Quantized Stabilization of Markov Jump Linear Systems via State Feedback

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Abstract—Motivated by quantized feedback control over an unreliable packet-based network, this paper addresses the quantized stabilization problem for single-input Markov jump systems. Mode-dependent and mode-independent quadratic control Lyapunov functions based on the availability of mode information at controller/quantizer are considered for the quantized feedback. Similar to the linear time-invariant case, it is shown that a mode-dependent (respectively, mode-independent) logarithmic quantizer is optimal (coarsest) in the mean square quadratic stability (respectively, strongly mean square quadratic stability) sense for Markov jump systems. Moreover, the sector bound approach is shown to be nonconservative in investigating the corresponding quantized state feedback problem. Under an appropriate definition of quantization coarseness, we also present a method of optimal quantizer design in terms of linear matrix inequalities. Several examples including applications in networked control systems are given to demonstrate the results.

I. INTRODUCTION

Starting from Kalman [1], quantization has been known to have an undesirable effect on system performance or even stability and thereby many works have been done in mitigating the effect. In modern networked systems, quantization is an indispensable step which aims at saving limited bandwidth and energy consumption. Elia and Mitter [2] first pointed out that quantization is “useful, if not essential, instead of undesirable”, and also indicated that the coarsest quantizer is logarithmic in the sense of quadratic stability for single-input linear time-invariant (LTI) systems. A relationship between the optimal quantization density and unstable eigenvalues of the plant under consideration has been established. Fu and Xie [3] showed that under quadratic stability, quantized stabilization is equivalent to the robust stabilization of an associated system with sector-bound uncertainty and extended the results to multiple-input-multiple-output (MIMO) systems and output feedback. Based on the result in [3], quantized stabilization is considered in [4] where a quantization error dependent Lyapunov function is adopted which offers a less conservative design. Other relevant papers on quantization include [5], [6], [7].

The packet-drop behavior of an unreliable communication channel is another important issue in networked control systems (NCSs) as it induces information loss and consequently affects the performance or even stability of the closed-loop system. There have been many interesting studies on the packet-loss issue; see, e.g., [8], [9], [10], [11] for networked control, [12], [13], [14] for networked estimation, and [15] for a review of recent results. In [8], the stability robustness of NCSs is addressed, where the packet losses are modeled according to an i.i.d. Bernoulli distribution and the control input becomes zero when the data are lost (so-called zero-control strategy). The Markov jump systems (MJSs) theory is applied to the $H_\infty$ control of NCSs with binary stochastic packet losses in [9]. In [10], bounded consecutive switching or Markovian packet losses are assumed. [11] considered the mean square stabilization over fading channel in the framework of robust control for deterministic systems with stochastic structured model uncertainties. One of the interesting discoveries in [11] is that the supremum of allowable packet-loss rate (probability of erasure) can be given in terms of the unstable poles of the single-input plant under investigation.

As quantization and packet losses co-exist in a NCS, it is natural and reasonable to take them into consideration simultaneously. The stabilization problem over channel containing both quantization and packet losses was first addressed in [16], where the packet-loss process is driven by a binary i.i.d. process. It was shown in [16] that the upper bound of the coarseness can be given by the packet-loss rate and the unstable eigenvalues of the plant.

Note that the results of [16] for the binary i.i.d. packet dropouts are no longer applicable for binary Markovian losses. In this case, the networked systems can be modeled as a general MJS. Therefore, this paper is to answer the following fundamental questions arising from quantized networked control: a) Is the logarithmic quantizer still optimal (coarsest) for MJSs? b) Is the sector bound approach still nonconservative in dealing with quantized stabilization of MJSs? c) How to design the optimal quantizer? This paper reveals that for MJSs with a given quadratic control Lyapunov function (QCLF), the optimal quantizer can be approached by adopting a logarithmic law operating on a linear state feedback, similar to that of the LTI systems [2], [3]. Again, the sector bound approach is shown to be nonconservative in investigating the quantized feedback control problem. A linear matrix inequality (LMI) approach is then presented to derive the optimal quantizer under some proper coarseness definition.

