Distributed Self Localization for Relative Position Sensing Networks in 2D Space

Zhiyun Lin, *Senior Member, IEEE*, Minyue Fu, *Fellow, IEEE*, and Yingfei Diao, *Member, IEEE*

***Abstract*—This paper studies the 2D localization problem of a sensor network given anchor node positions in a common global coordinate frame and relative position measurements in local coordinate frames between node pairs. It is assumed that the local coordinate frames of different sensors have different orientations and the orientation difference with respect to the global coordinate frame are not known. In terms of graph connectivity, a necessary and sufficient condition is obtained for self-localizability that leads to a fully distributed localization algorithm. Moreover, a distributed verification algorithm is developed to check the graph connectivity condition, which can terminate successfully when the sensor network is self-localizable. Finally, a fully distributed, linear, and iterative algorithm based on the complex-valued Laplacian associated with the sensor network is proposed, which converges globally and gives the correct localization result.**

***Index Terms*—Sensor networks, self localization, localizability, distributed algorithm.**

I. Introduction

MANY existing localization schemes for sensor networks utilize pairwise distance measurements between sensor nodes to compute the position of each sensor node in a global coordinate frame [1]–[6]. With the development of micro-electromechanical systems (MEMS), it is possible and relatively inexpensive to add measurements such as bearing angles [7]–[11], which together with the distance measurements lead to the availability of relative positions. However, a key technical difficulty for sensor network localization is that the bearing angles are typically made in sensor nodes' local coordinate frames without the knowledge of their true orientations.

Depending on whether a central processor exists or not, localization schemes can be divided into centralized schemes [12] and distributed schemes [13]. The former asks every sensor node to transmit its information to a central processor, which then computes the positions of all the nodes in the entire sensor network, while the latter allows every sensor node to exchange information only with its neighbors and conduct the computation of its own position locally. The distributed approach is more preferable since it is energy efficient and can avoid communication bottlenecks at and near the centralized processor.

This paper aims at solving the localization problem for sensor networks in a distributed manner using relative position measurements in local coordinate frames. There are a number of related works based on such relative measurements. In [14], [15], the localization task is modeled as solving a linear estimation problem, in which pairwise relative position measurements are used to estimate the position in a distributed manner. In [16], a randomized algorithm is developed to recursively estimate the position based on the differences of local measurements between pairwise neighbors. The works of [17], [8], [18] also consider the relative measurements to address the localization problem in a distributed manner.

Our work differs from the literature in the sense that a global coordinate frame is not required, whereas the necessity of a global coordinate frame is implicitly assumed in the aforementioned works. Note that the use of a global coordinate frame is not in line with the spirit of distributed processing because a piece of global information is needed, especially the common sense of direction, which for example requires each sensor node to have potentially sophisticated orientation sensing capability. The same concern is considered in [19], in which the authors develop a scheme to align the orientations of all local coordinate frames while doing localization. But towards the objective of aligning the orientations, mutual relative position measurements are assumed in [19]. In this paper, the measurement model is expressed in terms of a directed graph with the directed edges corresponding to the relative position measurements. Such a directed graph model represents the more general case as relative position measuring may be often unidirectional. For example, a camera commonly has a cone-like field of view, so depending on the orientations of the cameras, two sensor nodes may not be able to sense each other mutually. In this paper, we adopt the novel idea of using complex barycentric coordinates developed in formation control [20], [21] such that the local coordinate frames at different sensor nodes are allowed to have different orientations and mutual relative position measurements are not required. In our early work [22], the localization problem in such a setup is addressed under the assumption that every sensor node has exactly two neighbors. This paper presents a more general result by removing this restrictive assumption.

Manuscript received October 29, 2014; revised January 27, 2015 and April 14, 2015; accepted April 28, 2015. Date of publication May 12, 2015; date of current version June 09, 2015. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Paolo Banelli. This work was supported by the Zhejiang Provincial Natural Science Foundation of China under Grant LR13F030002. (*Corresponding author: Minyue Fu.*)

Z. Lin is with the State Key Laboratory of Industrial Control Technology, College of Electrical Engineering, Zhejiang University, Hangzhou, Zhejiang 310007, China (e-mail: linz@zju.edu.cn).

M. Fu is with the School of Electrical Engineering and Computer Science, University of Newcastle, Callaghan, NSW 2308, Australia, and also with the Department of Control Science and Engineering and State Key Laboratory of Industrial Control Technology, Zhejiang University, Zhejiang 310007, China (e-mail: minyue.fu@newcastle.edu.au).

Y. Diao is with Huawei Shanghai R&D Center, Huawei Technologies Co. Ltd., Shanghai 201206, China (e-mail: yfdiao@gmail.com).

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Digital Object Identifier 10.1109/TSP.2015.2432739

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The localization problem is characterized as a graph embeddability problem [23], i.e., whether there exists a unique embedding for the graph, given a set of distance measurements. For a general network, this graph-embeddability problem has been proved to be NP-hard. Thus, people turn to first study which kinds of graphs have the unique and generic embedding [24]. This leads to the fundamental issue in sensor network localization, called *localizability* [25], which checks whether all the sensor nodes in a sensor network are localizable based on the available relative measurements. For range-based localization, a sensor network is usually characterized by a distance graph [26] and then the graph rigidity theory is applied to tackle the localizability problem. For example, [27] shows that a sensor network in the 2D plane can be uniquely localized in the global coordinate frame if and only if it contains at least three location-known anchor nodes and its distance graph is globally rigid.

However, there is no such known result for the localizability of sensor networks based on relative position measurements, for which the ground graph is directed, i.e., the relative position between two sensor nodes may be known by only one sensor node. This paper develops a fundamental result to answer this question. Specifically, in this paper, we consider rigorously the so-called *self-localizability* problem, which refers to the capability of determining each sensor's position by the sensor node itself rather than others. This notion naturally leads to a distributed solution for the localization algorithm. A necessary and sufficient condition is obtained to characterize the self-localizability of a sensor network with relative position measurements. It is shown that a sensor network in the plane with relative position measurements in the local coordinate frames of individual sensor nodes is self-localizable if and only if the sensor network contains at least two location-known anchor nodes and its sensing digraph holds a 2-reachability property, i.e., every sensor node has two disjoint paths from the anchor nodes.

After addressing the fundamental self-localizability problem, this paper then develops a distributed verification algorithm to check whether a given sensor network is self-localizable. The algorithm can terminate successfully in a finite number of steps. Moreover, this paper also develops a distributed localization algorithm to calculate the position of each sensor node based on its own local measurements and some information exchanged from its neighbors. The proposed localization algorithm is in an iterative and linear form with guaranteed global convergence. This is more attractive compared with those distance-based localization algorithms that typically require solving a nonlinear optimization problem and may be trapped into local optima.

The organization of this paper is as below. In Section II, we introduce the preliminary knowledge of graphs and state the problems of self-localizability and localization based on relative position measurements. In Section III, we give a necessary and sufficient graph connectivity condition for self-localizability and then present a distributed verification algorithm to check self-localizability of a sensor network. In Section IV, we propose a fully distributed localization algorithm to compute the positions of sensor nodes. A simulation is provided in Section V to show the effectiveness of our method. We conclude our work in Section VI.

Notation: $\mathbb{C}$ denotes the set of complex numbers. $\iota = \sqrt{-1}$ denotes the imaginary unit.$\mathbf{I}$ denotes the identity matrix of appropriate dimension. For a complex number c, $|c|$ and $\bar{c}$ represent the modulus and the conjugate respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of the set. For a complex vector or matrix $\mathbf{A}, \mathbf{A}^T$ represents the conjugate transpose. $\lambda_{\max}(\mathbf{A})$ denotes the maximal eigenvalue of a Hermitian matrix $\mathbf{A}$ while $\text{rank}(\mathbf{A})$ represents the rank of $\mathbf{A}$.

