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Output Feedback Control for Output Tracking of Nonlinear Uncertain Systems

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Abstract. Despite recent advances in robust control, robust tracking in linear and nonlinear uncertain systems remains an important and challenging task. Existing results for robust tracking usually rely on state feedback due to the difficulty of constructing robust observers for uncertain systems. Furthermore, state tracking has been emphasized much more than output tracking. In this paper, we tackle the problem of robust output tracking via dynamic output feedback for nonlinear systems which are subject to both uncertainty in system functions and input noise. The notion of ß-tracking is adopted which refers to the property of the system output to follow a reference trajectory within a ß neighborhood in finite time. With some matching conditions, minimum phase condition, Lipschitz bounds and some other mild assumptions, we show that *if ß-tracking of a nominal system can be achieved via a linear-cone bounded state feedback controller, then similar tracking can be achieved for the nonlinear uncertain system via dynamic output feedback.* The resulting controller is linear-cone bounded, or can be purely linear with the tradeoff of high gain if desired. Our approach utilizes and generalizes existing results on robust tracking and stabilization via state feedback and robust observer design via loop transfer recovery theory.

1 Introduction

Recent developments in robust control have brought about significant results covering the issue of stablization of uncertain systems. However, research on robust tracking in linear and nonlinear uncertain systems remains an important and challenging area. The problem of robust tracking involves designing either a state or dynamic output feedback control law so that the output of an uncertain system follows a given reference trajectory in certain sense.

There have been many attempts at state feedback control for state tracking. A stream of results pertinent to this paper are given by Corless, Leitmann, Ryan and Coodall, see [2, 3, 4] and references thereof. These results deal with a class of nonlinear systems satisfying the so-called "matching conditions" concerning the nonlinearity and uncertainty in the system. They show that under a certain feasibility condition on the reference model, the so-called ß-tracking can be achieved for the state of the system for any neighborhood ß of the origin, i.e., the state of the system will follow the state of the reference model within ß after a finite period of time. Their method is based on earlier results on robust stabilization of uncertain systems with matching conditions initiated by Leitmann [13]. The matching conditions are also used by Schmitendorf and Barmish [16] to achieve robust asymptotic tracking of step functions via state feedback control.

The aim of tracking for uncertain systems via output feedback has previously been hindered by the difficulties in observer design, which forbade the recovery of full state feedback properties and robustness. However, recent progress has begun to approach this solution. Hollot and Galimidi [12] consider an uncertain system which is quadratically stabilizable by using a state feedback law and show that an observer-based stablizer can be constructed for a *nearby* uncertain system provided that the transfer function from the uncertainty to the output is of minimum phase. The design of a robust observer for stabilization has been achieved by Petersen and Hollot [14], Esfandiari and Khalil [7], and Saberi and Sannuti [15]. In [14], it is shown by using the interconnection between H_∞ disturbance attenuation and quadratic stabilization that an observer-based stabilizer can be constructed when a state feedback controller exists and certain minimum phase condition holds. Similar results are also reported in [7] and [15], the former uses a two-time scale structure and the latter utilizes the so-called "asymptotic time-scale and eigenstructure assignment" method for loop transfer recovery. The result of [15] is slightly more general than those in [7, 14] in the sense that nonlinear state feedback with a linear bound is allowed in [15].

The motivation of our paper stems from the following weaknesses of the exisiting results. First, a state reference is often used rather than an output reference. Because reconstruction of state dynamics for a given output trajectory may be a difficult task and that state dynamics may not be unique, especially when the output is contaminated by noises, the usage of state reference is limited. Secondly, many results on robust tracking rely on nonlinear state feedback. Although nonlinear control is sometimes unavoidable, especially for nonlinear systems, and may also have the advantage of requiring lower gains if carefully constructed, linear control is often preferred in the practice due to its simplicity and the richness of the linear control theory. Thirdly and perhaps most seriously, the state feedback control is usually required in spite of the fact that this is often inapplicable due to the lack of full state measurement. Although results such as in [7, 14, 15] are available for constructing observers for robust stabilization, no counterpart of these results exists for robust tracking.