Notation: $\equiv$ means "defined as". The superscript $'$ denotes the transpose of vector or matrix. $M^{-1}$ represents the inverse of square matrix $M$. $\mathbb{R}^n$ and $\emptyset$ stand for the $n$-dimensional...
Euclidean space and the empty set, respectively. When $X$ and $Y$ are real symmetric matrices, the notation $X \geq Y$ (respectively, $X > Y$) indicates that $X - Y$ is positive semidefinite (positive definite). $I$ is the identity matrix, and 0 denotes the zero matrix or zero vector. All matrices and matrix functions are assumed to be compatible for algebraic operations whenever their dimensions are not explicitly stated. Furthermore, let $E(\cdot)$ stand for the mathematical expectation operator. $\| \cdot \|$ represents the Euclidean norm for vectors.

II. System Description

As we can see from Fig. 1, a quantized feedback control system comprises three parts: a system to be controlled ($G$), a controller ($K$) and a quantizer ($Q$).

![Fig. 1. Typical Quantized Feedback Control System.](image)

In this paper, we consider a single-input MJS as follows:

$$G: \quad x_{t+1} = A_\theta x_t + B_\theta u_t,$$  
(1)

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}$ is the control input and $\theta_t \in \Theta \equiv \{0, 1, \cdots, N\}$ is the system mode governed by a time-homogeneous Markov chain with transition probability matrix

$$\Pi = (\pi_{ij})_{i,j \in \Theta} \equiv (\text{Pr}(\theta_{t+1} = j | \theta_t = i))_{i,j \in \Theta}.$$  
(2)

The quantized state feedback can be denoted by

$$K: \quad v_t = g(x_t, \gamma_t),$$  
(3)

$$Q: \quad u_t = f(v_t, \gamma_t),$$  
(4)

where $\gamma_t \in \Gamma$ is a mode-related input signal at the controller/quantizer side at time step $t$, which would further determine the desired form of stochastic Lyapunov function and the underlying quantization strategy. In the following, we deal with two cases indicated in the definition below.

**Definition 2.1:** The state feedback law ($K$) and quantizer ($Q$) defined in (3)(4) is said to be mode-dependent, if $\gamma_t = \theta_t$; it is mode-independent, if there is no mode-related input signal available at $K/Q$ side, i.e., $v_t = g(x_t), u_t = f(v_t)$.

III. Mode-Dependent Quantizer

A. Preliminaries

Before formulating the quantized feedback problem, we shall introduce the mean square stability. Let $F_t = \sigma\{x_0, \theta_0, \cdots, x_t, \theta_t\}$ for $\forall t \geq 0$, be the $\sigma$-algebra generated by $\{(x_k, \theta_k) : 0 \leq k \leq t\}$.

**Definition 3.1:** For system

$$x_{t+1} = h(x_t, \theta_t)$$  
(5)

with possibly nonlinear mapping $h(\cdot, \cdot)$, if for every initial condition $x_0, \theta_0$, $E[\|x_t\|^2 | x_0, \theta_0]$ is well-defined for any $t$, then the equilibrium point at the origin is mean square (MS) stable when

$$\lim_{t \to +\infty} E[\|x_t\|^2 | x_0, \theta_0] = 0.$$  

To avoid the trivialness, we assume that the system (1) with $u_t \equiv 0$ is not MS stable. In fact, the MS stability of the open-loop system would require only the existence of an uniform quantizer with zero quantization density since zero control input could be adopted all along; also see Corollary 2.1 in [2] for LTI systems.

**Definition 3.2:** The equilibrium point at the origin of system (5) is mean square quadratically (MSQ) stable, if there exist a positive-definite function $V_1(x_t, \theta_t) \equiv x_t'P_\theta x_t$ stable, if and positive-definite matrices $Q_\theta$, such that

$$\nabla V_1(x_t, \theta_t) \equiv E[V_1(x_{t+1}, \theta_{t+1}) - V_1(x_t, \theta_t)]F_T$$  

$$= E[V_1(x_{t+1}, \theta_{t+1}) | x_t, \theta_t] - V_1(x_t, \theta_t)$$  

$$< -x_t'Q_\theta x_t, \forall x_t \in \mathbb{R}^n, \theta_t \in \Theta, x_t \neq 0.$$  

**Remark 3.1:** In order to clearly reveal the connection between MSQ stability and coarsest quantization, we use ”<” in (7), without loss of generality, instead of ”\leq” as in Definition 2.2 of [16].

Following a similar proof of Theorem 1 in [17], we can get the lemma below.

**Lemma 1:** The MSQ stability of the equilibrium point at the origin of system (5) implies the corresponding MS stability.