II. Preliminary and Problem Formulation

A. Basic Notions From Graph Theory

First, we are going to introduce some basic notions from graph theory and algebraic graph theory. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty node set $\mathcal{V}$ and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If (j, i) is an edge in a directed graph $\mathcal{G}$, then node j is called an in-neighbor of node i while node i is called an out-neighbor of node j. We define $\mathcal{N}_i$ as the in-neighbor set of node i, i.e., $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. A *walk* in a directed graph $\mathcal{G}$ is an alternating sequence $\mathcal{W} : v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$ of nodes v_i and arcs e_i such that $e_i = (v_i, v_{i+1})$ for every $i = 1, \ldots, k-1$. For notation simplicity, we denote a walk as $\mathcal{W} : v_1 \to v_2 \to \cdots \to v_k$. We say that $\mathcal{W}$ is a walk from v_1 to v_k, and v_1 is the starting node of the walk $\mathcal{W}$. For a directed graph $\mathcal{G}$, if there is a walk from one node u to another node v, then v is said to be *reachable* from u. If the nodes of a walk $\mathcal{W}$ are distinct, $\mathcal{W}$ is a *path*. Moreover, a node $v \in \mathcal{V}$ is said to be *2-reachable* from a non-singleton set $\mathcal{U}$ of nodes if there exists a path from a node in $\mathcal{U}$ to v after removing any one node except node v or equivalently if there are two disjoint paths from $\mathcal{U}$ to v [28].

Next, we introduce two notions, called complex Laplacian matrix and Dirichlet matrix, which are associated with directed graphs. For a directed graph, associated each edge $(j, i) \in \mathcal{E}$ with a complex value w_{ij} called the weight of the edge, its complex Laplacian matrix $\mathbf{L}$ is defined with its (i, j)-th component being

$$\mathbf{L}_{ij} = \begin{cases} -w_{ij}, & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \\ 0, & \text{if } i \neq j \text{ and } j \notin \mathcal{N}_i \\ \sum_{j \in \mathcal{N}_i} w_{ij}, & \text{if } i = j. \end{cases} \tag{1}$$

Similarly to the one defined in [29], the Dirichlet matrix $\mathbf{H}$ is the matrix obtained from the Laplacian matrix $\mathbf{L}$ by deleting all rows and columns that correspond to a subset of specific nodes. Th Dirichlet matrix is also well known in the literature on distributed estimation and control with another name called *basis Laplacian* (see [30]).

B. Problem Formulation

Let Σ_g be a common *global coordinate frame*. We consider a sensor network consisting of two subsets of of sensor nodes. One subset of sensor nodes are called *anchor nodes* (labelled $1, \ldots, m$), whose positions in Σ_g, denoted by $\mathbf{p}_a = [p_1, \ldots, p_m]^T$, are known. The other subset of sensor nodes are called *free nodes* (labelled $m+1, \ldots, m+n$), whose positions in Σ_g, denoted by $\mathbf{p}_s = [p_{m+1}, \ldots, p_{m+n}]^T$, are not

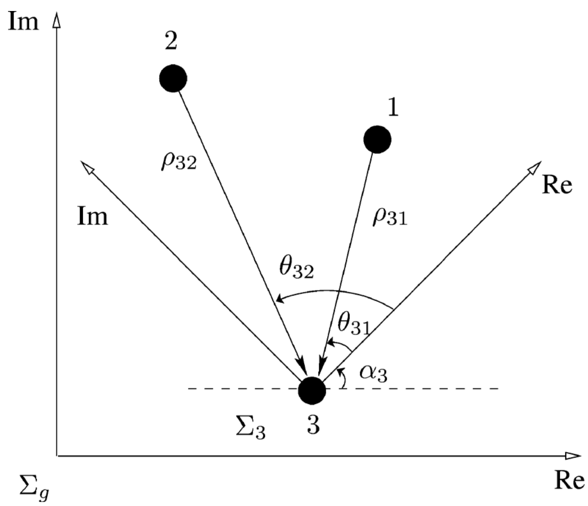


Fig. 1. An illustration of relative position measurements.

known and need to be determined. For notation convenience, the position p_i in the plane is represented by a complex number throughout the paper. We let $\mathbf{p} = [\mathbf{p}_a^T, \mathbf{p}_s^T]^T$ denote the aggregate position vector of all the sensor nodes and call it the *configuration* of the sensor network. In this paper, we have an assumption for the configuration.

Assumption 1: The configuration p is generic.

The configuration p is said to be *generic* if the coordinates $p_1, \ldots, p_{m+n}$ do not satisfy any nontrivial algebraic equation with integer coefficients [31]. Intuitively speaking, a generic configuration has no degeneracy, i.e., no three points staying on the same line, no three lines go through the same point, etc. The justification for this assumption is that the generic configurations form a set of full measure.

Suppose that each free node i holds a *local coordinate frame* Σ_i, for which the origin is set at the position of node i and the orientation is fixed but the offset angle α_i with respect to the global coordinate frame Σ_g is not known. Each free node is assumed to measure the relative distances and bearing angles of its neighboring nodes in its local coordinate frame Σ_i. That is, if sensor node j is a neighbor of sensor node i, then the distance ρ_{ij} from node j to node i and the bearing angle θ_{ij} of node j with respect to Σ_i are measured by node i. An illustrative example with three nodes 1, 2 and 3 is given in Fig. 1. In the figure, the arrowed line pointing from node 1 and 2 to node 3 are the edges, meaning that node 3 can measure the relative position of 1 and 2, namely, (ρ_{31}, θ_{31}) and (ρ_{32}, θ_{32}), in its local coordinate frame Σ_3. Mathematically, the relative position measurement in the local coordinate frame Σ_i can be expressed as

$$y_{ij} := \rho_{ij} e^{\iota \theta_{ij}}. \tag{2}$$

We construct a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, m+n\}$ to describe the neighboring topology. That is, an edge $(j, i) \in \mathcal{E}$ indicates that node i can measure the relative position y_{ij}. Thus, the sensor network will be denoted as a tuple $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ in what follows.

The relative position (RP) localization problem of a sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ is stated as follows.

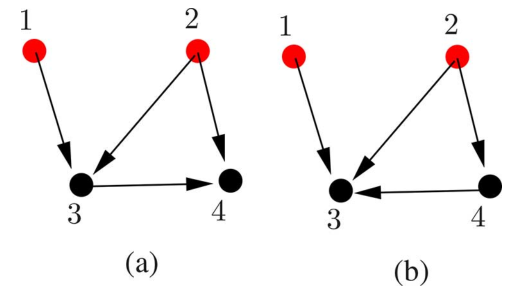


Fig. 2. Illustration of self-localizability and joint localizablity. (a) self-localizable, (b) jointly localizable.

Definition 1: (**RP localization problem**) Given a sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ and the local coordinate frame based relative position measurements y_{ij} for $(j, i) \in \mathcal{E}$, find the position $\mathbf{p}_s$ of the free nodes such that for any $i = m+1, \ldots, n$, there exists α_i satisfying

$$y_{ij} = (p_j - p_i) e^{-\iota \alpha_i}, \text{ for any } j \in \mathcal{N}_i.$$

Measurements generated from a given sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ guarantee that at least one solution exists.

Definition 2: A sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ is called *jointly localizable* if the RP localization problem, given the anchor positions $\mathbf{p}_a$ and the local coordinate frame based relative position measurements y_{ij} for $(j, i) \in \mathcal{E}$, admits a unique solution $\mathbf{p}_s$.

Definition 2 may imply the need of collecting all anchor positions information and all relative position measurements together in order to solve the RP localization problem. However, in order to reduce the information exchange and make each free node capable of computing its own position related to distributed computation, we introduce another notion called *self-localizability*. Some notation will be used in the following definition. Let $\mathcal{U}_s$ be any subset of free nodes. Then the set of all the rest sensor nodes (including the anchor nodes) is denoted as $\mathcal{U}_a := \mathcal{V} \setminus \mathcal{U}_s$. Moreover, we denote $\mathbf{p}_{\mathcal{U}_a}$ and $\mathbf{p}_{\mathcal{U}_s}$ the corresponding position vectors for sensor nodes in $\mathcal{U}_a$ and $\mathcal{U}_s$ respectively.

