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A Unified Framework for Mean Square Stability of Kalman Filters with Intermittent Observations

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Abstract— **This paper presents a unified framework for analysis of the stability of the expected error covariance (EEC) of Kalman filters subject to intermittent observations. A brief literature review summarizing some of the most important results in the area is provided. We state a method in the most general form, making no assumptions on the network model and only minor assumptions on the system. Then, as we adopt particular network models and assume some particular system structures, we recover most of the known results in the literature, which can be seen as a special case of our approach. Tight necessary and sufficient conditions for the EEC to be bounded are given for most cases, except for general degenerate systems, where only sufficient conditions are given in a closed form.**

I. Introduction

State estimators with missing measurements have attracted great attention in the last few years. This is partly motivated by the recent advances on network systems, allowing its use in data acquisition and control. Whilst the basic Kalman filter with constant parameters is well understood and has been used since the 1960s, its counterpart with missing measurements is still an active research topic. In [1], the authors pointed out some basic properties of the Kalman filter with intermittent measurements. Since then, many authors have studied similar problems, aiming for a better understanding of the effects of missing data on the estimation performance.

It was shown in [1] that the Kalman filter is still the optimal state estimator for linear systems with missing measurements. In [2], the author showed that if the sensor is allowed to run a local Kalman filter, and send its estimate instead of the raw measurements, the overall performance is improved. Similarly, the authors in [3] propose sending several measurements in each packet, allowing the centralized Kalman filter to obtain a bounded state estimation covariance whenever a packet is received.

In this paper we restrict our analysis to systems with limited resources, where only one (possibly vector) measurement is sent in each packet. We will further assume that there is no delay in the communication, and that each vector of measurements is either completely received or completely lost. Two network models have dominated the analysis of Kalman filtering with intermittent observations. In [1], the authors consider an independent and identically distributed (i.i.d.) erasure communication channel, i.e., the probability to receive a measurement is independent of the availability of previous measurements. As an attempt to model some channel phenomenon, like fading and interferences, the authors of [4] consider the Gilbert-Elliot [5] model, in which the transmission of a measurement is described by a Markov chain. More recently, some authors have considered more general network models [6], [7].

The covariance of the estimation error is the most commonly studied property of a Kalman filter with intermittent observations. Bounds for the expected value of the error covariance (EEC) were studied in [1], [8], while higher order statistics were studied in [7]. In [9], [10], the authors develop bounds on the cumulative distribution function of some norm of the error covariance. The question of whether the EEC is bounded is the subject of current research. In [1], the authors showed that for any unstable system, and for an i.i.d. channel model, there exists a minimum measurement arrival rate that results in a bounded EEC. Since then, several authors have provided necessary and sufficient conditions for different classes of systems and channels. However, stating the necessary and sufficient condition for the boundedness of the EEC, under general assumptions on the channel and system models, remains an open problem.

In this paper we summarize some results concerning necessary and/or sufficient conditions for the boundedness of the EEC. These conditions are derived under simplifying assumptions on the system and network models. We then provide a unified framework for these conditions. In this framework, we start by stating a necessary and a sufficient condition for a very general classes of systems, and for any network model. We then show how these conditions are equivalent to known results, for each particular assumption on the network and the system model.

II. Problem Statement

Consider the discrete-time linear system:

$$\begin{cases} x_{t+1} &= Ax_t + w_t \\ y_t &= Cx_t + v_t \end{cases} \tag{1}$$

where the state vector $x_t \in \mathbb{R}^n$ has initial condition $x_0 \sim N(0, P_0)$, $y_t \in \mathbb{R}^p$ is the measurement, $w_t \sim N(0, Q)$ is the process noise and $v_t \sim N(0, R)$ is the measurement noise. We assume that the measurements are sent to the Kalman estimator through a network subject to random packet losses. The binary random variable γ_t describes the arrival of a measurement at time t, i.e., $\gamma_t = 1$ when y_t was received at the estimator and $\gamma_t = 0$ otherwise.

It was shown in [1] that even when the measurements are subject to random losses, the Kalman filter still obtains

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the best estimate $\hat{x}_t$ of the state x_t. In this case, however, the covariance matrix P_t of the estimation error becomes a random variable.