In order to overcome these weaknesses, we propose to solve in this paper, the problem of *ß-output tracking for nonlinear uncertain systems via dynamic output feedback control.* That is, given a nonlinear uncertain system, an output reference trajectory and a neighborhood of the origin ß, we need to design a dynamic output feedback so that the output of the uncertain system tracks the reference trajectory within ß in a finite period of time (see Section 2 for precise definition of ß-output tracking). More specifically, we consider nonlinear systems with bounded input noise and uncertainty in system functions which satisfy some matching conditions, a minimum phase condition, some Lipschitz bounds for the uncertainty and nonlinearity and some other mild assumptions. *We show that if ß-output tracking for a nominal system can be achieved by using a state feedback controller for a given bounded output reference trajectory and a neighborhood ß of the origin, then $ß^\epsilon$-output tracking for the uncertain system can be guaranteed by using an dynamic output feedback controller for any $ß^\epsilon$ "larger" than ß.* Moreover, the resulting controller is linear-cone bounded, or can be purely linear with the tradeoff of high gain if desired. Our approach is based on the existing results on robust tracking and stabilization via state feedback [2, 3, 4] and robust observer design via loop transfer recovery theory [7, 14, 15]. We take the liberty of using the matching conditions, minimum phase and Lipschitz conditions in view of the fact that these properties are common to a large class of physical systems.

The design of our controller is carried out in three steps. The first step involves the design of the state feedback law for the nominal model to achieve ß-output tracking of the given reference trajectory. This can be done by any method of the designer's choice (e.g., see [1, 6, 18]), except for the constraint that the controller needs to be linear-cone bounded. The second step is to apply the matching conditions to design a state feedback law for the uncertain system to track the nominal model. The state feedback controller is modified from the nominal controller by including an additional linear feedback term to attenuate the nonlinearity, uncertainty and input noise. The final step is to construct an observer for the uncertain system by utilizing the minimum phase condition so that the observer-based design will "recover" the state feedback law. As a result, the observer-based closed-loop system will ß $^\epsilon$-track the given reference for some ß $^\epsilon$ slightly "larger" but arbitrarily close to ß. The rest of the paper is organized as follows. Section 2 formulates the ß-output tracking problem. Section 3 provides the main results of the paper. This section consists of state feedback design, observer design and construction of the output feedback controller. The results on state feedback design and observer design are independent so that they can be applied separately. Some conclusions are reached in Section 4.

2 Problem Formulation

We consider nonlinear systems with structured uncertainty of the following general form, **S** :

$$\begin{aligned} \dot{x}(t) &= f(t,x) + \Delta f(t,x) + [B + \Delta B(t,x,u)](u+w) &(1)\\ y(t) &= Cx &(2)\\ y_m(t) &= C_m x &(3) \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^r$ is the controlled output, $y_m(t)$ is the measured output, $w(t) \in \mathbf{R}^m$ is the input noise, B,C and C_m are constant matrices of appropriate dimensions, the mapping $f(\cdot) : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ is a known function, the mappings $\Delta f(\cdot) : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ and $\Delta B(\cdot) : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^{n\times m}$ are unknown functions representing the system uncertainties.

Suppose we want system (2) to track a given reference trajectory $y_{ref} : \mathbf{R}^+ \to \mathbf{R}^r$ in certain sense. Then we choose a dynamic feedback control law of the form

$$\begin{aligned} \dot{\xi} &= \Theta(t, \xi, y_m, y_{ref}) &(4)\\ u &= \Phi(t) = \Xi(t, \xi, y_m, y_{ref}) &(5) \end{aligned}$$

where $\Theta : \mathbf{R}^+ \times \mathbf{R}^l \times \mathbf{R}^r \times \mathbf{R}^r \to \mathbf{R}^l$ and $\Xi : \mathbf{R}^+ \times \mathbf{R}^l \times \mathbf{R}^r \times \mathbf{R}^r \to \mathbf{R}^m$. The resulting closed loop system can be written as, **T** :

$$\begin{aligned} \dot{x}(t) &= f(t,x) + \Delta f(t,x) + [B + \Delta B(t,x,\Phi)](\Phi + w) &(6)\\ y(t) &= Cx &(7) \end{aligned}$$

We define the tracking error as the difference between the closed-loop system output (7) and the reference trajectory:

$$y_e(t) = y(t) - y_{ref}(t) \qquad (8)$$

In the sequel, we denote by a closed "ball" $\mathbf{B}_\epsilon$ as being the closed neighbourhood of radius $\epsilon > 0$ centred at the origin. The pointwise sum of two sets **A** and **B** will be abbreviated by $\mathbf{A} + \mathbf{B}$, i.e.,

$$\mathbf{A} + \mathbf{B} = \{c : c = a + b, a \in \mathbf{A}, b \in \mathbf{B}\} \ . \qquad (9)$$

We use the standard 2-norm for vectors in $\mathbf{R}^n$ and the associated induced norm for matrices. The set of nongenative real numbers is denoted by $\mathbf{R}^+$.

We adopt the term ß-tracking from [2, 3] to describe the behaviour of the system output (7) as being *convergent* to a given neighborhood ß of y_{ref}. The following definition is slightly different from [2, 3] to cope with the output tracking, but the basic principle remains.