Definition 3: A sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ is called *self-localizable* if for any subset of free nodes $\mathcal{U}_s$, the sensor network $(\mathcal{G}, \mathbf{p}_{\mathcal{U}_a}, \mathbf{p}_{\mathcal{U}_s})$ is jointly localizable, given the positions $\mathbf{p}_{\mathcal{U}_a}$ in the global coordinate frame Σ_g and the local coordinate based relative position measurements y_{ij} for $i \in \mathcal{U}_s$ and $(j, i) \in \mathcal{E}$.

Definition 3 means that any subset of free nodes are able to compute their positions in the global coordinate frame Σ_g by themselves if the positions of their in-neighbors in Σ_g are known. The reason for coining this notion self-localizability is that it will lead to a fully distributed localization algorithm, as we will show later.

This paper concentrates on the self-localizability problem and aims to develop a distributed algorithm to compute the positions of all free nodes in the global coordinate frame Σ_g if the sensor network is self-localizable.

An example to explain the differences between the two notions of localizability is shown in Fig. 2. In both Fig. 2(a) and (b), nodes 1 and 2 are the anchor nodes while the others are free nodes.

The example in Fig. 2(a) is self-localizable as it can be checked that any subset of free nodes is jointly localizable given the relative position measurements and the positions of other nodes in the global coordinate frame Σ_g. More specifically, we can see that node 3 is able to determine the position of itself in Σ_g when it uses the relative position measurements about nodes 1 and 2 in Σ_3 and the positions of nodes 1 and 2 in Σ_g. Moreover, node 4 is also able to determine the position of itself in Σ_g when it uses the relative position measurements about nodes 2 and 3 in Σ_4 and the positions of node 2 and 3 in Σ_g.

However, the example in Fig. 2(b) is not self-localizable but jointly localizable. This is because node 4 has only one in-neighbor (namely, node 2) and is not able to determine its own position in Σ_g by using the relative position measurement about its in-neighbor (node 2) in Σ_4 and the position information of node 2 in Σ_g since different orientations of node 4's local coordinate frame lead to different position for node 4. However, as discussed above, node 3 in Fig. 2(b) is able to determine its own position in Σ_g. Because node 3 has a relative position measurement about nodes 1 and 2 and it now knows the its own position in Σ_g as well as node 1 and 2's positions in Σ_g, as a byproduct, node 3 can determine the orientation of its own local coordinate frame Σ_3 with respect to the global coordinate frame Σ_g. Then by means of this extra information, node 3 is able to determine the position of node 4 in Σ_g as it has the relative position measurement about node 4. Thus, the positions of all sensor nodes in Σ_g can be uniquely determined, which means the sensor network in Fig. 2(b) is jointly localizable.

III. Self-Localizability

In this section, we develop a necessary and sufficient graph connectivity condition for self-localizability and then present a verification method to check self-localizability of a sensor network.

A. Necessary and Sufficient Localizability Condition

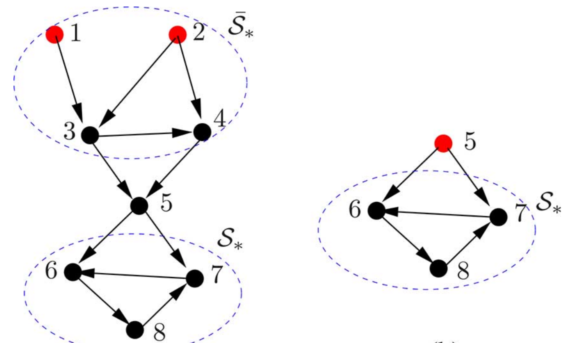
First, we provide a necessary and sufficient condition for self-localizability of a sensor network in terms of graph connectivity.

Theorem 1: Under Assumption 1, a sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ is self-localizable if and only if

(NS-1): the number of anchor nodes $m \geq 2$, and

(NS-2): every free node is 2-reachable from the set of anchor nodes.

Proof: (**Necessity**) Firstly, we prove the necessity of $m \geq 2$. When $m = 0$, the whole sensor network can freely rotate and translate in the plane without any constraint and thus is not localizable. When $m = 1$, suppose there is a set of locations satisfying the relative position measurements within their local coordinate frames. Now suppose their local coordinate frames are all rotated around this anchor node by the same angle. Then it leads to another set of locations for the sensor nodes satisfying the relative position measurements within their local coordinate frames. Hence, the positions of these nodes are not unique in the global coordinate frame Σ_g, i.e., these nodes are not localizable. In conclusion, there must be at least two anchor nodes.



(a) (b)

Fig. 3. Illustration for the necessity of (NS-2).

Secondly, we show the necessity of (NS-2). Suppose on the contrary that there is one node, say v_i, which is not 2-reachable from the anchor set, denoted as $\mathcal{S}_a$. By the definition of 2-reachability, it is then known that there exists another node v_* such that when node v_* is removed, node v_i is not reachable from $\mathcal{S}_a$. Denote by $\mathcal{S}_*$ the set of free nodes that are not reachable from the anchor set $\mathcal{S}_a$ after removing node v_* and denote by $\bar{\mathcal{S}}_*$ the set of nodes that are reachable from the anchor set $\mathcal{S}_a$ after removing node v_*. Then it follows that there is no edge from any node in $\bar{\mathcal{S}}_*$ to any node in $\mathcal{S}_*$ after removing v_*. An illustrative example is given in Fig. 3(a), where node 5 is the node v_*. We now consider the subset of free nodes $\mathcal{S}_*$ and denote $\mathcal{U}_a = \bar{\mathcal{S}}_* \cup \{v_*\}$. It can be seen that the joint localizability of the subset $\mathcal{S}_*$, given the positions of the nodes in $\mathcal{U}_a$ in the global coordinate frame Σ_g and the relative position measurements y_{ij} for $i \in \mathcal{S}_*$ and $(j, i) \in \mathcal{E}$ (the one shown in Fig. 3(a)), is equivalent to the joint localizability of the subset $\mathcal{S}_*$ of free nodes, given the position of node v_* in the global coordinate frame Σ_g and the relative position measurements y_{ij} for $i \in \mathcal{S}_*$ and $(j, i) \in \mathcal{E}$ (the one shown in Fig. 3(b)). Note that the latter case has only one anchor node. Then by the necessary condition (NS-1), the subset $\mathcal{S}_*$ of free nodes is not jointly localizable, given the positions of the nodes in $\mathcal{U}_a$ in the global coordinate frame Σ_g and the relative position measurements y_{ij} for $i \in \mathcal{S}_*$ and $(j, i) \in \mathcal{E}$. Thus, by Definition 3, the sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ is not self-localizable.

(Sufficiency) According to (NS-2) that every free node is 2-reachable from the set of anchor nodes, it can be known that every free node i has at least two in-neighbors. Then it is clear that with the local measurements $y_{ij}, j \in \mathcal{N}_i$, the following equation

$$\sum_{j \in \mathcal{N}_i} w_{ij} y_{ij} = 0 \tag{3}$$

has at least two unknowns and therefore it must have a solution $w_{ij}, j \in \mathcal{N}_i$. In other words, node i can solve for w_{ij} for $j \in \mathcal{N}_i$ from (3). Thus, $w_{ij}, j \in \mathcal{N}_i$, is a function of the relative position measurements $y_{ij}, \ j \in \mathcal{N}_i$, denoted as $w_{ij}(y_{ij}|_{j \in \mathcal{N}_i})$. Note that (3) implies

$$\sum_{j \in \mathcal{N}_i} w_{ij}\left(y_{ij}|_{j \in \mathcal{N}_i}\right)(p_j - p_i) = 0. \tag{4}$$

Recall that the anchor nodes do not have in-neighbors, indicating that they do not need to measure the relative positions about other nodes. So we can aggregate all the nodes including the anchor nodes and write (4) in a matrix form, that is,

$$\mathbf{L}\mathbf{p} = 0 \tag{5}$$

where $\mathbf{p}$ is the aggregate position $[\mathbf{p}_a^T, \mathbf{p}_s^T]$ and $\mathbf{L}$ is the Laplacian of $\mathcal{G}$ with the weights $w_{ij}(y_{ij}|_{j\in\mathcal{N}_i})$. Moreover, $\mathbf{L}$ is of the following form

$$\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{H} \end{bmatrix}$$

where $\mathbf{B} \in \mathbb{C}^{n\times m}$ and $\mathbf{H} \in \mathbb{C}^{n\times n}$.

