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Stability conditions for the time-varying linear predictor

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Abstract

The stability of the inverse of the optimum forward prediction error filter obtained when the input data is nonstationary is investigated. Due to this nonstationary character, the resulting system (which is obtained assuming optimality on a sample-by-sample basis) is time-varying. It turns out that an extension of the Levinson recursion still provides a means to order-update the prediction error filters, leading to asymmetric lattice realizations of the filters. Sufficient conditions on the input process are given in order to ensure exponential asymptotic stability of the corresponding inverse system. Thus this work extends the well-known result from linear prediction theory which states that the transfer function of the optimum forward prediction error filter for a stationary process is minimum phase.

1 Introduction

This paper considers certain aspects of optimum prediction of nonstationary sequences. To place the problem in context consider the linear prediction of wide sense stationary (WSS) signals. In particular suppose $u(k)$ is a zero mean WSS signal. The optimal m-th order linear prediction problem for such a process seeks to find $\{a_i^m\}_{i=1}^m$ to minimize the variance $E[f_m^2(k)]$ of the forward prediction error $f_m(k)$ given by

$$f_m(k) = u(k) + \sum_{i=1}^{m} a_i^m u(k-i). \tag{1}$$

The classical solution to this problem is through the celebrated Levinson-Durbin recursions, which simultaneously find both the optimal forward predictor above and the optimum backward predictor:

$$b_m(k) = u(k-m) + \sum_{i=1}^{m} c_i^m u(k-m+i)$$

that minimizes $E[b_m^2(k)]$, $b_m(k)$ being the backward prediction error. The optimizing coefficients of the backward predictor are the same as those of the forward predictor in reversed order. Associated with the Levinson-Durbin recursions is the forward lattice of figure 1, where each $f_i(k)$ and $b_i(k)$ represents the corresponding optimum signal and the α_i are the so called reflection coefficients. As long as there is no perfect m-th order predictor, i.e. the m-dimensional autocorrelation matrix of $u(k)$ is positive definite, these obey $|\alpha_i| < 1$ subject to the boundedness of the autocorrelations.

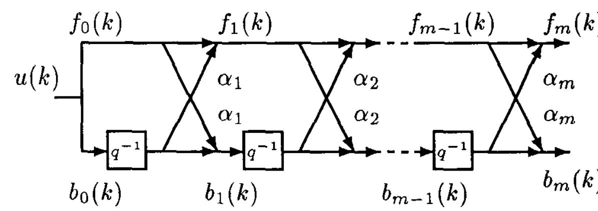

Figure 1: Lattice implementation of the prediction-error filter for WSS input.

Quite often in speech communication one transmits only $f_m(k)$ and the reflection coefficients, as the original signal can be asymptotically recovered by the Gray-Markel lattice of figure 2. Furthermore, as long as the reflection coefficients are less than one in magnitude, the structure in fig. 2 is exponentially stable. Alternatively one transmits $f_m(k)$ and the coefficients of the

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forward predictor (1), and resynthesizes the signal by implementing the inverse system of (1). Under the conditions stated above this inverse system is stable as well. This whole approach finds application in such areas as speech modelling [6], differential pulse code modulation (DPCM) [4] or spectral analysis [5].

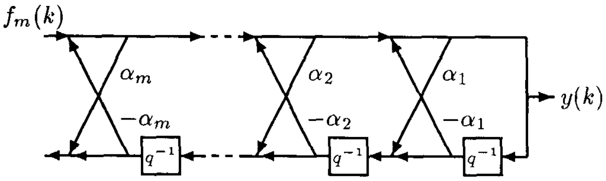

Figure 2: Recursive lattice structure.

However, most often in real-life problems the data $u(\cdot)$ are not stationary. In such a case one usually adaptively estimates the lattice coefficients and/or the predictor coefficients, on the basis of the WSS solution. However in this case neither the resulting inverse predictor nor the time varying structure in fig. 1 is guaranteed to be stable. For example even if the adaptively estimated lattice coefficients are magnitude bounded by 1, stability of figure 2 cannot in general be guaranteed if the α_i's vary with time [3], and in fact the basic structures of figures 1 and 2 are simply not optimal for non-WSS processes. Accordingly this paper considers optimal m-th order linear prediction problem for a nonstationary signal. Specifically, the goal is the minimization of $E[f_m^2(k)]$ where

$$f_m(k) = u(k) + \sum_{i=1}^{m} a_i^m(k)u(k-i), \tag{2}$$

by deriving both the Levinson-Durbin recursions, and the corresponding lattice implementations (i.e. the analogues of fig. 1 and 2). Particular attention is paid to the stability of both the inverse predictor and its lattice implementation. Observe in this case the predictor coefficients are obviously time varying.