The update equation of the error covariance matrix depends on the availability of the measurements. When a measurement is available, both measurement and time updates are performed. When a measurement is not available, only the time update can be computed. The update equation of P_t can then be written as follows:

$$P_{t+1} = \begin{cases} \Phi_1(P_t), & \gamma_t = 1 \\ \Phi_0(P_t), & \gamma_t = 0 \end{cases} \tag{2}$$

with

$$\begin{aligned} \Phi_1(P_t) &= AP_tA' + Q + \\ & \quad -AP_tC'(CP_tC' + R)^{-1}CP_tA' \qquad (3) \\ \Phi_0(P_t) &= AP_tA' + Q. \qquad (4) \end{aligned}$$

When all the measurements are available, and the Kalman filter reaches its steady state, the EC is given by the solution of the following algebraic Riccati equation

$$\underline{P} = A\underline{P}A' + Q - A\underline{P}C'(C\underline{P}C' + R)^{-1}C\underline{P}A'. \tag{5}$$

We are interested in finding necessary and sufficient conditions for the stability of the Kalman filter with missing measurements. To this end, different criteria have been used to define stability. In general, these conditions can be written as either,

$$\lim_{t\to\infty} E\{M_t\} < \infty \tag{6}$$

or

$$\sup_t E\{M_t\} < \infty, \tag{7}$$

where M_t is defined as either $M_t = \|P_t\|$ or $M_t = \mathrm{trace}(P_t)$. Nevertheless, it is straightforward to verify that all these convergence criteria are equivalent.

III. Review of Available Results

In this section we summarize the available results concerning necessary and sufficient conditions for the boundedness of the EEC. We organize them in subsections according to the network model adopted.

A. i.i.d. network model

In this section we summarize the results assuming an i.i.d. network model, i.e., γ_t and γ_k are uncorrelated for $t \neq k$. Let the probability that a measurement is available be given by

$$\lambda = \Pr(\gamma_t = 1). \tag{8}$$

Also, let $\alpha_\star = \max_i |\alpha_i|$, where $\alpha_i,\ i = 1, \cdots, n$ are the eigenvalues of A.

In [1], the following conditions were obtained:

a) Necessity: If
$$\lambda < 1 - \alpha_\star^{-2}, \tag{9}$$
then, $\lim_{t\to\infty} E\{P_t\} = +\infty$.

b) Sufficiency: If
$$\lambda > \overline{\lambda}, \tag{10}$$
then, $\lim_{t\to\infty} E\{P_t\} \leq M$, where $\overline{\lambda}$ is obtained from a quasi-convex optimization problem.

For the special cases where C is invertible or A has only one unstable eigenvalue, the conditions above become equivalent, and therefore necessary and sufficient.

In [11], the condition $\lambda > 1 - \alpha_\star^{-2}$ is shown to be also sufficient for systems in which the part of the matrix C corresponding to the observable subspace is invertible.

B. Gilbert-Elliot network model

The i.i.d. packet loss model is not accurate for modeling situations in which measurements can be lost or received in blocks. For example, when a network is subject to an external interference that persists for a significant time, the assumption that γ_t and γ_k are uncorrelated for $t \neq k$ is no longer valid. Motivated by this, some authors use a two-state Markov chain to model the packet losses. The general assumption is that the packet loss process is a time-homogeneous Markov chain, with the range set $\mathcal{S} = \{0, 1\}$. The transition probability matrix of γ_t is then assumed to be

$$\mathcal{P}(\gamma_{t+1} = j | \gamma_t = i)_{i,j\in\mathcal{S}} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \tag{11}$$

where p and q are the failure and recovery rates, respectively. Notice that the i.i.d. model is a special case of the Gilbert-Elliot network model, in which $q = 1 - p = \lambda$.

In [4], the authors use the fact that the sojourn times are i.i.d. random variables, and give a sufficient condition for the EEC to be bounded. For first order systems, the following condition was shown to be necessary and sufficient:

$$q > 1 - \alpha_\star^{-2}. \tag{12}$$

The sufficient condition for higher-order systems is technically involved, and we omit it here because it is superseded by a condition derived later in [7], [12].

