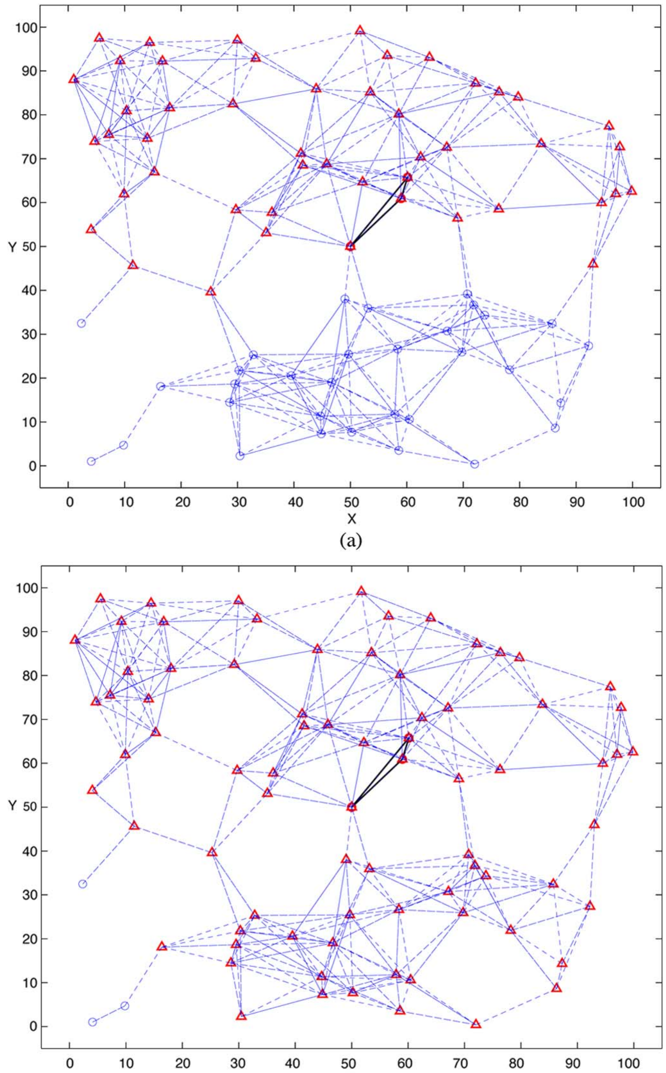


Fig. 6. Localizability test results with the WHEEL extension method and our proposed method on a specific network. (a) WHEEL extension. (b) Proposed condition.

C. Effect of Communication Radius

In this subsection we examine the effect of communication radius on localizable ratio for our proposed scheme.

As we can check from the row indicated by $r_{\text{loc}}^{(O)}$ in Table II, the localizable ratio increases when the communication radius increases. When the communication radius reaches 23, the average connectivity degree is about 10, for which the localizable ratio is rather satisfactory for practical applications. To balance the connectivity requirement and energy consumptions,



(a)

(b)

Fig. 7. Localizability test results with the Sweeps method and our proposed method on a specific network. (a) Localizable nodes detected by Sweeps. (b) Localizable nodes detected by the method.

as suggested in [23] and [24], the magic number of the average connectivity degree is about 6 or 8 in practical networks, corresponding to communication radius 17 and 20 in our example, for which we can see from Table II that the improvement of our scheme is significant in comparison with the trilateration and WHEEL extension methods.

D. Effect of Anchor Nodes

In this subsection we examine how the number and distribution of anchor nodes affect the localizable ratio using our proposed scheme.

In the above simulations, the number of anchor nodes is fixed at the least required number, 3. An intuitive idea is that more anchor nodes will lead more nodes to be localizable. So we provide another Monte Carlo simulation to verify this intuition. As before, a 80-node network is deployed in a 100×100 area in each round, with the communication radius fixed at 20. We compare four cases of anchor node number $N_a = 3$, 4, 6, and 10. The anchor nodes are distributed in three ways: central, random and corner. For the central case, we select the three anchor nodes in the center area and then randomly select the remaining anchor nodes. For the random case, all the anchor nodes are randomly selected. For the corner case, three anchor nodes are selected near the corners of the network and the remaining anchor nodes are randomly selected. The results are shown in Fig. 8 where the horizontal axis is N_a and the vertical axis is the localizable ratio $r_{\rm loc}^{(O)}$. We see that $r_{\rm loc}^{(O)}$ increases as N_a increases. We also notice that different distributions of anchor nodes lead to different localizable ratios, with the central distribution being the best, especially for small N_a.

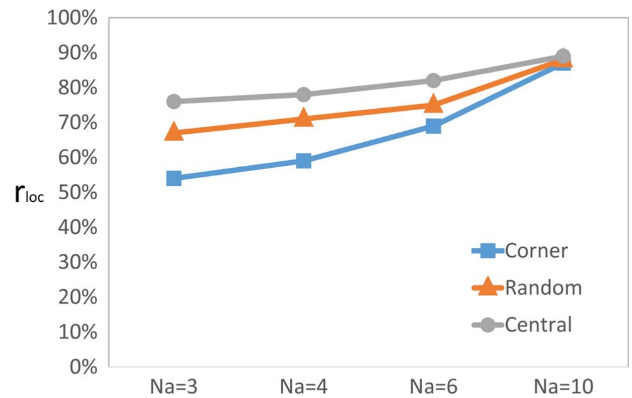


Fig. 8. Localizable ratio $r_{\rm loc}^{(O)}$ versus the number and distribution of anchor nodes.

VII. Conclusion

In this paper, we have provided new localizability conditions for sensor networks. Our conditions can be used to detect up to four localizable nodes in each step in a sequential manner. Our conditions are shown to be more general than the well-known trilateration, bilateration, and WHEEL extension conditions. Compared with the 3-connected and redundant rigid condition, our proposed conditions require only local connectivity information between two clusters of nodes. Our localizability test scheme can be implemented in either a centralized manner or a distributed manner. Simulation results show that our localizability conditions are more effective for detecting localizable nodes than known methods, yet remain to be computationally efficient. We have also explored the effect of communication radius as well as the number and the distribution of anchor nodes to localizability.

On the one hand, our sequential cluster-based approach can be developed to actual computation of the locations of sensor nodes cluster by cluster by solving a set of distance constrained nonlinear equations or an equivalent optimization problem. On the other hand, after identifying all localizable nodes in a network using our approach, centralized localization algorithms such as MDS [25] and SDP [26] or distributed concurrent localization algorithms such as DILOC [27] and ECHO [28] can be applied to compute the locations of these localizable nodes. It will certainly improve the localization accuracy as the iterated estimation will not be led away by wrong estimates of the locations of those nonlocalizable nodes.

Though localizability test in a sequential manner with each cluster up to four nodes each time gives a very good performance in terms of the localizable ratio, it will be still interesting to find out finite patterns of clusters such that all localizable nodes in a network can be determined by checking all such patterns sequentially without the constraint on the number of nodes in each cluster. In addition, natural extensions to the results in this paper include sequential cluster-based localizability for sensor networks in the 3-D space and for sensor networks in the case of unit disk graphs. For the latter case, each pair of nodes within a given distance threshold is connected by means of an edge and, thus, it is possible to take into account not only distance information but also *a priori* information about not being connected.

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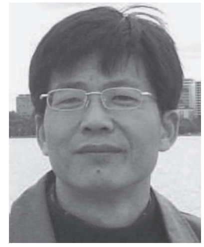

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