Section 2 describes the recursions and the forward and reverse lattice. Section 3 gives certain key properties that are useful in the stability analysis. Section 4 gives this analysis of both inverse predictor and its lattice implementation, and shows stability under mild conditions.

A key technical difficulty in the stability analysis resolved here is the following. It is known that in the WSS case should an m-th order predictor be perfect, i.e. *the optimum forward prediction error have zero variance*, then the inverse predictor is no longer asymptotically stable. This arises if the $(m+1)$-th dimensional autocorrelation matrix is singular. The difficulty in the nonstationary case is that the $(m+1)$-th dimensional autocorrelation matrix is no longer constant and can occasionally become singular even though it may be positive definite at other times. Some of the machinery developed in Sections 3 and 4 is geared toward coping with this occasional singularity problem.

2 Finding the optimum predictor

We shall assume that $u(\cdot)$ is a zero-mean real stochastic process. The autocorrelation coefficients are denoted by $r_n(k) \triangleq E[u(k)u(k-n)]$. It is convenient to introduce the backward linear prediction problem: minimize $E[b_m^2(k)]$ where

$$b_m(k) = u(k-m) + \sum_{i=1}^{m} c_i^m(k)u(k-m+i)$$

is the m-order backward prediction error. If we define the vectors

$$\begin{aligned}
\mathbf{a}_m(k) &= [\ a_1^m(k) \ \cdots \ a_m^m(k)\ ]',\\
\mathbf{c}_m(k) &= [\ c_m^m(k) \ \cdots \ c_1^m(k)\ ]',\\
\mathbf{u}_m(k) &= [\ u(k) \ \ u(k-1) \ \cdots \ u(k-m+1)\ ]',
\end{aligned}$$

then we can express the prediction errors more compactly as

$$\begin{aligned}
f_m(k) &= u(k) + \mathbf{u}_m(k-1)'\mathbf{a}_m(k), && (3)\\
b_m(k) &= u(k-m) + \mathbf{u}_m(k)'\mathbf{c}_m(k). && (4)
\end{aligned}$$

Let us introduce the notation

$$\begin{aligned}
&\mathbf{R}_m(k) \triangleq E[\mathbf{u}_m(k)\mathbf{u}_m(k)']\\
&= \begin{bmatrix} r_0(k) & r_1(k) & \cdots & r_{m-1}(k)\\ r_1(k) & r_0(k-1) & \cdots & r_{m-2}(k-1)\\ \vdots & \vdots & \ddots & \vdots\\ r_{m-1}(k) & r_{m-2}(k-1) & \cdots & r_0(k-m+1) \end{bmatrix},
\end{aligned}$$

(autocorrelation matrix) and

$$\begin{aligned}
&\mathbf{r}_m(k) \triangleq E[\mathbf{u}_m(k-1)u(k)]\\
&= \quad [\ r_1(k) \ \ r_2(k) \ \ r_3(k) \ \cdots \ r_m(k)\ ]',\\
&\mathbf{s}_m(k) \triangleq E[\mathbf{u}_m(k)u(k-m)]\\
&= [\ r_m(k) \ \ r_{m-1}(k-1) \ \cdots \ r_1(k-m+1)\ ]',
\end{aligned}$$

(autocorrelation vectors). This allows to write the variances of the prediction errors as

$$\begin{aligned}
E[f_m^2(k)] &= [\ 1 \ \ \mathbf{a}_m(k)'\ ]\mathbf{R}_{m+1}(k)\begin{bmatrix} 1\\ \mathbf{a}_m(k)\end{bmatrix} && (5)\\
E[b_m^2(k)] &= [\ \mathbf{c}_m(k)' \ \ 1\ ]\mathbf{R}_{m+1}(k)\begin{bmatrix} \mathbf{c}_m(k)\\ 1\end{bmatrix} && (6)
\end{aligned}$$