The authors in [7] introduce the concept of degenerate systems, and provided a necessary and sufficient condition for the EEC to be finite in systems whose unstable part is non-degenerate. The definition of degeneracy applies to systems whose matrix A is diagonalizable. After putting A in diagonal form, a quasi-equiblock is defined as a subsystem containing all the eigenvalues with the same magnitude. *A system is said to be degenerate if there exists at least one quasi-equiblock whose associated sub-matrix C does not have full column rank (FCR). It is non-degenerate otherwise.* For systems whose unstable part is non-degenerate, the condition in (12) was shown to be necessary and sufficient for the EEC to be bounded. Using a different approach, the authors in [12] showed the same result for non-degenerate systems, and they further obtained the following necessary and sufficient condition for second order degenerate systems:

$$\left(1 + \frac{pq}{(1-q)^2}\right)\left(\alpha_\star^2(1-q)\right)^d < 1, \tag{13}$$

where d is the smallest nonzero integer such that (C, A^d) is not observable.

C. General network model

The present authors considered degenerate systems with a single equiblock in [6]. Under no simplifying assumption on the packet drop model, the following necessary and sufficient condition was given:

$$\alpha_\star^2 \limsup_{T\to\infty} \Pr(\overline{\mathcal{M}_T})^{1/T} < 1 \tag{14}$$

where $\Pr(\overline{\mathcal{M}_T}) = 1 - \Pr(\mathcal{M}_T)$ and $\Pr(\mathcal{M}_T)$ denotes the probability of receiving a sequence of measurements guarantying observability, up to time T. The computation of the limit in (14) is explained in [6]. The condition above requires A to satisfy the following:

Condition W: The matrix A is diagonalizable, and the phases of the its complex eigenvalues are rational multiples of 2π.

Notice that condition W is weak in the sense that the set of matrices satisfying it is dense.

IV. A UNIFIED FRAMEWORK FOR NECESSARY AND SUFFICIENT CONDITIONS

In this section we provide a unified framework for necessary and sufficient conditions for the boundedness of the EEC. We start by providing general necessary and sufficient conditions which are valid for any network model, and for systems whose matrix A satisfy condition W, i.e., for "almost any" system. Since no assumption is made on the network model, these conditions are expressed in terms of the probability of arrival sequences having certain properties, as in (14). We then show how these conditions become equivalent (i.e., simultaneously necessary and sufficient), for non-degenerate systems and systems with only one quasi-equiblock, with Gilbert-Elliot and i.i.d. network models.

In the rest of the paper, we assume that all quasi-equiblocks of the system under analysis are unstable. This is without loss of generality, since the estimation error covariance is bounded on the stable states. As pointed out in [7], the bounded error covariance of the stable part can be interpreted as measurement noise in the unstable part.

A. The general case

Consider the system described in (1), and let A satisfy condition W. Following the argument in [7], we can assume, without loss of generality, that A is diagonal. Also, as pointed out in [7], the stable eigenvalues of A can be ignored. Then, the stability analysis can be restricted to that of the subsystem formed by the unstable eigenvalues of A, and the associated sub-matrix of C. Hence, to simplify the presentation, in the rest of this section we assume that A is diagonal and all its eigenvalues have magnitude strictly greater than one.

We decompose the system as follows:

$$x_t = \begin{bmatrix} x_t^{\prime(1)} & x_t^{\prime(2)} & \dots & x_t^{\prime(J)} \end{bmatrix}', \tag{15}$$

$$A = \operatorname{diag}(A_1, A_2, \dots, A_J), \tag{16}$$

$$C = \begin{bmatrix} C_1 & C_2 & \dots & C_J \end{bmatrix}, \tag{17}$$

where each subsystem (A_j, C_j) is a quasi-equiblock, i.e.,

$$A_j = \alpha_j \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \exp(2\pi i\theta_{j,2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(2\pi i \theta_{j,n_j}) \end{bmatrix}, \tag{18}$$

with $\alpha_j \in \mathbb{C}$.

We introduce the following notation.

Notation 4.1: For given $T \in \mathbb{N}$ and $1 \le m \le 2^T$, let S_m^T denote the binary sequence of length T formed by the binary representation of $m-1$. We also use $S_m^T(i)$, $i = 1, \cdots, T$ to denote the i-th entry of the sequence, i.e.,

$$S_m^T = \{S_m^T(1),\ S_m^T(2), \dots,\ S_m^T(T)\}. \tag{19}$$

For a given sequence S_m^T, and a matrix $P \in \mathbb{R}^{n\times n}$, we define the map

$$\phi(P, S_m^T) = \Phi_{S_m^T(T)} \circ \Phi_{S_m^T(k-1)} \circ \dots \Phi_{S_m^T(1)}(P) \tag{20}$$

where $\circ$ denotes function composition (i.e. $f \circ g(x) = f(g(x))$).