The controller (3) and the quantizer (4) can be further described as

$$K: \quad v_t = g(x_t, \theta_t), \quad Q: \quad u_t = f(v_t, \theta_t).$$  
(8)

**Definition 3.3 (QCLF):** A positive-definite quadratic function $V_1(x_t, \theta_t)$ of the form (6) is called a quadratic control Lyapunov function (QCLF) for system (1), if a memoryless quantized state feedback (8) exists such that the closed-loop system

$$x_{t+1} = A_\theta x_t + B_\theta f(g(x_t, \theta_t), \theta_t)$$  
(9)

admits $V_1(x_t, \theta_t)$ as a parameterized Lyapunov function, i.e., the condition (7) holds for some $P_\theta, Q_\theta, \forall \theta \in \Theta$.

It is also worthy mentioning that system (9) is generally nonlinear, since the control signal $u_t$ is a nonlinear function of $x_t$.

We introduce the following assumption for system (1).

**Assumption 3.1:** The unquantized system ($G$) is assumed to be MS stabilizable via linear mode-dependent state-feedback law:

$$v_t = K_{\theta_t} x_t.$$  
(10)
B. Solution

The main purpose of this paper is to solve the following problem.

**Problem 3.1:** For system (1) with a given QCLF \( V(x_t, \theta_t) \), find a function \( f(\cdot, \cdot) \) that is odd with respect to the first independent variable and a function \( g(\cdot, \cdot) \) such that the resultant quantizer is coarsest for the given \( V(x_t, \theta_t) \), i.e., there is no other set of \( \bar{\eta}_j(i) \), such that \( \bar{\eta}_j(i) \leq \eta_j(i), \forall i \in \Theta \) with at least one strict inequality and the condition (7) is satisfied for the predefined \( P_i, Q_i, \forall i \in \Theta \).

A mode-dependent quantizer is said to be logarithmic if for any \( i \in \Theta \), the corresponding set of quantized level \( U_i \) has the following form:

\[
U_i = \{ \pm u_i(i) : u_i(i) = \rho(i)u_0, \quad 0 > u_0 > 0, \quad \text{for } i \in \pm1, \pm2, \ldots \} \cup \{ \pm u_0 \} \cup \{ 0 \},
\]

where

\[
\rho(i) = \frac{1 - \delta(i)}{1 + \delta(i)}.
\]

Note that there is no loss of generality by choosing the same \( u_0 \) for every \( i \in \Theta \); see Lemma 2.1 in [2].

Further define

\[
\delta_m(i) = \begin{cases} 
+\infty, & \text{for } B_i = 0, \\
1 \sqrt{K_{mi}M_{i}}, & \text{for } B_i \neq 0,
\end{cases}
\]

where

\[
K_{mi} = \frac{\sum_{j=0}^{N} \pi_{ij} A_{i}^{j} P_{j} A_{i}}{\sum_{j=0}^{N} \pi_{ij} B_{i}^{j} P_{j} B_{i}}.
\]

Then the next theorem gives a solution to Problem 3.1.

**Theorem 3.1:** If the system (1) with quantized state feedback (8) admits (6) as a QCLF, i.e., the condition (7) holds for some \( P_i, Q_i, \forall i \in \Theta \), then the coarsest quantizer can be approached by a linear unquantized feedback law \( v_t = K_{\theta_t}x_t \) and the following mode dependent logarithmic form with (11):

- if \( \delta_m(\theta_t) < 1 \), then

\[
f(v_t, \theta_t) = \begin{cases} 
u_t(\theta_t), & \text{if }\frac{1}{1 + \delta(\theta_t)}u_t(\theta_t) < v_t \\
0, & \text{if } v_t = 0, \\
-f(-v_t, \theta_t), & \text{if } v_t < 0,
\end{cases}
\]

with

\[
\delta(\theta_t) = \delta_m(\theta_t), \quad K_{\theta_t} = K_{\theta_0};
\]

- if \( \delta_m(\theta_t) = 1 \), then

\[
f(v_t, \theta_t) = \begin{cases} 
u_t, & \text{if } v_t > \frac{1}{2}u_0, \\
0, & \text{if } 0 \leq v_t \leq \frac{1}{2}u_0, \\
-f(-v_t, \theta_t), & \text{if } v_t < 0;
\end{cases}
\]

\[
f(v_t, \theta_t) = \begin{cases} 
u_t, & \text{if } v_t > \frac{1}{2}u_0, \\
0, & \text{if } 0 \leq v_t \leq \frac{1}{2}u_0, \\
-f(-v_t, \theta_t), & \text{if } v_t < 0;
\end{cases}
\]