Definition 2.1 *(ß-tracking)*
Given an output reference trajectory $y_{ref}(\cdot) : \mathbf{R}^+ \to \mathbf{R}^r$ and a neighborhood ß of the origin, the output of the system T is said to ß-track the reference trajectory y_{ref} if the following conditions are satisfied:

1. Existence and Continuation of Solutions*: For all initial conditions $(t_0, x_0) \in \mathbf{R}^+ \times \mathbf{R}^n$, there exists a solution $x : [t_0, t_1) \to \mathbf{R}^n$ of system T (a function satisfying (6) which is absolutely continuous almost everywhere) and every solution can be extended to a solution defined on $[0, \infty)$.*

2. Uniform Bounded-Input-Bounded-Output Stability*: For any bounded $y_{ref}(\cdot)$ and initial condition $x(t_0) = x_0$, there exists $d > 0$ such that if $x : [t_0, \infty) \to \mathbf{R}^n$ is any solution to (6) with $y : [t_0, \infty) \to \mathbf{R}^m$ being the associated output trajectory, then $\|y(t)\| \le d$ for all $t \in [t_0, \infty)$.*

3. Uniform Ultimate Boundedness of the Tracking Error within ß*: For any initial condition $x(t_0) = x_0$, there exists a $T(ß, x_0) > 0$ such that if $y : [t_0, \infty) \to \mathbf{R}^r$ is any solution to (6), then $(y(t) - y_{ref}(t)) \in ß \ \forall \ t \ge t_0 + T$.*

To achieve robust tracking for the uncertain system, we make the following assumptions throughout the paper.

Assumptions:

A1 *Matching Conditions:* There exists a known matrix $A \in \mathbf{R}^{n\times n}$, a known scalar $0 < \beta < 1$, a known Carathéodory function[1] $g : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^m$ and unknown Carathéodory functions $h : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^m$ and $E : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^{m\times m}$ such that the following hold for all $(t, x, u) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^m$:

$$\begin{aligned} f(t,x) &= Ax + Bg(t,x) &(10)\\ \Delta f(t,x) &= Bh(t,x) &(11)\\ \Delta B(t,x,u) &= BE(t,x,u) &(12)\\ \|E(t,x,u)\| &\le \beta. &(13) \end{aligned}$$

A2 *Left Invertibility and Minimum Phase Condition:* The system defined by (C_m, A, B) is left invertible and of minimum phase.

A3 *Boundedness of System Functions:* There exist known positive scalars $\bar{g}$ and $\bar{h}$ such that

$$\begin{aligned} \|g(t,x)\| &\le \bar{g}\|x\| &(14)\\ \|h(t,x)\| &\le \bar{h}\|x\| \quad \forall \ (t,x) \in \mathbf{R}^+ \times \mathbf{R}^n. &(15) \end{aligned}$$

[1]A function $v : \mathbf{R} \times \mathbf{R}^p \to \mathbf{R}^q$ is called *Carathéodory* if: i) $v(\cdot, z)$ is Lebesgue measurable for each $z \in R^p$; ii) $v(t, \cdot)$ is continuous for each $t \in \mathbf{R}$; iii) for each compact set $U \subset \mathbf{R} \times \mathbf{R}^p$, there exists a Lebesgue integrable function $m_U(\cdot)$ such that $\|v(t,z)\| \le m_U(t)$ for all $(t,z) \in U$. This type of functions are needed primarily for assuring the existence and continuation of the solution to a differential equation; see [2] and references thereof.

A4 *Boundedness of Input Noise:* There exists a known scalar $\bar{w} \in \mathbf{R}^+$ such that

$$||w(t)|| \leq \bar{w} \quad \forall\, t \in \mathbf{R}^+. \tag{16}$$

A5 *Boundedness of Reference Signal:* There exists a known scalar $\bar{y}_{\text{ref}} \in \mathbf{R}^+$ such that

$$||y_{\text{ref}}(t)|| \leq \bar{y}_{\text{ref}} \quad \forall\, t \in \mathbf{R}^+. \tag{17}$$

We define the nominal system corresponding to (1) and (2) as that which is devoid of all uncertainty and input noise:

$$\begin{aligned} \dot{x}_0(t) &= f(t, x_0(t)) + Bu_0(t) & (18)\\ y_0(t) &= Cx_0(t)\ . & (19) \end{aligned}$$

The controller construction involves three steps:

(i) Design of state feedback to make the output of the nominal system ß-track the reference trajectory,

(ii) Design of state feedback to make the uncertain system follow the nominal system,

(iii) Design of an observer to make available a reconstructed state for feedback in (ii).