For given $T \in \mathbb{N}$, let Γ_t^T be the binary sequence describing the arrival of measurements between time $t-T$ and $t-1$, i.e.,

$$\Gamma_t^T = \{\gamma_{t-1},\ \gamma_{t-2},\ \dots,\ \gamma_{t-T}\}. \tag{21}$$

Notice that if m is chosen so that $S_m^T = \Gamma_t^T$, then $\phi(P_{t-T}, S_m^T)$ updates P_{t-T} according to the measurement arrivals in the last T sampling times, i.e.,

$$P_t = \phi(P_{t-T}, S_m^T) = \Phi_{\gamma_{t-1}} \circ \Phi_{\gamma_{t-1}} \circ \dots \Phi_{\gamma_{t-T}}(P_{t-T}). \tag{22}$$

Let $t_i, i = 1, 2, \dots, I$ be such that the measurements are available at the instants $t - t_i$. Then, the vector of received measurements up to time t is

$$Y_t^T = Y_t(\Gamma_t^T) = \begin{bmatrix} y_{t-t_1} \\ y_{t-t_2} \\ \vdots \\ y_{t-t_I} \end{bmatrix} \tag{23}$$

$$= O(\Gamma_t^T)x_t + V_t^T, \tag{24}$$

where the associated observability matrix $O(\Gamma_t^T)$ is given by

$$O(\Gamma_t^T) = \begin{bmatrix} CA^{-t_1} \\ CA^{-t_2} \\ \vdots \\ CA^{-t_I} \end{bmatrix}. \tag{25}$$

The vector V_t^T is a function of the process and measurement noises and of the sequence of available measurements (see [10] for more details). Decomposing the matrix $O(\Gamma_t^T)$ according to the decomposition (15)-(18), we obtain

$$O(\Gamma_t^T) = \begin{bmatrix} O^{(1)} & O^{(2)} & \dots & O^{(J)} \end{bmatrix}. \tag{26}$$

Hence, we can write

$$Y_t^T = \begin{bmatrix} O^{(1)} & O^{(2)} & \dots & O^{(J)} \end{bmatrix} x_t + V_t^T. \tag{27}$$

For a fixed $T \in \mathbb{N}$, let $\mathcal{S}_T = \{S_m^T : \; m = 1, \ldots, 2^T\}$ denote the set of all possible sequences of length T. Let the subsets $\mathcal{M}_T^{(j)} \subset \mathcal{M}_T \subset \mathcal{S}_T$ be defined as

$$\mathcal{M}_T = \{S_m^T : \; O(S_m^T) \text{ has FCR}\} \quad (28)$$
$$\mathcal{M}_T^{(j)} = \{S_m^T : \; O^{(j)}(S_m^T) \text{ has FCR}\}. \quad (29)$$

We use the shorthand notation $\Pr(\mathcal{M}_T^{(j)})$ to describe the probability to observe a sequence $\Gamma_t^T \in \mathcal{M}_T^{(j)}$. Also, for an arbitrary set $\mathcal{N}$, we use $\overline{\mathcal{N}}$ to denote its complement.

The main result of the section is given in Theorem 4.1. Its derivation requires the following lemma.

Lemma 4.1: Let A satisfy condition W. Then,

1) If the sequence $\Gamma_t^T \notin \mathcal{M}_T^{(j)}$, then

$$\|P_t\| \geq |\alpha_j|^{2T} K_j, \quad (30)$$

for each j, where K_j is a constant independent of T.

2) If the sequence $\Gamma^T \in \mathcal{M}_T$, then

$$\|P_t\| \leq \alpha_\star^{2T} \underline{K}, \quad (31)$$

where $\underline{K}$ is a constant independent of T.