- if \( \delta_m(\theta_t) > 1 \), then

\[
f(v_t, \theta_t) = 0.
\]

**Proof:** Suppose \( \theta_t = i, \ i \in \Theta \), and drop the time index \( t \) when no confusion is caused, then

\[
\nabla V_t(x, i) = \sum_{j=0}^{N} \pi_{ij} (A_{i}^{j} x + B_{i}^{j} u_{j})^{\prime} P_{j} (A_{i}^{j} x + B_{i}^{j} u_{j}) - x^{\prime} P_{i} x
\]

\[
= 2x^{\prime} \sum_{j=0}^{N} \pi_{ij} A_{i}^{j} P_{j} B_{j} u_{j} + u_{j}^{\prime} \sum_{j=0}^{N} \pi_{ij} \nabla P_{j} B_{j} u_{j}
\]

\[
+ \sum_{j=0}^{N} \pi_{ij} x^{\prime} (A_{i}^{j} P_{j} A_{i} - P_{i}) x.
\]

For Case 1: \( B_{i} = 0 \), the existence of the QCLF guarantees

\[
\nabla V_t(x, i) + x^{\prime} Q_{i} x
\]

\[
= \left\{ -x^{\prime} M_{i} x + (u - K_{mi} x)^{2} \right\} \sum_{j=0}^{N} \pi_{ij} B_{j}^{\prime} P_{j} B_{j},
\]

and therefore the given QCLF ensures \( M_{i} > 0 \). Then \( \nabla V_t(x, i) < -x^{\prime} Q_{i} x, \forall x \neq 0 \) if and only if \( u = f(v_t, i) \in (u_1(i), u_2(i)) \), where

\[
u_1(i) = K_{mi} x - x^{\prime} M_{i} x, \quad u_2(i) = K_{mi} x + x^{\prime} M_{i} x.
\]

Followed by the orthogonal decomposition method [3], \( M_{i}^{1/2} x \) can be decomposed into

\[
M_{i}^{1/2} x = \alpha(i) M_{i}^{1/2} K_{mi}^{\prime} + z(i),
\]

where \( \alpha(i) \) is a scalar and vector \( z(i) \) is orthogonal to \( M_{i}^{1/2} K_{mi}^{\prime} \). Therefore, \( u_1(i), u_2(i) \) can be rewritten with respect to the new coordinate system (18) as:

\[
u_1(i) = \alpha(i) \nabla (\delta(i))^{2} - \alpha(i)^{2} \nabla (\delta(i))^{2} + z(i),
\]

\[
u_2(i) = \alpha(i) \nabla (\delta(i))^{2} + \alpha(i)^{2} \nabla (\delta(i))^{2} + z(i).
\]

Moreover, in Case 2.a: \( \delta_m(i) > 1 \), we can again choose \( u = 0 \) similar to Case 1, since \( u = 0 \) belongs to the interval \( (u_1(i), u_2(i)) \); in Case 2.b: \( \delta_m(i) \leq 1 \), it can be proved that the optimal quantization strategy is logarithmic as shown in (14) and (15) with \( \delta(i) = \delta_m(i) \) [2].

Combining the above two cases gives us the coarsest mode-dependent quantizer stated in this theorem. The technique in the proof of Lemma 2.1 in [3] can still be used to prove that linear feedback law \( v_t = K_{\theta_t}x_t \) is sufficient to coarsest quantizer for Case 2.b, while for Case 1 and Case
2.a, the argument is trivial since \( u = 0 \) is adopted. This completes the proof.

The quantization error is mode-dependent:
\[
e_t \equiv u_t - v_t = f(v_t, \theta_i) - v_t = \Delta(v_t, \theta_i)v_t,
\]
where \( \Delta(v_t, \theta_i) \in [-\delta(\theta_i), \delta(\theta_i)] \) and \( \delta(\theta_i) \) is defined in Theorem 3.1. Then, the closed-loop quantized feedback system with \( v_t = K_\theta x_t \) becomes the following uncertain MJS:
\[
x_{t+1} = A_{\theta_t} x_t + B_{\theta_t} (1 + \Delta(K_{\theta_t}, \theta_t)) K_\theta x_t.
\]

The theorem below tells us that the sector bound approach is still valid.