It proves useful to introduce the following partitions:

$$\mathbf{R}_m(k) = \begin{bmatrix} r_0(k) & \mathbf{r}_{m-1}(k)' \\ \mathbf{r}_{m-1}(k) & \mathbf{R}_{m-1}(k-1) \end{bmatrix}, \quad (7)$$

$$\mathbf{R}_m(k-1) = \begin{bmatrix} \mathbf{R}_{m-1}(k-1) & \mathbf{s}_{m-1}(k-1) \\ \mathbf{s}_{m-1}(k-1)' & r_0(k-m) \end{bmatrix} \quad (8)$$

and

$$\mathbf{r}_m(k) = \begin{bmatrix} \mathbf{r}_{m-1}(k)' & r_m(k) \end{bmatrix}', \quad (9)$$

$$\mathbf{s}_m(k) = \begin{bmatrix} r_m(k) & \mathbf{s}_{m-1}(k-1)' \end{bmatrix}'. \quad (10)$$

Now the values of the coefficient vectors $\mathbf{a}_m(k)$, $\mathbf{c}_m(k)$ that minimize (5) and (6) respectively *at every time instant* are

$$\mathbf{R}_m(k-1)\mathbf{a}_m(k) = -\mathbf{r}_m(k), \quad (11)$$

$$\mathbf{R}_m(k)\mathbf{c}_m(k) = -\mathbf{s}_m(k). \quad (12)$$

These are the time-varying counterpart of the normal equations [7, 8]. Observe that although $\mathbf{R}_m(k)$ is symmetric, it need not be Toeplitz as in the WSS case; as a consequence, $\mathbf{a}_m(k) \neq \mathbf{c}_m(k)$ in general. The minimized values of the error variances are

$$\mathcal{F}_m(k) \triangleq r_0(k) + \sum_{i=1}^{m} a_i^m(k) r_i(k) = r_0(k) + \mathbf{r}_m(k)'\mathbf{a}_m(k), \quad (13)$$

$$\mathcal{B}_m(k) \triangleq r_0(k-m) + \sum_{i=1}^{m} c_i^m(k) r_i(k-m+i) = r_0(k-m) + \mathbf{s}_m(k)'\mathbf{c}_m(k). \quad (14)$$

This result could also have been obtained by applying the orthogonality principle [1]: the optimum vectors $\mathbf{a}_m(k)$, $\mathbf{c}_m(k)$ are such that the resulting prediction errors satisfy, for every k,

$$E[f_m(k)u(k-i)] = 0, \qquad i = 1, 2, \ldots, m, \quad (15)$$

$$E[b_m(k)u(k-j)] = 0, \qquad j = 0, 1, \ldots, m-1. \quad (16)$$

In the sequel the following assumption will always apply.

Assumption 1 *There exists a constant M such that for all k and m,*

$$\mathbf{0} \leq \mathbf{R}_m(k) \leq M\mathbf{I}.$$

This assumption guarantees that both $\mathcal{F}_m(k)$ and $\mathcal{B}_m(k)$ are nonnegative. Further they can be zero only if $\mathbf{R}_{m+1}(k)$ is singular. Now the equivalent of the Levinson recursion [1, 7, 8] can be obtained. To begin, let $\Delta_m(k)$ denote the cross-correlation between $f_m(k)$ and $b_m(k-1)$:

$$\Delta_m(k) \triangleq E[b_m(k-1)f_m(k)],$$

and define the *reflection coefficients* $\alpha_m(k)$, $\beta_m(k)$ as

$$\alpha_m(k) = \begin{cases} -\frac{\Delta_{m-1}(k)}{\mathcal{B}_{m-1}(k-1)}, & \text{if } \mathcal{B}_{m-1}(k-1) > 0, \\ 0, & \text{if } \mathcal{B}_{m-1}(k-1) = 0, \end{cases} \quad (17)$$

$$\beta_m(k) = \begin{cases} -\frac{\Delta_{m-1}(k)}{\mathcal{F}_{m-1}(k)}, & \text{if } \mathcal{F}_{m-1}(k) > 0, \\ 0, & \text{if } \mathcal{F}_{m-1}(k) = 0, \end{cases} \quad (18)$$