The tracking of the nominal system has been well studied; see [1, 6, 18]. It is not our intention to study this problem. We therefore assume that a controller satisfying (i) is already available. More precisely, we assume the following:

A6 Suppose there exists a Carathéodory function $\Phi_0 : \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}^m$ satisfying $||\Phi_0(t,x)|| \leq \bar{\Phi}_0 ||x||$ for some $\bar{\Phi}_0 \in \mathbf{R}^+$ such that the state feedback control law

$$u_0(t) = \Phi_0(t, x_0) \tag{20}$$

guarantees that the output of the nominal system will ß-track the reference trajectory y_{ref} for some given ß.

3 Control Design

Under Assumption 6, the design problem in Step ii) now becomes one of finding a state feedback law Φ to make the the uncertain system (1)-(2) track the nominal system arbitrarily closely. Furthermore, the design problem in Step iii) is to find a robust observer to reconstruct the state feedback law Φ. In this context, we consider separately the issues of the state feedback and the observer design before discussing the feedback of the resconstructed state.

A. State Feedback

We determine here a nonlinear control law which is linear-cone bounded. More specifically, the control law Φ will contain two parts: linear and nonlinear. The nonlinear part is used to emulate the nominal control and to cancel the known part of the system nonlinearity and the linear part is for attenuating the uncertainty and input noise and for stabilizing the system. If desired, a purely linear control can be developed with the tradeoff of higher feedback gain.

Observe that the nominal system (18)-(19) must be stabilizable due to the fact that the nominal system with the feedback law (20) can ß-track y_{ref}. This implies that the pair (A, B) must be stabilizable. Therefore, a feedback matrix K_0 can be selected so as to ensure that $A + BK_0$ is asymptotically stable. Moreover, for any positive definite symetric matrix $Q_1 = Q_1^T > 0$, there exists another such matrix $P_1 = P_1^T > 0$ (unique) such that

$$(A + BK_0)^T P_1 + P_1(A + BK_0) + Q_1 = 0\ . \tag{21}$$

Based on the selection of K_0 and P_1, the feedback control law Φ is chosen to be of the following form:

$$\begin{aligned} \dot{x}_a &= f(t, x_a) + B\Phi_0(t, x_a) & (22)\\ u(t) &= \Phi(t, x(t)) \\ &= \Phi_0(t, x_a) + \Psi(t, x(t)) - \gamma_\epsilon B^T P_1[x(t) - x_a(t)] & (23) \end{aligned}$$

where

$$\Psi(t, x) = K_0(x - x_a) - g(t, x) + g(t, x_a)\ , \tag{24}$$

and the scalar γ_ϵ in (23) is a design constant to be determined. The auxilary dynamics (22) is used to emulate the nominal system. The functions in (23) are similar as in [2] for ß-tracking of state reference except that the third term in (23) is linear rather than nonlinear as in [2]. This term, which is adopted from Thorp and Barmish [17], is used to faciliate the design of observer.

Theorem 3.1 *Consider the uncertain system (1) and an output reference trajectory $y_{\text{ref}} : \mathbf{R}^+ \rightarrow \mathbf{R}^r$ satisfying Assumptions A1,A3-A5. Given a neighborhood ß of the origin, suppose there exists a state control law Φ_0 such that Assumption A6 holds. Then, For each $\epsilon > 0$ and the corresponding neighborhood of the origin*

$$ß^{\,\epsilon} = ß + \mathbf{B}_\epsilon\ , \tag{25}$$

there exists a $\gamma_\epsilon > 0$ (sufficiently large) such that the state feedback control law (22)-(24) guarantees the output of the uncertain system (1)-(2) to ß$^{\epsilon}$ - track y_{ref} with the Lyapunov function $V_1(x - x_a) = (x - x_a)^T P_1 (x - x_a)$ satisfying

$$\frac{d}{dt} V_1(x(t) - x_a(t)) \leq \lambda[||x(t) - x_a(t)||^2 - \frac{\epsilon^2}{||C||^2}] \quad \forall\, t \in \mathbf{R}^+ \tag{26}$$

for some $\lambda > 0$. Moreover, there exists $M_\epsilon > 0$ such that

$$||\Phi(t,x)|| \leq M_\epsilon ||x||, \quad \forall\, x \in \mathbf{R}^n,\ t \in \mathbf{R}^+. \tag{27}$$

Proof: See [22].