**Theorem 3.2:** Given a mode dependent logarithmic quantizer (14)-(16) with a set of fixed \( \rho(i) \leq 1, \ i \in \Theta \), system (1) with quantized linear state feedback (8) has a QCLF (6) if and only if the following uncertain system:
\[
x_{t+1} = A_{\theta_t} x_t + B_{\theta_t} (1 + \Delta(\theta_t)) v_t,
\]
is robust MSQ stabilizable for uncertainty \( \Delta(\theta_t) \in [-\delta(\theta_t), \delta(\theta_t)] \) via linear feedback (10), where \( \rho(i) \) and \( \delta(i) \) are related by (12).

**Proof:** Again, suppose \( \theta_i = i, \ i \in \Theta \), then for closed-loop system (19), we have
\[
\nabla V_i(x, i) = \sum_{j=0}^{N} \pi_{ij}(A_i x + B_i (1 + \Delta(i)) K_i x) P_j x - x' P_i x.
\]
By defining the following matrix function
\[
\Omega(\cdot, i) = \Omega_0(i) + \Omega_1(\cdot, i) K_i + K_1' \Omega_1(\cdot, i) + K_1' \Omega_2(\cdot, i) K_i,
\]
with a constant matrix \( \Omega_0(i) \), a vector function \( \Omega_1(\cdot, i) \), and a matrix function \( \Omega_2(\cdot, i) \), Lemma 2.2 in [3] indicates that \( \nabla V_i(x, i) \leq -x' Q_i x \), \( \forall x \neq 0 \) is equivalent to
\[
\sum_{j=0}^{N} \pi_{ij}(A_i x + B_i (1 + \Delta(i)) K_i x) P_j x - x' P_i x < -x' Q_i x
\]
for \( x \neq 0 \). This kind of equivalence is true for any \( i \in \Theta \), and thus by Definition 3.2, inequality (22) is the condition for robust MSQ stabilizability of system (20).

Next, we formulate the robust MSQ stability of (20) into LMIs.

**Proposition 3.1:** The robust MSQ stability of the system (20) with a set of \( 0 < \delta(i) \leq 1, \ i \in \Theta \) and \( v_t = K_\theta x_t \) is equivalent to
\[
\begin{bmatrix}
-S_i & S_i & Y_i & \Phi_i \\
* & -W_i & 0 & 0 \\
* & * & -\tau(i) & 0 \\
* & * & * & \Xi_i
\end{bmatrix} < 0,
\]
for some variables \( S_i > 0, \ W_i > 0, \ Y_i, \Phi_i, \Xi_i \) are shown on the bottom of the next page.

**Proof:** By Schur complement, inequality (22) holds if and only if
\[
\begin{bmatrix}
-P_i + Q_i \sqrt{\pi_0(A_i + B_i K_i)} & \cdots & \sqrt{\pi_N(A_i + B_i K_i)} \\
* & -P_0^{-1} & \cdots & 0 \\
* & * & \ddots & \vdots \\
* & * & * & -P_N^{-1}
\end{bmatrix} > 0
\]
\[
\begin{bmatrix}
0 \\
\Delta(i) [K_i 0 \cdots 0] \\
K_i' \\
0
\end{bmatrix}
\]
\[
\begin{bmatrix}
\tau(i)^{-1} K_i' K_i & 0 & \cdots & \tau(i) \delta(i)^2 \pi_0 B_i B_i' \\
* & \tau(i) \delta(i)^2 \pi_0 B_i B_i' & \ddots & \vdots \\
* & * & \ddots & \vdots \\
* & * & * & \tau(i) \delta(i)^2 \pi_0 B_i B_i'
\end{bmatrix} < 0,(24)
\]
with \( \tau(i) > 0 \) representing the scaling variable. By setting \( S_i = P_i^{-1}, W_i = Q_i^{-1}, Y_i = K_i S_i \) and using Schur complement again, it is straightforward to see that (24) holds if and only if (23) is true for \( \forall i \in \Theta \).

Note that (23) is convex in \( S_i > 0, W_i > 0, Y_i \) and \( \tau(i) > 0 \) for any fixed set of \( \delta(i) \). It is thus possible to search desirable \( 0 < \delta(i) \leq 1 \) over constraints (23) based on some measures of data-rate. However, such kind of searching may be time-consuming especially when the number of system modes \( N \) is large.

**C. Examples**

In order to facilitate the quantization optimization, the measurement on sector bound could be chosen as
\[
\delta = \min_{i \in \Theta} \{\delta(i)\},
\]
which captures the worst-case quantizer among all system modes. In the following two examples, we try to characterize the quantizer with the largest sector bound \( \delta \) over all possible QCLFs.