By Lemma 1, all the principal minors of $\mathbf{H}$ are distinct from zero. Hence, the equation

$$\mathbf{B}\mathbf{p}_a + \mathbf{H}\mathbf{p}_s = 0, \tag{6}$$

which is equivalent to (5), has a unique solution p_s. Next, we consider any subset of free nodes $\mathcal{U}_s$. Denote by $\mathcal{U}_a := \mathcal{V} \setminus \mathcal{U}_s$ the set of all the rest sensor nodes (including the anchor nodes). In addition, we denote $\mathbf{p}_{\mathcal{U}_a}$ and $\mathbf{p}_{\mathcal{U}_s}$ the corresponding position vectors for sensor nodes in $\mathcal{U}_a$ and $\mathcal{U}_s$ respectively. Then the equations in (5) with the row indices corresponding to the nodes in $\mathcal{U}_s$ can be written as

$$\mathbf{B}_{\mathcal{U}_a}\mathbf{p}_{\mathcal{U}_a} + \mathbf{H}_{\mathcal{U}_s}\mathbf{p}_{\mathcal{U}_s} = 0, \tag{7}$$

where $\mathbf{H}_{\mathcal{U}_s}$ is the principal minor of $\mathbf{H}$ with the indices corresponding to the nodes in $\mathcal{U}_s$ and $\mathbf{B}_{\mathcal{U}_a}$ is the corresponding block in $\mathbf{L}$. Recall that the principal minors of $\mathbf{H}$ are distinct from 0, so $\mathbf{H}_{\mathcal{U}_s}$ is invertible. Thus (7) admits a unique solution, which implies that the subset of free nodes $\mathcal{U}_s$ is jointly localizable, given the positions $\mathbf{p}_{\mathcal{U}_a}$ in the global coordinate frame Σ_g and the local coordinate based relative position measurements y_{ij} for $i \in \mathcal{U}_s$ and $(j,i) \in \mathcal{E}$. As a result, by Definition 3, the sensor network $\mathcal{G}$ is self-localizable. ∎

Remark 1: To understand that Assumption 1 of generic configuration of a sensor network is important in proving the necessary and sufficient condition in Theorem 1, we consider an example in Fig. 4. In this example, nodes 5 and 6 are the anchor nodes and every free node $i = 1, \ldots, 4$ is 2-reachable from the set of anchor nodes. Suppose the sensor configuration is the one given in Fig. 4(a), i.e., the position vector of the six sensor nodes in the global coordinate frame Σ_g is

$$p = [1+\iota, 1, 0, 2+\iota, \iota 3, 3]^T,$$

where each complex number is the position of each node in the global coordinate frame Σ_g. Then it can be checked that $\mathrm{rank}(\mathbf{L}) = \mathrm{rank}(\mathbf{H})$ is less than 4 for any $w_{ij}(y_{ij}|_{j\in\mathcal{N}_i})$, meaning that the locations of the sensor nodes cannot be solved from the linear constraint (5). However, such situations are of zero measure due to finite zeros for any finite-order polynomials. In other words, when the configuration changes a little bit, for example, we shift a little bit the position of node 5 in the neighborhood of the original location (Fig. 4(b)), then the new Dirichlet matrix $\mathbf{H}$ is nonsingular and thus the locations of all sensor nodes can be solved from (5).

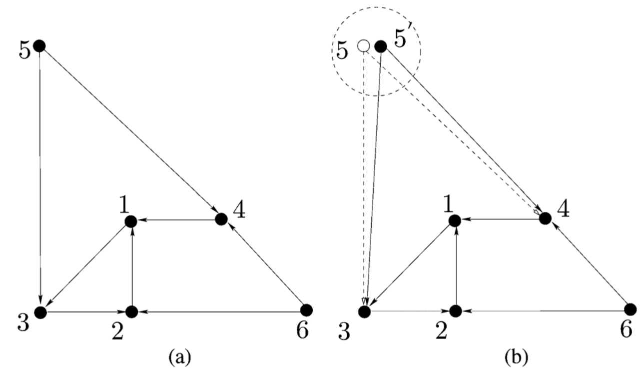


Fig. 4. Interpreaton of the generic configration assumption. (a) Not self-localizable. (b) self-localizable.

B. Verification of Self-Localizability Condition

Now, we provide a distributed algorithm to check whether a sensor network is self-localizable in terms of Theorem 1. We will mainly verify whether every sensor node is 2-reachable from the set of anchors by assuming that there are at least two anchor nodes in the network. In order to conduct a distributed verification, communications among the nodes are needed. We assume that the communication graph is also of the same topology as $\mathcal{G}$. This is reasonable as the communication range is usually larger than the sensing range. Though communications can be bidirectional, we consider only unidirectional communications in our distributed verification algorithm, namely, each node only receives some messages from its in-neighbors and sends some messages to its out-neighbors.

The idea of verifying the 2-reachability condition is simple. That is, each node receives path(s) from its in-neighbors and evaluates the paths:

— If there are two disjoint paths from the anchor set, it declares itself as an anchor node by transmitting the new path with its own ID to its out-neighbors.
— Otherwise, if there are multiple paths but not disjoint, it chooses the shortest one, adds its own ID at the end of this shortest path, and then transmits it to its out-neighbors (unless that this path has been transmitted before, in which case, there will be no repeated transmission); (Also if there is a loop in the path, there is no need to transmit it because the shorter path without the loop must have been transmitted before).

The result is that each node has new transmission only if it has just declared to be a new anchor node and re-initializes a new path with its own ID, or it has a new single shorter path from others. The precise description is presented in Algorithm 1 where $\Theta_i(t)$ is used to store the path information.

As an illustration of Algorithm 1, we take the sensor network given in Fig. 4 as an example, where nodes 5 and 6 are the two anchor nodes in the network. The transmitted information by the sensor nodes at each step is presented in Fig. 5, where $6 \to 2$ for example represents the path that is stored and transmitted by the sensor nodes, and the symbol '$-$' indicates that it is idle without any message transmission. In a little bit more details, at $t = 0$, nodes 1, 2, 3 and 4 do not have any path information to transmit as they are not anchors, so they remain idle, while nodes 5 and 6 are anchors, so they transmit the path information, namely, the paths with only one node, $\Theta_5(0) = 5$ and $\Theta_6(0) = 6$, to their out-neighbors. At $t = 1$, node 2 receives the path information 6 from its in-neighbor node 6 and nothing from the other in-neighbor node 3, so it adds its own ID to the end of this path (namely, $\Theta_3(1) = 6 \to 2$) and sends it to its out-neighbor. In the same time, node 4 at $t = 1$ receives two pieces of path information from node 5 and 6 and it checks that these two paths are disjoint, so it declares itself as a new anchor node and sends the new shortest path information (namely, $\Theta_4(1) = 4$) to its out-neighbors. The procedure continues until every node finds two disjoint paths to itself. The algorithm runs three steps for this example since the length of the longer path of the two shortest disjoint paths from the anchor set to node 3 is three.

Algorithm 1 Verification of self-localizability condition.

Initialization:
if node i is an anchor **then**
  $\Theta_i(0) = i$
else
  $\Theta_i(0) = \emptyset$
end if
Update at $t \geq 1$*:*
if node i does not receive any message from its in-neighbors **then**
  $\Theta_i(t) = \Theta_i(t-1)$
else
  Compare $\Theta_i(t-1)$ with the received message from its in-neighbors
  if two disjoint paths are found **then**
    $\Theta_i(t) = i$
  else
    $\Theta_i(t) =$ the shortest path (by comparing $\Theta_i(t-1)$ and all the paths received from its in-neighbors in this round)
  end if
end if
Message transmission at $t \geq 1$*:*
if $\Theta_i(t) = \emptyset$ **or** $\Theta_i(t) = \Theta_i(t-1)$ **then**
  remain idle
else
  transmit $\Theta_i(t)$ to its out-neigbhors
end if

Next, we show in the following theorem that Algorithm 1 is competent for checking self-localizability of a sensor network.