Proof:

1) Fix j, and suppose we know the exact value of all the states $x_t^{(i)}$, for all $i \neq j$. Suppose that we run a Kalman filter to estimate the state $x_t^{(j)}$, based on this assumption, and let $\tilde{P}_t^{(j)}$ be the estimation error covariance obtained. Doing so would lead to a better estimation than the one obtained from the Kalman filter under study. Hence

$$P_t \geq \operatorname{diag}\left(\tilde{P}_t^{(1)}, \; \cdots, \; \tilde{P}_t^{(J)}\right). \quad (32)$$

Then, the result follows from Lemma 4.1[b] in [6].

2) When O has FCR, from (24), one can obtain the unbiased estimate

$$\hat{x}_t = O^{-1} Y_t^T, \quad (33)$$

with error covariance

$$E\{(x_t - \hat{x}_t)(x_t - \hat{x}_t)'\} = O^{-1} \Sigma_V O'^{-1}, \quad (34)$$

where

$$\Sigma_V = E\{V_t^T V_t'^T\}. \quad (35)$$

Notice that this estimator produce a greater error covariance than the Kalman filter.

$$P_t \leq \overline{P_t} = O^{-1} \Sigma_V O'^{-1} \quad (36)$$

We have

$$\overline{P_t} = A^T (OA^T)^{-1} \Sigma_V (OA^T)'^{-1} A^T. \quad (37)$$

We will now show that there exists an upper bound $\underline{K}$ for $\|(OA^T)^{-1} \Sigma_V (OA^T)'^{-1}\|$. From (24) and (35), we have that the i,j-th entry of Σ_V is

$$[\Sigma_V]_{i,j} = \sum_{k=1}^{\min\{t_i,t_j\}} CA^{k-t_i-1} Q A'^{k-t_j-1} C' + R\delta(i,j) \quad (38)$$

with $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ if $i \neq j$. Notice that since all eigenvalues of A have magnitude greater that one, there exist an upper bound $vI \geq \overline{\Sigma_V} \geq \Sigma_V$ for all sequences, where $v = \|\overline{\Sigma_V}\|$. It follows that

$$\|(OA^T)^{-1} \Sigma_V (OA^T)'^{-1}\| \leq v \|(OA^T)^{-1} (OA^T)'^{-1}\| \quad (39)$$

Condition W requires that the phases of the eigenvalues of A are rational numbers. Let d_j be the least common multiple of the denominators of the phases of the quasi-equiblock A_j. Let D be the least common multiple of $d_j, \; j = 1, \ldots, J$. Notice that

$$A^D = \operatorname{diag}(\alpha_1^D I_{n_1}, \alpha_2^D I_{n_2}, \ldots, \alpha_J^D I_{n_J}). \quad (40)$$

Also, for any sequence, we have

$$O^{(j)} A_j^T = \begin{bmatrix} C_j A_j^{T-t_1} \\ C_j A_j^{T-t_2} \\ \vdots \\ C_j A_j^{T-t_I} \end{bmatrix} = \Psi_j \tilde{O}^{(j)} \quad (41)$$

with

$$\Psi_j = \operatorname{diag}(\alpha_j^{k_1 D}, \alpha_j^{k_2 D}, \ldots, \alpha_j^{k_I D}) \quad (42)$$

$$\tilde{O}^{(j)} = \begin{bmatrix} C_j A_j^{\tilde{t}_1} \\ C_j A_j^{\tilde{t}_2} \\ \vdots \\ C_j A_j^{\tilde{t}_I} \end{bmatrix} \quad (43)$$

where $\tilde{t}_n$ and k_n, for $n = 1, \ldots, I$ are such that

$$\begin{cases} T - t_n = k_n D + \tilde{t}_n \\ \tilde{t}_n < D. \end{cases} \quad (44)$$

We have $OA^T = \Psi \tilde{O}$, where

$$\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 & \ldots & \Psi_J \end{bmatrix} \quad (45)$$

and

$$\tilde{O} = \operatorname{diag}(\tilde{O}^{(1)}, \tilde{O}^{(2)}, \ldots, \tilde{O}^{(J)}). \quad (46)$$

Notice that there exist a finite set of natural numbers $\tilde{t}_n < D$ and hence a finite set of possible matrices $\tilde{O}$. Let $\delta = \min \|\tilde{O}^{-1}\|^{-1}$ and notice that we have $\delta > 0$ from the assumption that O has FCR. Notice that

$$OA^T (A^T O)' = \Psi \tilde{O} \tilde{O}' \Psi' \quad (47)$$
$$\geq \delta^2 \Psi \Psi' \quad (48)$$
$$= \delta^2 \sum_{j=1}^{J} \Psi_j^2 \quad (49)$$
$$\geq \delta^2 I_n. \quad (50)$$