**Example 3.1:** Consider a MJS (1) modified from [16]:
\[
A_0 = A_1 = \begin{bmatrix} 0 & 1 \\ 1.8 & -0.3 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
The system matrices $A_0, A_1$ contain two unstable poles \{1.2, -1.5\}. Suppose the transition probability matrix is given by

$$
\Pi_1 = \begin{bmatrix}
0.1 & 0.9 \\
0.3 & 0.7
\end{bmatrix},
$$

then the plant is MS stabilizable via (10), we can compute $\delta = 0.4783$ with state feedback (10) and gain matrices:

$$
K_0 = [-0.8846 - 0.3611], \quad K_1 = [-1.8 1.1507].
$$

Next, we assume the transition probability matrix is changed to

$$
\Pi_2 = \begin{bmatrix}
0.2 & 0.8 \\
0.2 & 0.8
\end{bmatrix},
$$

then we can get $\delta = 0.4698$ with gain matrices:

$$
K_0 = [0 0], \quad K_1 = [-1.8 0.675].
$$

IV. MODE-INDEPENDENT QUANTIZER

A. Preliminaries

Assumption 4.1: In this section, the unquantized system (3) is assumed to be MS stabilizable via linear mode-independent state-feedback law:

$$
v_t = K x_t,
$$

which is also said to be strongly MS stable in literature; see e.g. [19].

The controller and the quantizer now can be expressed as

$$
K : v_t = g(x_t), \quad Q : u_t = f(v_t).
$$

In contrast to (6), for the mode-independent scheme the Lyapunov function is chosen as

$$
V_2(x_t) \equiv x'_t P x_t.
$$

Definition 4.1: The equilibrium point at the origin of system (5) is strongly MSQ stable, if there exist a positive-definite function $V_2(x_t)$ in (30) and positive-definite matrix $Q$ such that

$$
\begin{align*}
\nabla V_2(x_t) &= E[V_1(x_{t+1})]x_t - V_1(x_t) \\
&> -x'_t Q x_t, \quad \forall x_t \in \mathbb{R}^n, x_t \neq 0.
\end{align*}
$$

Assumption 4.2: The transition probability matrix (2) satisfies $\pi_{ij} = \pi_j$ for $i, j \in \Theta$.

The above assumption implies that the system mode process is governed by an i.i.d. process (a special case of Markov chain), which would facilitate our further deduction. For autonomous jump linear system driven by a finite-state homogeneous Markov chain, the Lyapunov functions $V_i(x_t, \theta_t)$ is equivalent to $V_2(x_t)$ when the Markov functions degenerates to the i.i.d. process [18].

B. Solution

Based on (30) we can easily get

$$
\nabla V_2(x) = \sum_{j=0}^{N} \pi_j (A_j x + B_j u)' P (A_j x + B_j u) - x' P x
$$

$$
= \sum_{j=0}^{N} \pi_j x' (A_j' P A_j - P) x + 2 x' \sum_{j=0}^{N} \pi_j A_j' B_j u + u' \sum_{j=0}^{N} \pi_j B_j' B_j u.
$$

A mode-independent quantizer is called logarithmic if the set of quantized level $\mathcal{U}$ has the following form:

$$
\mathcal{U} = \{ \pm u_0 : u_l = \rho^l u_0, \quad u_0 > 0, \quad \text{for } l \in \{1, 2, \ldots, m\} \} \cup \{0\}.
$$

The corresponding quantizer is given by

$$
f(v_t) = \begin{cases}
    u_l, & \text{if } \frac{1}{1+\rho} u_0 < u_l < \frac{1}{1-\rho} u_0, \\
    0, & \text{if } v_t = 0, \\
    -f(-v_t), & \text{if } v_t < 0,
\end{cases}
$$

where

$$
\rho = \frac{1 - \delta}{1 + \delta}.
$$

Define

$$
\delta_m = \frac{1}{\sqrt{K_m M + 1 K_m}},
$$

where

$$
K_m = -\sum_{j=0}^{N} \pi_j B_j' P A_j, \quad M = \frac{\sum_{j=0}^{N} \pi_j B_j' P B_j A_j - \sum_{j=0}^{N} \pi_j A_j' P A_j - P^1 + Q}{(\sum_{j=0}^{N} \pi_j B_j' P B_j)^2}.
$$

The following theorem can be proved similarly to Theorem 3.1 and 3.2.