Now we can state:

Theorem 1 *Under assumption 1 the optimum coefficient vectors $\mathbf{a}_m(k)$, $\mathbf{c}_m(k)$ can be order-updated as per*

$$\mathbf{a}_m(k) = \begin{bmatrix} \mathbf{a}_{m-1}(k) \\ 0 \end{bmatrix} + \alpha_m(k) \begin{bmatrix} \mathbf{c}_{m-1}(k-1) \\ 1 \end{bmatrix}, \quad (19)$$

$$\mathbf{c}_m(k) = \begin{bmatrix} 0 \\ \mathbf{c}_{m-1}(k-1) \end{bmatrix} + \beta_m(k) \begin{bmatrix} 1 \\ \mathbf{a}_{m-1}(k) \end{bmatrix}, \quad (20)$$

with the initialization

$$\mathbf{a}_1(k) = \begin{cases} -\frac{r_1(k)}{r_0(k-1)}, & \text{if } r_0(k-1) > 0, \\ 0, & \text{if } r_0(k-1) = 0, \end{cases} \quad (21)$$

$$\mathbf{c}_1(k) = \begin{cases} -\frac{r_1(k)}{r_0(k)}, & \text{if } r_0(k) > 0, \\ 0, & \text{if } r_0(k) = 0, \end{cases} \quad (22)$$

Corollary 1 *With the initialization $f_0(k) = b_0(k) = u(k)$, the prediction errors obey the following recursions, for $i > 0$:*

$$f_i(k) = f_{i-1}(k) + \alpha_i(k) b_{i-1}(k-1), \quad (23)$$

$$b_i(k) = b_{i-1}(k-1) + \beta_i(k) f_{i-1}(k), \quad (24)$$

What corollary 1 says is that, as in the WSS case, all the forward and backward prediction error filters up to a certain order can be wound up in a lattice structure (see figure 3) which is now time-varying. More important, note that this structure is also *asymmetric*, i.e. in general $\alpha_i(k) \neq \beta_i(k)$, in contrast with the WSS case.

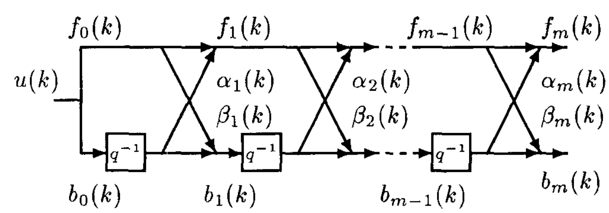

Figure 3: Lattice implementation of the prediction-error filter.

Eqs. (23), (24) can be rewritten as

$$f_{i-1}(k) = f_i(k) - \alpha_i(k) b_{i-1}(k-1), \quad (25)$$

$$b_i(k) = b_{i-1}(k-1) + \beta_i(k) f_{i-1}(k), \quad (26)$$

for $i > 0$. This provides a means to implement the inverse of the forward prediction error filter, by starting the recursions (25), (26) with $f_m(k)$ and closing the loop with $b_0(k) = f_0(k)$. This recursive, asymmetric, time-varying lattice structure is depicted in figure 4.

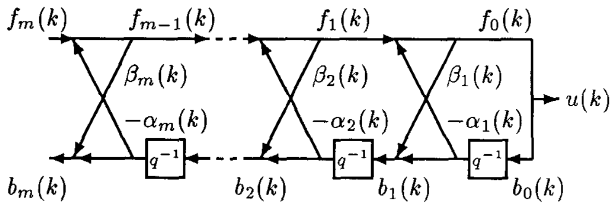

Figure 4: Recursive lattice structure for the inverse forward prediction error filter.

The natural question at this point is whether the time-varying system of figure 4 is stable. First we present some properties of the prediction error filters, which will help us take on the question of stability in section 4.

3 Properties of the linear predictor

Throughout assumption 1 is assumed to hold. Here we address the main properties of the time-varying linear predictor. In most cases they are extensions of well-known properties of the LTI filters obtained in the WSS case.