Theorem 2: A sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ with $m \geq 2$ anchor nodes is self-localizable if and only if Algorithm 1 terminates with every $\Theta_i(t) = i$ at some time t.

Proof: **(Necessity)** If a sensor network $(\mathcal{G}, \mathbf{p}_a, \mathbf{p}_s)$ with $m \geq 2$ anchor nodes is self-localizable, then it follows from Theorem 1 that every free node is 2-reachable from the set of anchors or equivalently there exist two disjoint paths from the set of anchors to every free node i. Then it is straightforward that every $\Theta_i(t)$ will become equal to i in a finite steps since

	node1	node2	node3	node4	node5	node6
t=0	-	-	-	-	5	6
t=1	-	$6 \to 2$	$5 \to 3$	4	-	-
t=2	1	2	-	-	-	-
t=3	-	-	3	-	-	-

Fig. 5. Illustration of Algorithm 1.

the lengths of the two disjoint paths from the set of anchor to every free node are finite.

(Sufficiency) If Algorithm 1 terminates with every $\Theta_i(t) = i$ at some time t, then two disjoint paths are found by node i, say $\mathcal{P}_1 : a_1 \to \cdots \to i$ and $\mathcal{P}_2 : a_2 \to \cdots \to i$. We consider three cases.

Case 1: the starting nodes a_1 and a_2 of the two disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are both anchors. For this case, it is concluded straightforward that node i has two disjoint paths from the set of anchors.

Case 2: the starting node of one of the two disjoint paths, say $\mathcal{P}_1$, is an anchor, but the other is not. Then according to Algorithm 1, two disjoint paths, say $\mathcal{P}_3 : b_1 \to \cdots \to a_2$ and $\mathcal{P}_4 : b_2 \to \cdots \to a_2$, are found by node a_2 as it is a starting node of a path transmitted to others. Without loss of generality, assume that the starting nodes b_1 and b_2 of the two disjoint paths found by node a_2 are anchors as otherwise we can repeat the same argument by considering three difference cases. Then by cutting off any one node on the path $\mathcal{P}_1$, there is still a path from the set of anchors to node i by going through either $\mathcal{P}_3$ or $\mathcal{P}_4$ followed by $\mathcal{P}_2$ since the node being cut off cannot be in $\mathcal{P}_3$ and $\mathcal{P}_4$ simultaneously and cannot be in $\mathcal{P}_2$ as well. Moreover, by cutting off any one node not in the path $\mathcal{P}_1$, then clearly, the path $\mathcal{P}_1$ still exists leading from an anchor to node i. So node i is 2-reachable from the set of anchors in this case.

Case 3: the starting nodes a_1 and a_2 of the two disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are neither anchors. In this case, it is clear that both a_1 and a_2 have two disjoint paths from others. For the same reason as in Case 2, we can assume that the starting nodes of these paths are anchors. Thus, by cutting off any one node in $\mathcal{P}_1$, there must exist another path from an anchor to node i by following one of the two disjoint paths from anchors to node a_2 followed by the path $\mathcal{P}_2$. The same counterpart conclusion can be obtained by cutting off any one node in $\mathcal{P}_2$. Moreover, by cutting of any node neither in $\mathcal{P}_1$ or $\mathcal{P}_2$, then there is a path from an anchor to either a_1 or a_2 and therefore there is a path from an anchor to node i. So node i is 2-reachable from the set of anchors in this case.

In conclusion, if Algorithm 1 terminates with every $\Theta_i(t) = i$ at some time t, then every free node i is 2-reachable from the set of anchors. Thus by Theorem 1, the conclusion follows. ■

IV. Distributed Localization Algorithm

In this section, we aim to develop a distributed algorithm for the computation of the position of each free node by itself in the global coordinate frame Σ_g when a sensor network is self-localizable. The distributed algorithm consists of two steps. Firstly, each free node uses its relative position measurements about its

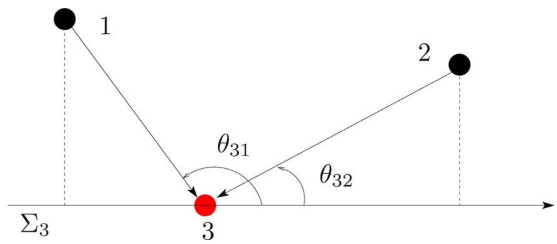


Fig. 6. An illustration of relative position measurements in node 3's local coordinate frame Σ_3.

in-neighbors to compute weights w_{ij}'s that will be used in iterations. Secondly, each free node updates its own estimate of the position of itself based on exchanged information from its neighbors. These two steps are addressed in the following two subsections respectively.

A. Weight Computation

As assumed in this paper, each free node can get the relative position measurement (2) in its own local coordinate frame. Moreover, by Theorem 1, we know that if a sensor network is self-localizable, then each free node is 2-reachable from the anchor set, which indicates that each free node has at least two in-neighbors. Thus, with respect to all its neighbors, each free node i can calculate the weights w_{ij}'s from the following equation

$$\sum_{j \in \mathcal{N}_i} w_{ij} y_{ij} = 0, \tag{8}$$

where $y_{ij},\ j \in \mathcal{N}_i$, are the measured data by node i.

Certainly, (8) has infinite number of solutions for w_{ij}'s. When $|\mathcal{N}_i| = 2$, the solution is of one-dimension scaled by complex numbers. So we can simply choose the complex weights w_{ij} that normalize the relative position vectors of its two in-neighbors and project them onto the positive and negative real axis of its local coordinate frame Σ_i. That is, if j is an in-neighbor of node i, then

$$w_{ij} = \frac{e^{-\iota\theta_{ij}}}{\rho_{ij}}, \tag{9}$$

and if k is the other in-neighbor of node i, then

$$w_{ik} = -\frac{e^{-\iota\theta_{ik}}}{\rho_{ik}}. \tag{10}$$

Taking Fig. 6 for example, we can choose the complex weights for arcs $(1, 3)$ and $(2, 3)$ as $w_{31} = \frac{e^{-\iota\theta_{31}}}{\rho_{31}}$, $w_{32} = -\frac{e^{-\iota\theta_{32}}}{\rho_{32}}$. Then (8) holds at node $i = 3$.

When $|\mathcal{N}_i| > 2$, the solution space of (8) is more than 1. Node i can randomly select one solution that does not contain zero entries. To be more specific, suppose that node i has in-neighbors, labelled $j_1, j_2, \ldots, j_k$ with $k > 2$. Then nonzero weights $w_{ij_1}, \ldots, w_{ij_k}$ should be selected to satisfy

$$\begin{bmatrix} y_{ij_1} & \cdots & y_{ij_k} \end{bmatrix} \begin{bmatrix} w_{ij_1} \\ \vdots \\ w_{ij_k} \end{bmatrix} = 0,$$

in which $y_{ij_1}, \ldots, y_{ij_k}$ are the measured data.

As shown in the sufficiency proof of Theorem 1, aggregating (8) leads to a linear equation (6), which is re-written in the following

$$\mathbf{H}\mathbf{p}_s + \mathbf{B}\mathbf{p}_a = 0. \tag{11}$$

Remark 2: If a sensor network is self-localizable, then from Theorem 1 and Lemma 1, it is almost sure that the matrix $\mathbf{H}$ is nonsingular by randomly selecting one solution w_{ij}'s from the solution space of (8).

B. Self-Localization Algorithm in Presence of Noisy Measurements

In this subsection, we will develop a distributed algorithm for localization.

Considering the noises in the relative position measurements and the round-off errors in computing the weights w_{ij}'s according to (8), the linear constraint (11) becomes

$$(\mathbf{H} - \mathbf{\Delta}_H)\mathbf{p}_s + (\mathbf{B} - \mathbf{\Delta}_B)\mathbf{p}_a = 0, \tag{12}$$

where $\mathbf{H}$ and $\mathbf{B}$ are the corresponding matrices calculated according to (8) while $\mathbf{\Delta}_H$ and $\mathbf{\Delta}_B$ are the error matrices due to the measurement noises and round-off errors. Denote $\mathbf{v} = \mathbf{\Delta}_H\mathbf{p}_s + \mathbf{\Delta}_B\mathbf{p}_a$. Then, (12) is re-written as

$$\mathbf{H}\mathbf{p}_s = -\mathbf{B}\mathbf{p}_a + \mathbf{v}. \tag{13}$$

For simplicity of analysis, we assume that $\mathbf{v} \sim \mathcal{CN}(0, \mathbf{\Sigma})$, which belongs to a complex Gaussian distribution with zero mean and covariance $\mathbf{\Sigma}$. Since the measurements and computation are independent processes carried out by individual nodes, $\mathbf{\Sigma}$ is diagonal and each node is supposed to know the i-th diagonal entry σ_i of $\mathbf{\Sigma}$.