Hence,

$$\|(OA^T)^{-1} (A^T O)'^{-1}\| \leq \delta^{-2}. \quad (51)$$

With (39), we have

$$\|(OA^T)^{-1} \Sigma_V (A^T O)'^{-1}\| \leq v\delta^{-2} \quad (52)$$

and the result follows making $\underline{K} \geq v\delta^{-2}$ and noting that

$$\|P_t\| \leq \|\overline{P_t}\| \leq \|A^{2T}\| \underline{K} \leq \alpha_\star^{2T} \underline{K}. \quad (53)$$

■

From Lemma 4.1, it follows that

$$\Pr(\|P_t\| \le K_j \alpha_j^{2T}) \le \Pr(\mathcal{M}_T^{(j)}) \quad (54)$$

for all j, and

$$\Pr(\|P_t\| \le \underline{K} \alpha_\star^{2T}) \ge \Pr(\mathcal{M}_T). \quad (55)$$

Then, following the steps in [6], we have that

$$\lim_{t\to\infty} E\{\|P_t\|\} \ \ge \ \max_j \left\{ \tilde{K}_j \sum_{T=0}^{\infty} \alpha_j^{2T} \Pr\left(\overline{\mathcal{M}_T^{(j)}}\right) \right\} \quad (56)$$

$$\lim_{t\to\infty} E\{\|P_t\|\} \ \le \ \tilde{K} \sum_{T=0}^{\infty} \alpha_\star^{2T} \Pr\left(\overline{\mathcal{M}_T}\right), \quad (57)$$

where $\tilde{K}_j = K_j(\alpha_j^2 - 1)$ and $\tilde{K} = \underline{K}(\alpha_\star^2 - 1)$.

The following theorem states a necessary and a sufficient condition for the EEC to be bounded.

Theorem 4.1: If, for some $1 \le j \le J$

$$\alpha_j^2 \limsup_{T\to\infty} \Pr\left(\overline{\mathcal{M}_T^{(j)}}\right)^{1/T} > 1, \quad (58)$$

then

$$\lim_{t\to\infty} E\{\|P_t\|\} = +\infty. \quad (59)$$

Also, if

$$\alpha_\star^2 \limsup_{T\to\infty} \Pr\left(\overline{\mathcal{M}_T}\right)^{1/T} < 1, \quad (60)$$

then

$$\lim_{t\to\infty} E\{\|P_t\|\} \le \mathcal{M}. \quad (61)$$

Proof: From the root convergence test [13], we have that (58) implies that $\tilde{K}_j \sum_{T=0}^{\infty} \alpha_j^{2T} \Pr\left(\overline{\mathcal{M}_T^{(j)}}\right) = \infty$. Hence, (59) follows from (57). Using the same argument, (61) follows from (60) and (56). ∎

B. Systems whose unstable part is non-degenerate

In this section we study the necessary and sufficient conditions given in Theorem 4.1, for systems whose unstable part is non-degenerate. Since nothing further than what is stated in Theorem 4.1 can be said about this kind of systems, we further assume that the network model is Gilbert-Elliot. In this case, we show that both conditions are equivalent, and equal to the condition (12) stated in [7], [12]. To this end, we need to derive expressions for $\limsup_{T\to\infty} \Pr\left(\overline{\mathcal{M}_T}\right)^{1/T}$ and $\limsup_{T\to\infty} \Pr\left(\overline{\mathcal{M}_T^{(j)}}\right)^{1/T}$.