Property 1 *The forward and backward prediction error variances satisfy*

$$\begin{aligned}\mathcal{F}_m(k) &= \mathcal{F}_{m-1}(k) - \alpha_m^2(k)\mathcal{B}_{m-1}(k-1), \quad (27)\\ \mathcal{B}_m(k) &= \mathcal{B}_{m-1}(k-1) - \beta_m^2(k)\mathcal{F}_{m-1}(k). \quad (28)\end{aligned}$$

Note that in general $\alpha_m(k) \neq \beta_m(k)$ and $\mathcal{F}_m(k) \neq \mathcal{B}_m(k)$. An inmediate consequence of property 1 is:

Property 2 *For any fixed time instant k, the sequences $\{\mathcal{F}_m(k)\}_{m\geq 0}$ and $\{\mathcal{B}_m(k+m)\}_{m\geq 0}$ are non-increasing:*

$$\begin{aligned}0 \leq \quad &\mathcal{F}_m(k) \quad \leq \mathcal{F}_{m-1}(k),\\ 0 \leq \quad &\mathcal{B}_m(k+m) \quad \leq \mathcal{B}_{m-1}(k+m-1),\end{aligned}$$

for all k and for all $m \geq 1$.

Note that it is not true in general that $0 \leq \mathcal{B}_m(k) \leq \mathcal{B}_{m-1}(k)$ for all k and all $m \geq 0$.

Property 3 *For all k and for all $m \geq 1$,*

$$0 \leq \alpha_m(k)\beta_m(k) \leq 1.$$

Further, there exist M_1, M_2 such that for all k,

$$\alpha_m(k) \leq M_1,$$

and

$$\beta_m(k) \leq M_2,$$

Property 3 is the extension to the well-known result for the WSS case that states that the reflection coefficients have magnitude no greater than one. Note, however, that in general $|\alpha_m(k)| \leq 1$ and/or $|\beta_m(k)| \leq 1$ need not hold.

We turn our attention now to predictable processes, whose definition is given next:

Definition 1 *The process $u(\cdot)$ is forward (resp., backward) perfectly predictable of order m at time k_0 if $\mathcal{F}_m(k_0) = 0$ (resp., if $\mathcal{B}_m(k_0) = 0$).*

In the WSS case, if the process $u(\cdot)$ is forward or backward perfectly predictable of order m at time k_0, then $\mathcal{F}_m(k) = \mathcal{B}_m(k) = 0$ for all k. In the nonstationary case, however, the process $u(\cdot)$ may become predictable at some time instants and not at others. Also note that perfect predictability of order m at k_0 can happen only if $\mathbf{R}_{m+1}(k_0)$ becomes singular (the reciprocal is not true in general though):

Lemma 1 *Assume that the autocorrelation matrix sequence $\mathbf{R}_{m+1}(\cdot)$ is uniformly positive definite (u.p.d.), i.e.*

$$c_1\mathbf{I} \leq \mathbf{R}_{m+1}(k) \leq c_2\mathbf{I} \qquad \forall k,$$

for some $c_1, c_2 > 0$. Then there exists a finite constant $c_3 > 0$ such that $c_1 \leq \mathcal{F}_m(k), \mathcal{B}_m(k) \leq c_3$ for all k.

Thus if $\mathbf{R}_{m+1}(\cdot)$ is u.p.d., the input process cannot be perfectly predictable (at least of order m) at any time instant. On the other hand, for the reflection coefficients of the *first* stage that achieves perfect prediction, the following holds:

Lemma 2 *Assume that $\mathbf{R}_m(\cdot)$ is u.p.d. Then, if $\mathcal{F}_m(k_0) = 0$ (resp. if $\mathcal{B}_m(k_0) = 0$) at a given k_0, the following holds true:*

1. *$\alpha_m^2(k_0)$ is bounded away from zero,*
2. *$\beta_m^2(k_0)$ is bounded away from zero,*
3. *$\alpha_m(k_0)\beta_m(k_0) = 1$,*

4. $\mathcal{B}_m(k_0) = 0$ *(resp.* $\mathcal{F}_m(k_0) = 0$*).*

Also if $\alpha_m(k_0)\beta_m(k_0) = 1$ *then* $\mathcal{F}_m(k_0) = \mathcal{B}_m(k_0) = 0$.