By the theory of weighted least squares (WLS) estimation, it is known that the optimal weighted least squares estimate for the linear measurement (13) is

$$\mathbf{p}_s = \mathbf{\Psi}^{-1}\boldsymbol{\theta}, \tag{14}$$

where $\mathbf{\Psi} = \mathbf{H}^T\mathbf{\Sigma}^{-1}\mathbf{H}$ and $\boldsymbol{\theta} = -\mathbf{H}^T\mathbf{\Sigma}^{-1}\mathbf{B}\mathbf{p}_a$.

Next, we present a distributed algorithm for localization. That is, the algorithm should converge to the optimal solution (14) starting from any initial conditions.

Denote

$$\hat{\mathbf{H}} = \mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{H} \text{ and } \hat{\mathbf{B}} = \mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{B}.$$

By introducing an auxiliary variable $\boldsymbol{\zeta} \in \mathbb{C}^n$, we propose the following iteration algorithm

$$\begin{cases} \zeta(t+1) = \hat{\mathbf{H}}\hat{\mathbf{p}}_s(t) + \hat{\mathbf{B}}\mathbf{p}_a \\ \hat{\mathbf{p}}_s(t+1) = \hat{\mathbf{p}}_s(t) - \epsilon\hat{\mathbf{H}}^T\zeta(t). \end{cases} \tag{15}$$

The iteration algorithm (15) can be implemented on each node in a distributed manner (Algorithm 2), for which bidirectional communications are required between neighbors. In practical applications, the communication range is usually larger than the sensing range, so it is reasonable to assume that if node i can sense node j, then it is able to communicate with node j bidirectionally.

Algorithm 2 The implementation of (15) at node i.

1: Node i receives the estimate $\hat{p}_j(t)$ from its in-neighbors.
2: Node i receives the weighted auxiliary state $\bar{w}_{ki}\zeta_k(t)$ from its out-neighbors.
3: Node i updates its auxiliary state and its estimate as follows.

$$\begin{cases} \zeta_i(t+1) = \frac{1}{\sqrt{\sigma_i}} \sum_{j\in\mathcal{N}_i} w_{ij}\hat{p}_j(t), \\ \hat{p}_i(t+1) = \hat{p}_i(t) - \epsilon\frac{1}{\sqrt{\sigma_i}} \sum_{i\in\mathcal{N}_k} \bar{w}_{ki}\zeta_k(t). \end{cases}$$

4: Node i sends its estimate $\hat{p}_i(t+1)$ to its out-neighbors.
5: Node i sends the weighted auxiliary state $\bar{w}_{ij}\zeta_i(t+1)$ to its in-neighbors.

Finally, we provide the convergence result for the iteration algorithm (15).

Theorem 3: Suppose that $\mathbf{H}$ is non-singular. If

$$0 < \epsilon < \frac{1}{\lambda_{\max}(\mathbf{\Psi})},$$

then the distributed algorithm (15) converges to the optimal solution (14) as t tends to ∞.

Proof: First of all, it can be checked that the optimal solution (14) is the unique equilibrium point of (15).

Writing (15) in the matrix form, we have

$$\begin{bmatrix} \zeta(t+1) \\ \hat{\mathbf{p}}_s(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{H}} \\ -\epsilon\hat{\mathbf{H}}^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \zeta(t) \\ \hat{\mathbf{p}}_s(t) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{B}} \\ \mathbf{0} \end{bmatrix} \mathbf{p}_a.$$

Let λ be an eigenvalue of $\mathbf{A} = [\begin{smallmatrix} \mathbf{0} & \hat{\mathbf{H}} \\ -\epsilon\hat{\mathbf{H}}^T & \mathbf{I} \end{smallmatrix}]$ and let $[\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}]$ be the associated eigenvector. Then we have

$$\begin{cases} \hat{\mathbf{H}}\mathbf{y} = \lambda\mathbf{x}, \\ -\epsilon\hat{\mathbf{H}}^T\mathbf{x} + \mathbf{y} = \lambda\mathbf{y}. \end{cases}$$

After several steps of mathematical manipulations, it is obtained that

$$\epsilon\mathbf{\Psi}\mathbf{y} = (-\lambda^2 + \lambda)\mathbf{y}.$$

So the eigenvalue λ must satisfy

$$\lambda^2 - \lambda + \epsilon\gamma = 0$$

where γ is an eigenvalue of Ψ. That is,

$$\lambda = \frac{1 \pm \sqrt{1 - 4\epsilon\gamma}}{2}.$$

Note that γ is real and positive since $\mathbf{\Psi}$ is a Hermitian matrix. In order to make every λ lie strictly inside the unit disk, it requires $\epsilon < \frac{1}{\lambda_{\max}(\mathbf{\Psi})}$. Thus, under this condition, the distributed algorithm (15) converges to the optimal solution (14) as t tends to ∞. ■

Remark 3: To ensure the convergence of the distributed algorithm (15), ϵ has to be upper bounded as shown in Theorem 3. In this remark, we provide a distributed approach to find a feasible ϵ in a finite number of steps. Recall that

$$\lambda_{\max}(\mathbf{\Psi}) \leq \|\hat{\mathbf{H}}\|_1 \|\hat{\mathbf{H}}\|_\infty,$$

so we can take $\frac{1}{\|\hat{\mathbf{H}}\|_1\|\hat{\mathbf{H}}\|_\infty}$ as an upper bound for ϵ. Since each node i knows the entries in the i-th row of $\hat{\mathbf{H}}$ (namely, the weights w_{ij}'s with j being its in-neighbor and σ_i), it can then knows $r_i = \sum_{j=1}^n |\frac{1}{\sqrt{\sigma_i}} w_{ij}|$. Thus, $\|\hat{\mathbf{H}}\|_\infty = \max_i r_i$ can be computed in finite time by the maximum consensus algorithm [32]. Furthermore, the nonzero entries in the i-th column of $\hat{\mathbf{H}}$ are available to node i via communications from its out-neighbors. So each node i knows $l_i = \sum_{j=1}^n |\frac{1}{\sqrt{\sigma_j}} w_{ji}|$. Similarly, $\|\hat{\mathbf{H}}\|_1 = \max_i l_i$ can be computed in finite time by the maximum consensus algorithm. Then ϵ can be chosen to be any value in $(0, \frac{1}{\|\hat{\mathbf{H}}\|_1\|\hat{\mathbf{H}}\|_\infty})$, which ensures the convergence of the distributed algorithm (15).

V. Simulations

In this section we provide two simulations to illustrate our results. We consider a sensor network with 60 nodes, in which two nodes are anchor nodes. Simulations are carried out for both the noiseless case and noisy case. For the noisy case, the measurement noise v in (13) is assumed to satisfy satisfies $\mathbf{v} \sim \mathcal{CN}(0, \mathbf{I})$.

In the first simulation, we consider a directed sensing graph, for which only 28 nodes are localizable as checked by Algorithm 1. The configuration of the 60 nodes in the plane is shown in Fig. 7(d), where red circles represent the anchor nodes, blue stars represent the free nodes that are localizable, and green squares correspond to the free nodes that are unlocalizable.