Using the results on generalized Vandermonde determinants [14], it is straightforward to show that for systems with scalar measurements, all the sequences with J or more measurements are contained in the set $\mathcal{M}_T$. This property can be extended to systems with vector measurements, but the argument is omitted here due to space limitations. Let $|\Gamma_t^T|$ denote the number of arrivals in the sequence Γ_t^T. Define

$$W_t = \begin{bmatrix} \Pr(|\Gamma_t^T| = 0) \\ \Pr(|\Gamma_t^T| = 1, \Gamma_t^T(1) = 1) \\ \Pr(|\Gamma_t^T| = 1, \Gamma_t^T(1) = 0) \\ \Pr(|\Gamma_t^T| = 2, \Gamma_t^T(1) = 1) \\ \vdots \\ \Pr(|\Gamma_t^T| = J-1, \Gamma_t^T(1) = 0) \end{bmatrix}, \quad (62)$$

where $\Pr(|\Gamma_t^T| = x, \Gamma_t^T(1) = y)$ is the probability that the sequence Γ_t^T has x measurement arrivals and γ_{t-1} is $y \in \{0,1\}$. Now we define the probability transition matrix M such that $W_{t+1} = MW_t$. We have

$$M = \begin{bmatrix} 1-q & 0 & 0 & 0 & \dots & 0 \\ q & 0 & 0 & 0 & \dots & 0 \\ 0 & p & 1-q & 0 & \dots & 0 \\ 0 & 1-p & q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-q \end{bmatrix}. \quad (63)$$

Since no measurement is available when the Kalman iterations start, we have that $\Pr(\overline{\mathcal{M}_T}) = uM^T z$ with $u = [1\ 1\ 1\ \dots\ 1]$ and $z = [1\ 0\ 0\ \dots\ 0]'$. Now, since M is a lower triangular matrix, its eigenvalues are $1-q$ and 0. Hence, $uM^T z = f(1-q)^T$, for some polynomial f. Hence,

$$\limsup_{T\to\infty} \Pr(\overline{\mathcal{M}_T})^{1/T} = \limsup_{T\to\infty} (uM^T z)^{1/T} \quad (64)$$

$$= 1-q. \quad (65)$$

Also, for all $1 \le j \le J$, $\mathcal{M}_T^{(j)}$ contains all sequences with one or more measurement arrivals. Hence, using an argument similar to the one above, we have

$$\limsup_{T\to\infty} \Pr(\overline{\mathcal{M}_T^{(j)}})^{1/T} = 1-q, \ 1 \le j \le J. \quad (66)$$

Then, from (65), (66) and Theorem 4.1, it follows that

$$(1-q)\alpha_\star^2 < 1 \quad (67)$$

is a necessary and sufficient condition for the boundedness of the EEC, for a Gilbert-Elliot network model. For i.i.d. models, the same condition is valid making $q = \lambda$.

C. Degenerate Systems with one quasi-equiblock

For degenerate systems formed by only one quasi-equiblock, and a general network model, it follows immediately that both conditions in Theorem 4.1 are simultaneously necessary and sufficient. In this case, $\limsup_{T\to\infty} \Pr(\overline{\mathcal{M}_T})^{1/T}$ can be evaluated for any arbitrary stochastic network model, following the approach in [6]. For the particular case of a Gilbert-Elliot network model, and a second order system, this becomes

$$\limsup_{T\to\infty} \Pr(\overline{\mathcal{M}_T})^{1/T} = (1-q)\left(1 + \frac{pq}{(1-q)^2}\right)^{1/d}. \quad (68)$$

Hence, condition (13), derived in [12], is necessary and sufficient. See the example for details on how to compute (68).

V. EXAMPLE

Consider the system (1), whose measurements are transmitted through a network described by the two-state Markov chain in (11). Let

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \exp(2\pi i n/d) \end{bmatrix} \quad (69)$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad (70)$$

where $\alpha \in \mathbb{R}$ and n, d are integers and the fraction n/d is irreducible. Notice that $A^T = \alpha^T I_2$, where I_2 denotes the 2×2 identity matrix. Since the system has only one quasi-equiblock, we have $\Pr\left(\overline{\mathcal{M}_T}\right) = \Pr\left(\overline{\mathcal{M}_T^{(j)}}\right)$. Hence, the necessary and sufficient conditions (58) and (60) are equivalent. Following the method described in [6], we can express the probability to observe a sequence Γ^T such that $O(\Gamma^T)$ does not have FCR, as

$$\Pr(\overline{\mathcal{M}_T}) = uM^T z \tag{71}$$

where

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ p & 1-q & 0 & 0 & 0 & 0 \\ 1-p & q & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 1-q \\ 0 & 0 & p & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{Q} & 0 \end{bmatrix}, \tag{72}$$

with $\mathcal{Q} = (1-q)I_{d-2}$, $u = [1\ 1\ \dots\ 1]$ and $z = [1\ 0\ \dots\ 0]'$. The eigenvalues of M are given by $\{\lambda : \det(\lambda I - M) = 0\}$. To compute this determinant we use its cofactor expansion along the last row. Doing so, we obtain the characteristic equation

$$\lambda^2 \left(\lambda - (1-q)\right) \left(\lambda^d - \left(1 + \frac{pq}{(1-q)^2}\right)(1-q)^d\right) = 0. \tag{73}$$