Theorem 4.1: (a) If the system (1) under Assumption 4.2 with quantized state feedback (29) admits (30) as a QCLF, i.e., the condition (31) holds for some $P, Q$, then the coarsest
quantizer can be approached by a linear unquantized feedback law \( v_t = Kx_t \) and the mode-independent logarithmic form (34) with \( K = K_m \) and \( \delta = \delta_m \).

(b) Given a mode-independent logarithmic quantizer (34) with a fixed \( \rho \in [0, 1] \), system (1) with quantized linear state feedback (29) has a QCLF (30) if and only if the following uncertain system:

\[
x_{t+1} = A_0x_t + B_0(1 + \Delta)u_t,
\]

is robust strongly MSQ stabilizable for uncertainty \( \Delta \in [-\delta, \delta] \) via linear state-feedback law (28), where \( \rho \) and \( \delta \) are related by (35).

Obviously, the density for quantizer (34) is \( \eta_f = -\frac{2}{m(\rho)} \) with \( \rho \) defined in Theorem 4.1.

**Theorem 4.2:** The least density \( \eta_f = -\frac{2}{m(\rho)} \) can be obtained through the optimization \( \bar{\delta}^2 \equiv \max_{S,W,Y} \delta^2 \) over the constraint

\[
\begin{bmatrix}
-S & S & Y' & \Phi \\
* & -W & 0 & 0 \\
* & * & -1 & 0 \\
* & * & * & \Xi
\end{bmatrix} < 0,
\]

with variables \( S > 0, W > 0, Y \) and \( \delta^2 \), where \( \Phi, \Xi \) are shown on the bottom of the next page. Note that \( \eta_f \) and \( \delta \) are related by (35) and \( K = YS^{-1} \).

**Proof:** It is easy to get the result by taking \( S = P^{-1}/\tau, W = Q^{-1}/\tau \) and \( Y = KS \), where \( \tau \) is the scaling variable. In contrast to (23) in the form of \( N+1 \) coupled constraints, Theorem 4.2 only contains one single constraint without linear searching, and thus the computational burden would be eased.

**Example 4.1:** Suppose \( A_0, A_1, B_0, B_1, \Pi_2 \) are given in Example 3.1. By Theorem 4.2, \( \delta = 0.3873 \) with state feedback (28) and gain matrix

\[
K = [-1.682 \ 0.8854].
\]

**Remark 4.1:** By comparing Example 3.1 with 4.1, we can see that mode-dependent pattern presents a less conservative but more computationally demanding result with larger \( \delta \). It is understandable, as the quantizer (14)-(16) takes into account the system mode information.

**C. Application to NCS with UDP Channel**

UDP here means that there is no acknowledgment signal with respect to data transmission through unreliable networks, which falls into the mode-independent pattern. Two NCS structures are considered as follows.

1) **Discrete Plant with Zero-Control Strategy and Binary Dropouts:** A quantized feedback NCS with UDP-like channel is shown in Fig. 2, where the network (N) is an analog multiplicative memoryless channel associated to \( \theta \):

\[
N: \ w_t = \theta_t u_t,
\]

where \( \theta_t \in \Theta = \{0, 1\} \) is an i.i.d. random variable:

\[
\Pr(\theta_t = 0) = \alpha, \ \Pr(\theta_t = 1) = 1 - \alpha.
\]

As a result, the system adopted an unreliable network with packet-dropout rate \( \alpha \) and a zero-control strategy.

![Fig. 2. Quantized Networked Control System with Binary Packet Losses.](image)

The LTI plant (\( \mathbb{P} \)) is described as

\[
\mathbb{P}: \ x_{t+1} = Ax_t + Bw_t,
\]

and the jump system (\( \mathbb{G} \)) is a combination of network and LTI plant with the following system matrices:

\[
A_0 = A_1 = A, \ B_0 = 0, \ B_1 = B.
\]

From Theorem 4.1, we have

\[
K_m = -\frac{B'PA}{B'PB},
\]

\[
M = \frac{(1-\alpha)A'PBB'PA}{(1-\alpha)B'PB} - \frac{A'PA - P + Q}{(1-\alpha)B'PB}.
\]

Under this situation, the inequality (37) or the condition \( \nabla V_2(x) < -x'Qx, \forall x \neq 0 \) for system (36) with control law (28) can be rewritten in the following modified Riccati inequality:

\[
A'PA - P + Q - (1-\alpha)(1-\delta^2)A'PB(B'PB)^{-1}B'PA < 0.
\]

Based on Lemma 5.4 [15] for modified algebraic Riccati equation, \( \alpha \) and \( \delta \) should satisfy the following condition in order to ensure the existence of \( P > 0 \) to (43):

\[
(1-\alpha)(1-\delta^2) < 1 - \frac{1}{\Pi_1|\lambda_1^n(A)|^2},
\]

where \( \lambda_1^n(A) \) denote the unstable poles of \( A \). It is easy to check the above result is consistent with Theorem 2.1 of [16], which can be seen as a special case of Theorem 4.2 in this paper.