Remark 3.1: The state feedback control law used above, although linear-cone bounded, is nonlinear. If desired, a purely linear control law can be developed to achieve the same robust tracking result. This can simply be done by extracting the linear portions of the nominal controller Φ_0 and $g(\cdot)$ and grouping them together with the "deterministic" part of the system and leaving the rest in the uncertain part. The boundedness on the nominal controller and $g(\cdot)$ means that the regrouping will not affect the boundedness of the uncertain part. The resulting controller, as derived from Theorem 3.1, will obviously be linear. Although a linear controller is often advantageous, one must be cautious about the tradeoff of high gain.

B. Observer Design

We now turn to the problem of observer design for robust tracking of nonlinear uncertain systems. We aim at developing a general result (Theorem 3.2) rather than restricting ourselves to the class of systems in (1)-(2). In other words, this result is applicable to systems which do not satisfy matching conditions and some other restrictions. For this purpose, we consider the following uncertain system:

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}v(t) \\ &\quad + \mathcal{D}[U(t, z(t), v(t), \omega(t)) + N(t, z(t), v(t))] \\ \eta(t) &= \mathcal{C}z(t) & (28) \end{aligned}$$

where $z(t) \in \mathbf{R}^n$ is the state, $v(t) \in \mathbf{R}^m$ is the control, $\eta(t) \in \mathbf{R}^r$ is the output, $\omega(t) \in \mathbf{R}^k$ is an unknown input noise, $U : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k \to \mathbf{R}^k$ and $N : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^k$ are unknown and known Carathéodory functions, respectively, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are known constant matrices of appropriate dimensions. The following assumptions are imposed on the system:

A3.1 *Left Invertibility and Minimum Phase*: The system defined by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is left invertible and of minimum phase;

A3.2 *Boundedness of System Functions*: There exists positive constants $k_i, i = 1, \ldots, 6$ such that

$$
\begin{aligned}
||U(t,z,v,\omega)|| \leq k_1||z|| + k_2||v|| &+ k_3||\omega|| &(29)\\
||U(t,z,v_1,\omega) - U(t,z,v_2,\omega)|| &\leq k_4||v_1 - v_2|| &(30)\\
||N(t,z_1,v) - N(t,z_2,v)|| &\leq k_5||z_1 - z_2|| &(31)\\
||N(t,z,v_1) - N(t,z,v_2)|| &\leq k_6||v_1 - v_2|| &(32)
\end{aligned}
$$

for all $t \in \mathbf{R}^+, z, z_1, z_2 \in \mathbf{R}^n, v, v_1, v_2 \in \mathbf{R}^m, \omega \in \mathbf{R}^k$.

A3.3 *Boundedness of Input noise*: There exists a positive constant $\bar{\omega}$ such that

$$||\omega(t)|| \leq \bar{\omega} \tag{33}$$

for all $t \in \mathbf{R}^+$.

Let a given continuous state trajectory $z_{\text{ref}} : \mathbf{R}^+ \to \mathbf{R}^n$ satisfy

A3.4 *Boundedness of Reference*: There exists a positive constant $\bar{z}_{\text{ref}}$ such that

$$||z_{\text{ref}}(t)|| \leq \bar{z}_{\text{ref}} \tag{34}$$

for all $t \in \mathbf{R}^+$.

Suppose for a given $\epsilon > 0$, there exists a continuous control law

$$v(t) = \mathcal{F}(\zeta) \tag{35}$$

where

$$\zeta = z - z_{\text{ref}} \tag{36}$$

such that the state of the closed-loop system

$$\dot{z} = \mathcal{A}z + \mathcal{B}\mathcal{F}(\zeta) + \mathcal{D}[U(t,z,\mathcal{F}(\zeta),\omega) + N(t,z,\mathcal{F}(\zeta))] \tag{37}$$

$\mathbf{B}_\epsilon$-tracks z_{ref} with a Lyapunov function $W(\zeta)$ satisfying the following assumption.

A3.5 *Boundedness of Lyapunov Function*: There exists positive constants $k_i, i = 7, \cdots, 10$ such that

$$
\begin{aligned}
k_7||\zeta||^2 &\leq W(\zeta) \leq k_8||\zeta||^2 \;\; \forall\, \zeta \in \mathbf{R}^n &(38)\\
||\frac{dW}{d\zeta}|| &\leq k_9||\zeta|| \;\; \forall \zeta \in \mathbf{R}^n &(39)
\end{aligned}
$$

$$
\begin{aligned}
W_d &= \frac{d}{d\zeta}W(\zeta)\{\mathcal{A}z + \mathcal{B}\mathcal{F}(\zeta) + \mathcal{D}[U(t,z,\mathcal{F}(\zeta),\omega) \\
&\quad + N(t,z,\mathcal{F}(\zeta))] - \dot{z}_{\text{ref}}\} \\
&< -k_{10}(||\zeta||^2 - \epsilon^2) \;\; \forall\, t \in \mathbf{R}^+, z \in \mathbf{R}^n &(40)
\end{aligned}
$$

and for all admissible ω.