Thus if one prediction error becomes identically zero for some k, so must the other (showing that backward and forward perfect predictability are equivalent concepts), and then the product of the reflection coefficients equals one. Finally we give the converse result:

Lemma 3 *Assume that* $\mathbf{R}_m(\cdot)$ *is u.p.d. Then, if* $\alpha_m(k_0) = 0$, *(resp. if* $\beta_m(k_0) = 0$*) at a given* k_0, *the following holds true:*

1. $\beta_m(k_0) = 0$ *(resp.* $\alpha_m(k_0) = 0$*),*
2. $\mathcal{F}_m(k_0)$ *is bounded away from zero,*
3. $\mathcal{B}_m(k_0)$ *is bounded away from zero.*

Therefore we conclude that under the u.p.d. assumption, the prediction error variances and the reflection coefficients of the same lattice stage cannot vanish simultaneously.

4 Stability of the inverse system

We pick now the question from the end of section 2 of whether the inverse forward prediction error filter, whose lattice implementation was shown in figure 4, is stable. We shall show first that the *direct form* implementation of this system is stable; this in turn will imply stability of the lattice structure. First the definition of exponential stability is given:

Definition 2 *The linear time-varying system* $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k)$ *is exponentially asymptotically stable (e.a.s.) if for any bounded initial condition* $||\mathbf{x}(k_0)|| < \infty$, *with arbitrary* k_0, *the resulting state vector sequence* $\mathbf{x}(\cdot)$ *obeys the exponential bound*

$$||\mathbf{x}(k)|| < cd^{k-k_0}||\mathbf{x}(k_0)|| \qquad \text{for all} \quad k \geq k_0,$$

where c is some fixed constant and $0 < d < 1$.

We shall use also the concept of uniform detectability [2], which we shall define next. First let us introduce the state transition matrix for the system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k)$ as

$$\Phi(k,l) = \begin{cases} \mathbf{I} & \text{if} \quad k = l, \\ \mathbf{A}(k-1)\mathbf{A}(k-2)\cdots\mathbf{A}(l) & \text{if} \quad k > l. \end{cases}$$

Definition 3 [2] *The pair* $[\mathbf{A}(k), \mathbf{c}(k)]$ *is* uniformly detectable *if there exist integers* $s,\ t \geq 0$ *and constants* $d,\ b$ *with* $0 \leq d < 1$, $0 < b < \infty$ *such that whenever*

$$||\Phi(k+t,k)\mathbf{w}|| \geq d||\mathbf{w}||$$

for some $\mathbf{w}$ *and* k, *then*

$$\mathbf{w}'\mathbf{M}(k+s,k)\mathbf{w} \geq b\mathbf{w}'\mathbf{w},$$

where $\mathbf{M}(k+s,k)$ *(the observability Gramian) is given by*

$$\mathbf{M}(k+s,k) \triangleq \sum_{i=k}^{k+s} \Phi(i,k)'\mathbf{c}(i)\mathbf{c}(i)'\Phi(i,k).$$

We are now in position to study the stability of the system of interest. Assume that the coefficients $\{a_i^m(k)\}$ have been obtained as the solution to the forward linear prediction problem for some random process $u(\cdot)$. The inverse forward prediction error filter can be implemented then in direct form as

$$y(k) = v(k) - \sum_{i=1}^{m} a_i^m(k)y(k-i), \tag{29}$$

where $v(\cdot)$ is the input and $y(\cdot)$ is the output. In the WSS case, (29) reduces to an LTI all-pole filter which is e.a.s. if and only if the WSS process $u(\cdot)$ was not fully predictable [7, 8].

Eq. (29) has the companion form state-space representation: $\{\mathbf{F}(k), \mathbf{G}, \mathbf{H}(k), I\}$:

$$\mathbf{F}(k) = \begin{bmatrix} -a_1^m(k) & -a_2^m(k) & \cdots & -a_{m-1}^m(k) & -a_m^m(k) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\mathbf{G} = [\ 1\ \ 0\ \ 0\ \ \cdots\ \ 0\ ]',$$

$$\mathbf{H}(k) = [\ -a_1^m(k)\ \ -a_2^m(k)\ \ \cdots\ \ -a_m^m(k)\ ]'.$$

This realization has the following property:

Lemma 4 *The matrix sequence* $\mathbf{F}(\cdot)$ *satisfies the following time-varying equation:*

$$\mathbf{R}_m(k) - \mathbf{F}(k)\mathbf{R}_m(k-1)\mathbf{F}(k)' = \mathbf{v}(k)\mathbf{v}(k)' \qquad \forall k, \tag{30}$$

with

$$\mathbf{v}(k) = \left[\ \sqrt{\mathcal{F}_m(k)},\ \ 0,\ \ \cdots,\ \ 0\ \right]'.$$

Eq. (30) is the key for establishing exponential stability of of the above realization. We can now give sufficient conditions for the direct form above to be e.a.s..