**Example 4.2:** Consider a LTI system (39) with

\[
A = \begin{bmatrix} 0 & 1 \\ 1.8 & -0.3 \end{bmatrix}, \ \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Then the system matrices \( A_0, A_1, B_0, B_1 \) are the same as in Example 3.2. Assume \( \alpha = 0.2 \), which is consistent with \( \Pi_2 \) in (27). The plant is strongly MS stabilizable, and based on Theorem 4.2 or (44), it follows that \( \delta = 0.3685 \) with \( K = [-1.8 \ 0.675] \).

**Remark 4.2:** For system (39) with i.i.d. packet losses and zero-control strategy, it is redundant to know any system mode information. Furthermore, although Theorem 3.1 exhibits theoretical importance in quantization of MJSs, it is unsuitable for solving the quantization problem over unreliable channel, since it is unrealistic for the quantizer to know whether the current packet will be lost or not before the packet is sent over the network, which is also true for the following bounded packet loss case.
2) Continuous Plant with ZOH and Bounded Dropouts: In Fig. 3, the sampler is clock-driven, while the zero-order hold (ZOH) is event-driven. The continuous-time plant together with the ZOH and the sampler can be expressed in the general discrete-time form (39).

Let \( \{i_k, k \geq 0\} \) be a strictly monotonically increasing subset of \( \{0, 1, \cdots\} \), representing the sequence of time instants at which the data packets are successfully transmitted through the network (\(N\)). Without loss of generality, set \( t_0 = 0\). The packet-loss process is defined as 
\[
\eta(i_k) = i_{k+1} - i_k, \quad k \geq 0.
\]

Here, \( \eta(i_k) \) is driven by an i.i.d. process with finite-state range set \( \Theta = \{0, 1, \cdots, N\} \), where \( \eta(i_k) = 0 \) indicates that there is no packet dropout during the time interval \( [i_k, i_{k+1}) \) and \( N \) represents the maximal length of consecutive packet losses. The transition probability matrix is the same as in Assumption 4.2. Then, from Theorem 9 of [10] the system \( \mathcal{G} \) in Fig. 3 can be modeled as a MJS (1) with
\[
A_i = A^i, \quad B_i = \sum_{r=0}^{i-1} A^r B.
\]

Thus, Theorems 4.1 and 4.2 can be applied as shown in the numerical example borrowed from [10].

**Example 4.3:** The discretized system in Fig. 3 is (39) with
\[
A = \begin{bmatrix}
0.6065 & 0 & -0.2258 \\
0.3445 & 0.7788 & -0.0536 \\
0 & 1 & 1.2840
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.0582 \\
-0.0093 \\
0.5681
\end{bmatrix}.
\]

Suppose the maximum consecutive packet dropouts of the underlying network is \( N = 4 \) and the transition probability matrix (2) under Assumption 4.2 is given by:
\[
\pi_0 = 0.5, \quad \pi_1 = 0.2, \quad \pi_2 = 0.1, \quad \pi_3 = 0.1, \quad \pi_4 = 0.1.
\]

By using Theorem 4.2 with system matrices (47), it follows that \( \delta = 0.4085 \) with \( K = [0 \ 0 \ -0.8452] \).

**V. CONCLUSIONS**

This paper has shown that for linear systems with Markovian jump parameters, mode-dependent (or mode-independent) logarithmic quantizer is still optimal in MSQ (or strongly MSQ) stability sense, and the sector bound approach again provides a nonconservative way for studying the corresponding design problems. We have also presented methods via LMIs to find the optimal quantizer under some definition of coarseness. It is also worthy noting that the mode-dependent or mode-independent QCLF in this paper can be modified into other complicated forms, e.g. quantization-error-dependent form or polytopic form, in order to achieve less conservative results (smaller \( \theta \) or larger \( \delta \)) at the expenses of additional computational complexity. Possible future work includes output feedback stabilization problem, \( H_\infty \) and quadratic performance analysis, and generalization to the MIMO system case.

**REFERENCES**