According to Theorem 3, the parameter ϵ in Algorithm 2 is selected in the interval $(0, \frac{1}{\lambda_{\max}(\Psi)})$. By running Algorithm 2, the estimated positions of the 28 localizable nodes for the noisy case are shown in Fig. 7(a), (b), and (c), which plot three snapshots of $\hat{\mathbf{p}}_s(t)$ at $t = 1, t = 1000$, and $t = 15000$ respectively. Moreover, the ratio of the estimation error at time t to the initial estimation error, described by $\frac{\|\hat{p}_s(t) - p_s\|}{\|\hat{p}_s(0) - p_s\|}$, is plotted in Fig. 8 for both the noiseless and noisy case. From both the snapshots in Fig. 7 and the estimation error ratio in Fig. 8, the estimations converge to the true positions in the global coordinate frame. But from Fig. 8, we can see that with the noisy measurements, the convergence is slower than the one without noisy measurements.

In the second simulation, we consider a directed sensing graph with less edges, for which we check by Algorithm 1 that only 21 free nodes are localizable. The true positions of these 21 localizable nodes are plotted as blue stars in Fig. 9(d). Running Algorithm 2 with the same parameter ϵ, the estimated positions of the 21 localizable nodes for the noisy case are shown in Fig. 9(a), (b), and (c), which plot three snapshots of $\hat{\mathbf{p}}_s(t)$ at $t = 1, t = 1000$, and $t = 15000$ respectively. Moreover, the ratio of the estimation error at time t to the initial estimation error is plotted in Fig. 10 for both the noiseless and noisy case. It can be seen that the estimation converge to the true positions, too. But with less number of nodes by comparing with the first simulation, the convergence becomes faster.

VI. Conclusion

This paper concentrates on the self-localization problem based on relative position measurements for sensor networks, aiming to let each sensor node determine its unique position

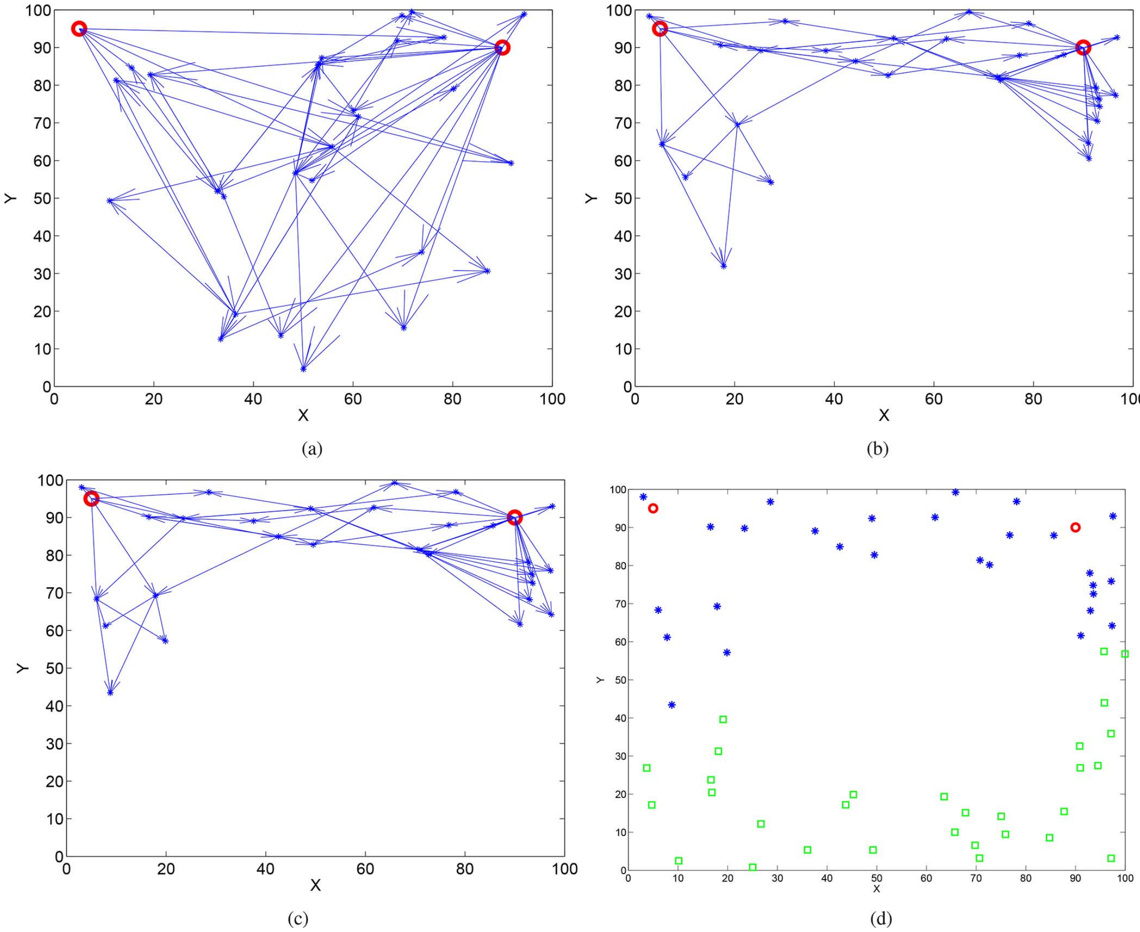


Fig. 7. Simulation I: Three snapshots and true positions. In these figures, the lines with arrows indicate the directed edges of the sensing graph. (a) The snapshot of position estimation at $t = 1$. (b) The snapshot of position estimation at $t = 1000$. (c) The snapshot of position estimation at $t = 15000$. (d) True positions.

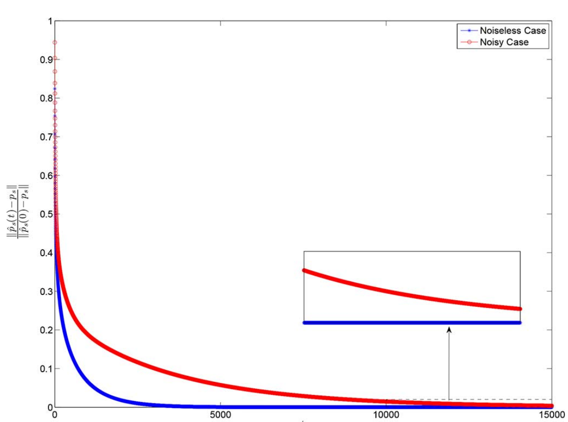


Fig. 8. Simulation I: The estimation error ratio with respect to t.

using its relative position measurements about its neighbors and the received estimates of the positions of its neighbors. Assume that the relative position measurements are obtained by sensor nodes in their own local coordinate frames that do not share a common orientation. A necessary and sufficient condition is presented for self-localizability of a sensor network in terms of 2-reachability of the directed sensing graph. Moreover, a distributed algorithm is developed to verify the self-localizability of a sensor network. A rigorous analysis is provided showing that the verification algorithm can terminate with every node being able to find two disjoint paths from the anchor nodes if and only if the whole sensor network is self-localizable. Finally, a fully distributed self-localization algorithm is developed to iteratively compute the position of each node itself in the presence of measurement noises and round-off errors. The iterative localization algorithm is linear and thus ensures global convergence.

Many interesting problems arising from this research deserve further investigation. Examples include optimal selection of anchors and Laplacian weights that lead to the fastest localization iterations, distributed localization for mobile sensor networks, finding joint-localizability conditions for sensor networks with relative position measurements on local coordinate frames with different orientations; devising distributed methods for self-localization or joint-localization by considering time-varying communication networks that may or may not be of the same topology as the sensing graph; and generalizing the work to the localization problem in the 3D space.