A3.6 *Boundedness of Control Law*: There exists a positive constant k_{11} such that

$$||\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)|| \leq k_{11}||\zeta_1 - \zeta_2|| \tag{41}$$

for all $\zeta_1, \zeta_2 \in \mathbf{R}^n$.

Under Assumptions A3.1-A3.6, we take the observer-based feedback control of the following form:

$$
\begin{aligned}
\dot{\hat{z}} &= \mathcal{A}\hat{z} + \mathcal{B}\hat{z} + \mathcal{D}N(t,\hat{z},v) + \mathcal{L}(\eta - \mathcal{C}\hat{z}) &(42)\\
v &= \mathcal{F}(\hat{z} - z_{\text{ref}}) \;. &(43)
\end{aligned}
$$

Theorem 3.2 *Suppose Assumptions A3.1-A3.6 hold for the system (28), some given reference z_{ref} and a scalar $\epsilon > 0$. Then, for any $\epsilon_1 > \epsilon$, there exists an observer gain matrix $\mathcal{L}$ such that with the observer-based control (42)-(43) the closed-loop system $\mathbf{B}_{\epsilon_1}$-tracks z_{ref}, which is equivalent to selecting the matrix $\mathcal{L}$ such that $||(sI - \mathcal{A} + \mathcal{L}\mathcal{C})^{-1}\mathcal{D}||_\infty$ is sufficiently small.*

Proof: See [22].

Remark: The result above shows that we simply need to ensure that $||(sI - \mathcal{A} + \mathcal{L}\mathcal{C})^{-1}\mathcal{D}||_\infty$ is sufficiently small. This is actually a standard loop transfer recovery problem when $\mathcal{D} = \mathcal{B}$. See, for example, [14, 7, 15, 8] for various design techniques.

C. Dynamic Output Feedback Control

With the results on state feedback design and observer design, we are to develop a dynamic output feedback controller for robust output tracking of the system (1)-(2). The observer-based control design for robust tracking will of the following form:

$$
\begin{aligned}
\dot{x}_a &= f(t,x_a) + B\Phi_0(t,x_a) &(44)\\
\dot{\hat{x}} &= f(t,\hat{x}) + Bu + L(y_m - C_m\hat{x}) &(45)\\
u(t) &= \Phi(t,\hat{x}(t)) &(46)\\
&= \Phi_0(t,x_a) + \Psi(t,\hat{x}(t)) - \gamma_\epsilon B^T P_1(\hat{x}(t) - x_a(t))
\end{aligned}
$$

where

$$\Psi(t,\hat{x}) = K_0(\hat{x} - x_a) - g(t,\hat{x}) + g(t,x_a) \;. \tag{47}$$

Theorem 3.3 (Main Result) *Consider the uncertain system (1)-(2) and an output reference trajectory $y_{\text{ref}} : \mathbf{R}^+ \to \mathbf{R}^r$ satisfying Assumptions A1-A5. Given a neighborhood β of the origin, suppose there exists a state control law $\Phi_0 : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^m$ such that Assumption A6 holds. Then, for each $\epsilon > 0$ (arbitrarily small) and the corresponding neighborhood of the origin*

$$\beta^{\,\epsilon} = \beta + \mathbf{B}_\epsilon \;, \tag{48}$$

there exists a $\gamma_\epsilon > 0$ (sufficiently large) and an observer gain matrix L (to guarantee that $||(sI - A + LC_m)^{-1}B||_\infty$ is sufficiently small) such that the control law (44)-(47) guarantees the uncertain system (1)-(2) to $\beta^{\,\epsilon}$-track y_{ref}. Moreover, there exists $M_\epsilon \in \mathbf{R}^+$ such that

$$||\Phi(t,\hat{x}_1(t))|| \leq M_\epsilon||\hat{x}_1(t)||, \quad \forall\, \hat{x}_1(t) \in \mathbf{R}^n,\; t \in \mathbf{R}^+. \tag{49}$$

Proof: See [22].

Remark: An obvious implication of Theorem 3.3 is that if there exists a control law (20) such that the nominal system *asymptotically* tracks y_{ref}, then there also exists an observer-based design such that the output of the uncertain system (1) will $\mathbf{B}_\epsilon$-track y_{ref} for any $\epsilon > 0$.

Remark: It is clear from Theorem 3.3 that the design of the dynamic output feedback controller simply involves tuning the scalar parameter γ_ϵ and the observer gain matrix. The former can be calculated by using the given data, or by computer simulation of the closed-loop system. The design of the observer, as mentioned earlier, can be done by using the loop transfer recovery methods.