It follows from (73) that $\{0, 0, (1-q)\}$ are eingenvalues of M. Also, all the other eigenvalues have magnitude $(1-q)\left(1+\frac{pq}{(1-q)^2}\right)^{1/d}$.

As pointed out in [6], $uM^T z$ can be written in a polynomial form as

$$uM^T z = \sum_{l=1}^{2} \Lambda_l^T \Psi_l(T) \tag{74}$$

where $\Lambda_1 = (1-q)\left(1+\frac{pq}{(1-q)^2}\right)^{1/d}$, $\Lambda_2 = 1-q$, and $\Psi_1(T)$ and $\Psi_2(T)$ are polynomials in T of order less than the size of the Jordan blocks associated with eigenvalues with the corresponding magnitude. In this case, since the eigenvalue $1-q$ has multiplicity one, we have $\Psi_2 = K$, where K is a constant independent of T. We will now show that $\Psi_1(T) \neq 0$ by contradiction. Suppose that $\Psi_1(T) = 0$. Then, we have

$$\Pr(\mathcal{M}_1) = uMz = K(1-q) \tag{75}$$

$$\Pr(\mathcal{M}_2) = uM^2 z = K(1-q)^2. \tag{76}$$

The probability of having a sequence $\Gamma_t^T \in \mathcal{M}_T$ is easy to compute using basic probabilities

$$\begin{aligned} \Pr(\mathcal{M}_1) &= \Pr(\Gamma^1 = \{0\} \cup \Gamma^1 = \{1\}) && (77) \\ &= p + (1-p) = 1 && (78) \\ \Pr(\mathcal{M}_2) &= 1 - \Pr(\Gamma^2 = \{1,1\}) && (79) \\ &= 1 - (1-p)^2 = 2p - p^2. && (80) \end{aligned}$$

Putting (75) to (80) together, we have

$$\begin{cases} \Pr(\mathcal{M}_1) = K(1-q) = 1 \\ \Pr(\mathcal{M}_2) = K(1-q)^2 = 2p - p^2 \end{cases} \tag{81}$$

Notice that the solution for (81) is $K = (1-q)^{-1}$ and it requires $q = p^2 - 2p + 1$. The inclusion of $\Pr(\mathcal{M}_3) = K(1-q)^3$ further requires that $2p^4 - 6p^3 + 5p^2 = 0$, for $d > 2$ and $p^4 - 3p^3 + 2p^2 + p = 0$ for $d = 2$. In both cases, none of the solutions allows $0 \le p \le 1$. Hence we conclude that $uM^T z \neq K(1-q)^T$ and therefore $\Psi_1(T) \neq 0$.

Now, since $\Lambda_1 > \Lambda_2$, we have

$$\begin{aligned} \limsup_{T\to\infty} (uM^T z)^{1/T} &= \limsup_{T\to\infty} (\Lambda_1^T \Psi_1(T))^{1/T} && (82) \\ &= \Lambda_1 \limsup_{T\to\infty} (\Psi_1(T))^{1/T} && (83) \\ &= (1-q)\left(1 + \frac{pq}{(1-q)^2}\right)^{1/d} && (84) \end{aligned}$$

Substituting (82) in (58) (or (60)), we obtain the following necessary and sufficient condition for the boundedness of the EEC:

$$\alpha^2 (1-q)\left(1 + \frac{pq}{(1-q)^2}\right)^{1/d} < 1. \tag{85}$$

Notice that this condition is equivalent to (13), reported in [12]. If we consider i.i.d. packet losses, then $q = 1 - p = \lambda$ and the condition (85) becomes

$$\alpha^2 (1-\lambda)^{\frac{d-1}{d}} < 1. \tag{86}$$

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