Theorem 2 *Suppose that the matrix sequence $\mathbf{R}_m(\cdot)$ is bounded above and that there exist an integer S and a constant $\epsilon > 0$ such that for all k, there exists n_k satisfying:*

1. $n_k \geq k$,
2. $n_k + m - 1 \leq k + S$,
3. $\mathcal{F}_m(n_k + i) \geq \epsilon$ *for* $i = 0, 1, \ldots m - 1$.

Then the system $\mathbf{x}(k+1) = \mathbf{F}(k)\mathbf{x}(k)$ *is exponentially asymptotically stable.*

The conditions of the theorem are met if on any time window of size S it is possible to find m consecutive time instants in which the forward prediction error variance is bounded away from zero. This is the case if for example $\mathcal{F}_m(k) \geq c$ for some constant $c > 0$ and for all k.

To see that the same condition implies the exponential stability of the recursive lattice implementation of figure 4, first observe that the state vector $\mathbf{x}(k)$ of the direct form realization consists simply of delayed outputs:

$$\mathbf{x}(k) = [\ y(k-1) \quad \cdots \quad y(k-m)\ ]'.$$

Under the conditions of theorem 2, the matrix sequence $\mathbf{R}_m(\cdot)$ is bounded; we showed above that this implies $\mathbf{R}_{m+1}(\cdot)$, $\mathbf{a}_m(\cdot)$,$\mathbf{c}_m(\cdot)$ bounded. Then in view of (5) and (6), $\mathcal{F}_i(\cdot)$, $\mathcal{B}_i(\cdot)$ are also bounded, $i = 1, \ldots, m$. Thus in view of (27) and (28), the reflection coefficients $\alpha_i(\cdot)$, $\beta_i(\cdot)$ remain bounded, $i = 1, \ldots, m$. Also, because the direct form is e.a.s., $v(\cdot)$ bounded yields $y(\cdot)$ bounded.

Now look at the recursive lattice structure in figure 5. Since we know that this system implements the same input-output relation as (29), if the is $v(k)$ then the output is $y(k)$. The state vector of this system is

$$\mathbf{z}(k) = [\ z_1(k) \quad \cdots \quad z_m(k)\ ]'.$$

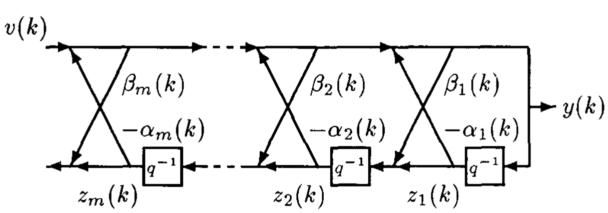

Figure 5: The recursive time-varying lattice structure.

From figure 5, it turns out that

$$z_i(k) = y(k-i) + LC\{y(k-i+1), \ldots, y(k-1)\}, \quad (31)$$

for $i = 1, \ldots, m$, where LC stands for 'linear combination'. The weights of these linear combinations are polynomic functions of the reflection coefficients and thus they are bounded. This shows that the state vector of the lattice structure remains bounded. Moreover, if $v(k) = 0$ for all k, $\mathbf{x}(k)$ goes to zero exponentially fast and in view of (31) so does $\mathbf{z}(k)$. Therefore the lattice structure is e.a.s.

5 Conclusions

An analysis of the properties of the prediction error filters obtained for nonstationary input processes has been developed. Many similarities with the stationary case were found, although the properties of the LTI filters obtained for WSS processes do not carry out directly to the nonstationary case. Nevertheless, an (asymmetric, time-varying) lattice structure for the filters still exists. Most important, under mild conditions on the input process, the time-varying system given by the inverse of the forward prediction error filter remains exponentially asymptotically stable.

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