4 Illustrative Example

We illustrate the above results with a practical example. The system treated here is a second order input-output representation of the longitudinal subsystem of the nonlinear dynamics of a fixed wing aircraft.

We wish to make the aircraft perform a coordinated turn, and assume that the lateral dynamics are suitably controlled such that the desired bank angle trajectory ϕ_c is achieved. Our objective then is to control the longitudinal dynamics of the aircraft described by the angle of attack $\alpha = \alpha_0 + \alpha_1$, where α_0 is the trim angle of attack (initial condition). Consider α_1 as the controlled variable. To illustrate the robustness of the proposed control design to the nonlinearities of the system, the simulation is of an aircraft performing an 80^o coordinated turn. The desired angle of attack trajectory is described by

$$\alpha_c = \alpha_0 / \cos\phi_c - \alpha_0. \tag{50}$$

where the reference bank angle is generated as follows:

$$\frac{d^3\phi_c}{dt^3} = -3\omega_n\frac{d^2\phi_c}{dt^2} - 3\omega_n^2\frac{d\phi_c}{dt} - \omega_n^3\phi_c + \omega_n^3\phi_{max} \tag{51}$$

and where $\omega_n = 3$ radians per second, $\phi_{max} = 80$ degrees, and ϕ_c and its derivatives are initially zero. The resulting reference trajectory α_c is shown in Figure 1.

The longitudinal dynamics can be reduced to the following input-output form

$$\ddot{\alpha_1} + a_3(p,\alpha_1,\phi)\dot{\alpha_1} + a_2(p)\alpha_1 + a_1(p,\alpha_1,\phi) = b_1(p)\delta_{e_1} \tag{52}$$

where $\delta_e = \delta_{e_0} + \delta_{e_1}$ is the elevator deflection, δ_{e_0} is the initial condition and δ_{e_1} the input, $p = [p_1, p_2, p_3, p_4, p_5, p_6] \subset \mathbf{P} \subset \mathbf{R}^6$ is the vector of uncertain parameters which is comprised of aerodynamic derivatives, $p_0 \in \mathbf{P}$ is the vector of nominal parameters and the coefficients of the equation are adapted from the equations of motion in wind axes developed in [21]:

$$\begin{aligned} a_1 &= -.05p_1(\cos(\alpha_1+\alpha_0) - \cos\alpha_0)\cos\phi_c && (53)\\ a_2 &= -113.5p_2 + .75p_3 && (54)\\ a_3 &= -\{\dot{\phi}_c + .04\cos\phi_c\}\sin(\alpha_1+\alpha_0) - .63p_4 - 1.19p_1 && (55)\\ b_1 &= 113.5p_5 - .75p_6 && (56)\end{aligned}$$

The parameter set $\mathbf{P}$ is defined as

$$\begin{aligned} &p_1 \in [\ -3.063, -2.552\ ], \quad p_2 \in [\ -0.435, -0.321\ ], && (57)\\ &p_3 \in [\ 6.021, 8.665\ ], \quad p_4 \in [\ -2.848, -2.378\ ], && (58)\\ &p_5 \in [\ -0.488, -0.360\ ], \quad p_6 \in [\ 1.775, 2.895\ ]. && (59)\end{aligned}$$

It can be seen that in addition to the relatively weak nonlinear effects due to the trigonometric terms in α_1 (usually $-10^o < \alpha_1 < 15^o$), a large bank angle ($60^o \le \phi < 90^o$) induces strong nonlinearity in the system equation. The system (52) can be represented in state space in the following manner:

$$\begin{aligned} \dot{x} &= f(p,x,\phi_c) + B(p)\delta_{e_1} && (60)\\ &= \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_{a_1} \\ x_{a_2} \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \end{bmatrix}\delta_{e_1} \\ y(t) &= Cx = \alpha_1(t). && (61)\end{aligned}$$

In this instance, we assume that the measured output and the controlled output are identical. We choose $y_m(t) = y(t)$.

Control Design: Defining the following auxiliary system:

$$\begin{aligned} \dot{x}_a &= f(p_0,x_a,\phi_c) + B(p_0)\delta_{e_{1_0}} && (62)\\ &= \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_{a_1} \\ x_{a_2} \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \end{bmatrix}\delta_{e_{1_0}} \end{aligned}$$

it has been found that the following control law guarantees good transient performance and zero steady state tracking error when applied to the nominal system.

$$\begin{aligned} \dot{\xi} &= x_1 - \alpha_c && (63)\\ \delta_{e_{1_0}} &= k_1x_{a_1} + k_2x_{a_2} + k_3\xi \triangleq \Phi_0(t,x_a) && (64)\end{aligned}$$

where $[k_1, k_2] = K_0 = [3.0, 0.3], k_3 = 15.0$. This controller places the closed loop poles well into the left half plane giving zero steady state tracking error with good transient performance. Figure 1 shows the response of the closed loop system under state feedback.