APPENDIX A

Lemma 1: [33] Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a subset of nodes $\mathcal{R} \subset \mathcal{V}$ with $|\mathcal{R}| \geq 2$, and a generic configuration $\xi \in \mathbb{C}^l$. Denote by $\mathbf{L}$ a complex Laplacian of $\mathcal{G}$ satisfying $\mathbf{L}\xi = 0$. Denote by $\mathbf{L}_{\mathcal{R}}$ the sub-matrix of $\mathbf{L}$ with the rows and

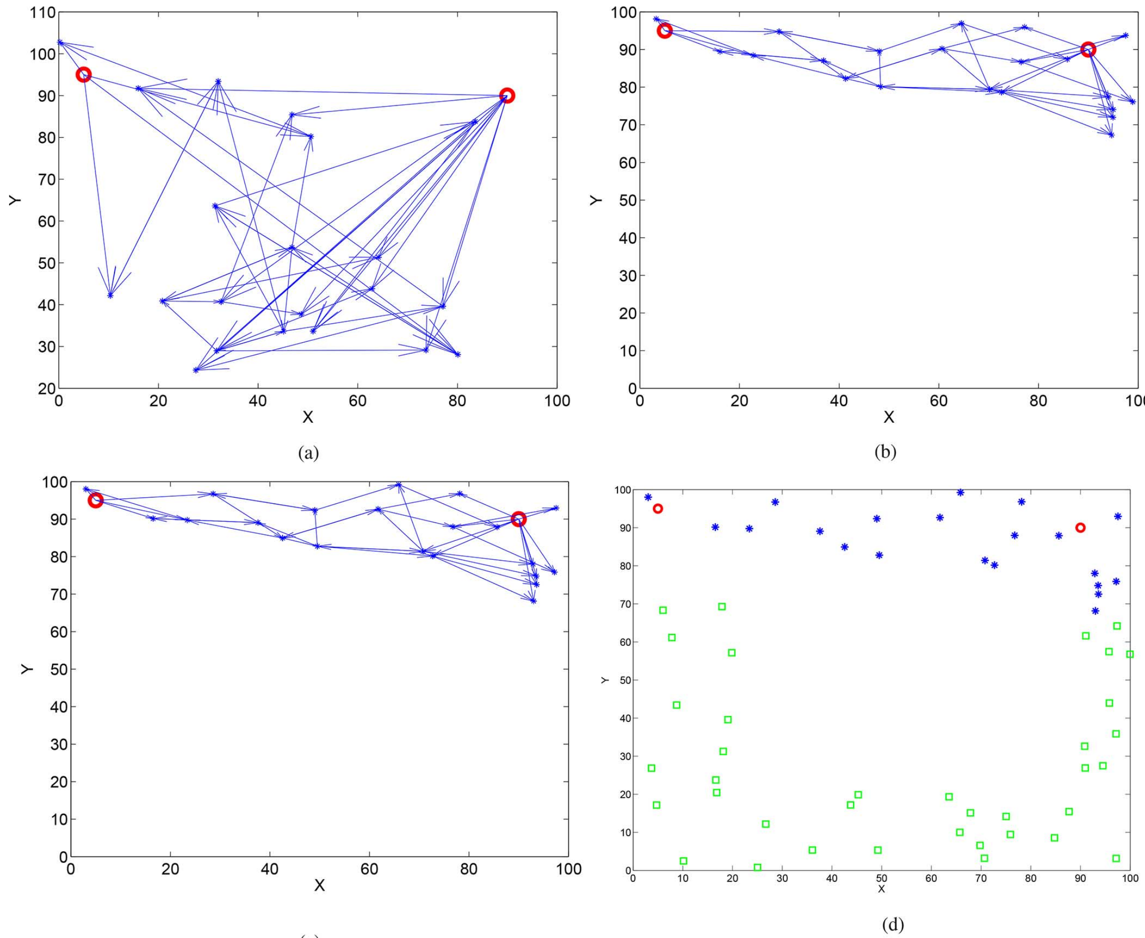


Fig. 9. Simulation II: Three snapshots and true positions. (a) The snapshot of position estimation at $t = 1$. (b) The snapshot of position estimation at $t = 1000$. (c) The snapshot of position estimation at $t = 15000$. (d) A network with 60 nodes and smaller communication radius.

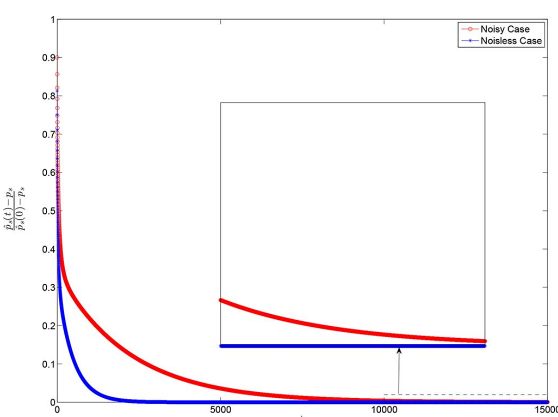


Fig. 10. Simulation II: The estimation error ratio with respect to t.

columns of $\mathbf{L}$ corresponding to the nodes in $\mathcal{R}$ crossed out. The following statements are equivalent.

1) Every node in $\mathcal{V} - \mathcal{R}$ is 2-reachable from $\mathcal{R}$.
2) For almost all $\mathbf{L}$ satisfying $\mathbf{L}\xi = 0, \det(\mathbf{L}_{\mathcal{R}}) \neq 0$.
3) For almost all $\mathbf{L}$ satisfying $\mathbf{L}\xi = 0$, the principal minors of $\mathbf{L}_{\mathcal{R}}$ are distinct from zero.

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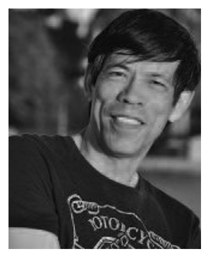
Zhiyun Lin (S'04–M'05–SM'10) received his bachelor degree in electrical engineering from Yanshan University, China, in 1998, master degree in electrical engineering from Zhejiang University, China, in 2001, and Ph.D. degree in electrical and computer engineering from the University of Toronto, Canada, 2005.

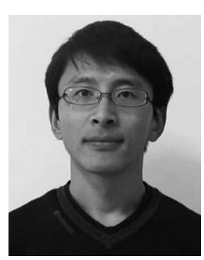
He was a Postdoctoral Research Associate in the Department of Electrical and Computer Engineering, University of Toronto, Canada, from 2005 to 2007. He joined the College of Electrical Engineering, Zhejiang University, China, in 2007. Currently, he is a Professor of Systems Control in the same department. He is also affiliated with the State Key Laboratory of Industrial Control Technology at Zhejiang University. He held visiting professor positions at several universities including The Australian National University (Australia), University of Cagliari (Italy), University of Newcastle (Australia), University of Technology Sydney (Australia), and Yale University (USA). His research interests focus on distributed control, estimation and optimization, coordinated and cooperative control of multi-agent systems, hybrid and switched control system theory, and locomotion control of biped robots. He is a senior member of IEEE. He is currently an associate editor for Hybrid systems: Nonlinear Analysis and International Journal of Wireless and Mobile Networking.



Minyue Fu (S'84–M'87–SM'94–F'04) received his Bachelor's Degree in electrical engineering from the University of Science and Technology of China, Hefei, China, in 1982, and M.S. and Ph.D. degrees in electrical engineering from the University of Wisconsin-Madison, in 1983 and 1987, respectively.

From 1987 to 1989, he served as an Assistant Professor in the Department of Electrical and Computer Engineering, Wayne State University, Detroit, Michigan. He joined the Department of Electrical and Computer Engineering, the University of Newcastle, Australia, in 1989. Currently, he is a Chair Professor in Electrical Engineering. He was a Visiting Associate Professor at University of Iowa in 1995–1996, a Senior Fellow/Visiting Professor at Nanyang Technological University, Singapore, 2002, and Visiting Professor at Tokyo University in 2003. He has held a ChangJiang Visiting Professorship at Shandong University, a visiting Professorship at South China University of Technology, and a Qian-ren Professorship at Zhejiang University in China.

He was elected to a Fellow of IEEE in late 2003. His main research interests include control systems, signal processing and communications. His current research projects include networked control systems, smart electricity networks and super-precision positioning control systems. He has been an Associate Editor for the IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Signal Processing, and Journal of Optimization and Engineering.



Yingfei Diao (M'14) is now a Research Engineer in Huawei Shanghai R&D Center, Huawei Technologies Co. Ltd., China. He received the Ph.D. degree from the School of Control Science and Engineering, Shandong University, China, in 2013, and Bachelor degree from the School of Information and Electrical Engineering, China University of Mining and Technology, China, in 2006.

He has been a visiting student in the University of Newcastle, Australia, from March 2012 to February 2013, and a Research Fellow in the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, from November 2013 to June 2014. His research interests include wireless communication, cooperative localization for sensor network, multi-agent systems, algebraic graph theory, distributed estimation and collective behavior of robotics.