State feedback: According to the development of the control laws (21)-(24), we can formulate the following state feedback based on the nominal control law.

$$\delta_e = \Phi(t,x) = \Phi_0(t,x_a) + \Psi(t,x) - \gamma B^T P_1(x - x_a) \tag{65}$$

where

$$\Psi(t,x) = K_0(x - x_a) - g(t,x) + g(t,x_a) \tag{66}$$

and P_1 has been chosen to satisfy

$$(A + BK_0)^T P_1 + P_1(A + BK_0) = -Q \tag{67}$$

where A and B are constant matrices as defined in (1),(10) and derived from the linearisation of (62). Q is chosen such that the closed loop poles of the linearised system are sufficiently left of the origin to guarantee good transient performance and to avoid numerical problems. We have

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 5.6857 & .0255 \\ .0255 & .0265 \end{bmatrix}. \tag{68}$$

The choice $\gamma = 1$ is sufficient to guarantee that the feedback law induces stability and zero steady-state error for all $p \in \mathbf{P}$.

Output Feedback: The observer-based control design implements the state feedback control law (65) - (67), based on the observer state rather than the true system state vector.

$$\begin{aligned} \dot{x}_a &= f(t,x_a) + B\Phi_0(t,x_a) && (69)\\ \dot{\hat{x}} &= f(t,\hat{x}) + B\delta_e + L(y_m - C_m\hat{x}) && (70)\\ \delta_e &= \Phi_0(t,x_a) + \Psi(t,\hat{x}) - \gamma B^T P_1(\hat{x} - x_a) && (71)\\ \Psi(t,\hat{x}) &= K_0(\hat{x} - x_a) - g(t,\hat{x}) + g(t,x_a) && (72)\end{aligned}$$

where $L = [l_1, l_2]^T$ is chosen to minimise the norm

$$||(sI - A + LC_m)^{-1}B||_\infty. \tag{73}$$

We find that $L = [1, 2000]^T$ maintains $||\cdot||_\infty < 2.843$, with which the output response is very close to that achieved by state feedback, see Figure 1. In this case, nonzerosteady state error is observed. However, the error will decrease as the H_∞-norm of $(sI - A + LC_m)^{-1}B$ is reduced.

5 Conclusions

A new method has been developed for robust output tracking of nonlinear uncertain systems. For the nonlinear uncertain system (1) satisfying Assumptions A1-A5, we have shown that if a nominal system can achieve ß-tracking of a given output reference trajectory via linear-cone bounded state feedback control, then the output of the system (1) can $ß^\epsilon$-track the reference trajectory via dynamic output feedback for any $ß^\epsilon$ "larger" than ß in the sense of (48). The controller is of a simple form and its construction consists of state feedback design and observer design, the former involves tuning of a single parameter (γ_ϵ) and the latter, of the H_∞-norm of an observer function $(sI - A + LC_m)^{-1}B$ which can be done, for example, by using loop transfer recovery techniques. Furthermore, the controller is linear-cone bounded while a purely linear controller can also be developed with a higher feedback gain. In view of the fact that a large class of nonlinear uncertain systems admit matching conditions, minimum phase and Lipschitz conditions, our results should promise good application in situations such as control of aircraft manoeuvres.

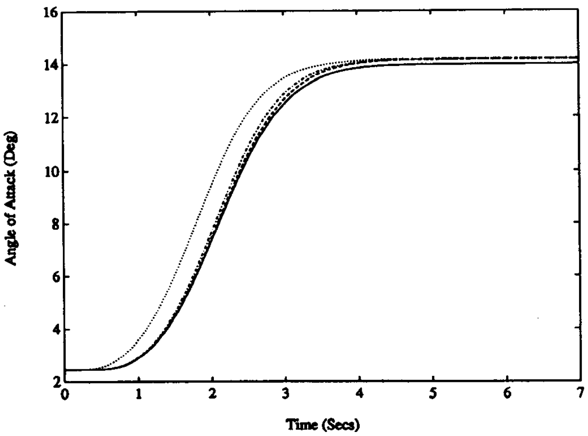

Figure 1. *Time history of Angle of Attack during a coordinated turn manoeuvre with an 80° Bank Angle.*
Dotted line: Reference trajectory.
Dot-Dash line: Nominal system response with state feedback.
Dashed line: True system response with state feedback.
Solid line: True system response with output feedback.

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