Proceedings of the 37th IEEE
Conference on Decision & Control
Tampa, Florida USA • December 1998

Robust Stabilization of a Class of Nonlinear Systems with an Up-triangular Structure

Weizhou Su and Minyue Fu
Department of Electrical and Computer Engineering
The University of Newcastle, NSW 2308, Australia

Abstract. In this paper, a class of nonlinear uncertain systems with an *up-triangular* structure is considered and a design method is proposed for robust stabilization via state feedback. The up-triangular structure has been considered in the literature of *forwarding design*. But our method has a unique feature, i.e., we allow uncertain parameters (of large size) to be present in the system. Also, our design method is conceptually different from those available in the literature. When specialized to linear systems, we recover an important result of Wei on quadratic stabilization.

Keywords: Nonlinear systems; Forwarding; Robust stabilization; State feedback.

1 Introduction

This paper studies the problem of robust stabilization for nonlinear systems which have an *up-triangular* structure:

$$\begin{aligned} \dot{x}_1 &= f_1(x_2, \cdots, x_n, q) \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_n, q) \\ \dot{x}_n &= f_n(x_1, \cdots, x_n, q) + u \end{aligned} \tag{1}$$

where $x_1, \cdots, x_n$ are state variables, q is an uncertain parameter vector in a compact set Ω and f_i, $i = 1, \cdots, n$ are smooth functions.

The *up-triangular* system (1) can be constructed recursively via a series of *up-augmentations* (see definition in Sections 2 and 3). The first person studying the up-triangular structure is perhaps Wei [6]. In [6], Wei proved an impressive result for quadratic stabilization of linear systems with this structure and uncertain parameters. A lot of work has also been in the nonlinear case; see, e.g., [5], [1], [2], [3] and [4]. However, all the existing methods for the nonlinear case require precise information of the system.

The purpose of this paper is to present a Lyapunov-based approach for robust stabilization of nonlinear systems with the up-triangular structure. That is, we allow the system to have uncertainty parameters of possibly large size. When specialized to linear systems, we recover the result of Wei [6].

2 Linear Systems

In this section, quadratic stabilization of a class of linear systems with the up-augmented structure will be discussed. This is the same problem that Wei considered in [6]. However, Wei's approach seems hard to be extended to nonlinear systems. Hence, a new proof will be given in this section for his quadratic stabilization result. This proof provides an explicit formula for constructing Lyapunov functions and controllers, serving the key to the nonlinear case in Section 3.

Up-Augmentation

Consider the uncertain linear system

$$\dot{x} = A(q)x + b(q)u \tag{2}$$

where $x \in \Re^n$ is the state, $u \in \Re$ is control input, q is an uncertain parameter vector belonging to a compact set Q, $A(\cdot)$ and $b(\cdot)$ are continuous functions. We will refer to (2) as the *base system.*

An *up-augmented* system has the following structure

$$\begin{aligned} \dot{x}_0 &= a(q)x \\ \dot{x} &= A(q)x + b[u + d(q)x_0] \end{aligned} \tag{3}$$

where x_0 is an extra state variable, $a(\cdot)$ and $b(\cdot)$ are continuous functions.

Assumption 2.1 (*System Structure*):

$$A(q) = \begin{bmatrix} 0 & A^-(q) \\ d_{n1}(q) & \star \end{bmatrix} \tag{4}$$

$$b(q) = \theta_n(q) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \theta_n(q) b \tag{5}$$

and

$$a(q) = [\ \theta_0(q) \quad \star\ ] \tag{6}$$

where $A^-(q) \in \Re^{(n-1)\times(n-1)}$, $d_{n1}(q) \in \Re$, $\star$ is an arbitrary uncertain term, $1 \geq \theta_0(q) \geq \underline{\theta}_0 > 0$ and $1 \geq \theta_n(q) \geq \underline{\theta}_n > 0$, $\forall q \in Q$. □

Assumption 2.2 (*Quadratic Stabilizability of the Base System*): There exist a symmetric and positive-definite matrix $P \in \Re^{n\times n}$ and a constant $\varepsilon > 0$ such that

$$PA(q) + A^T(q)P \leq -\varepsilon I, \qquad \forall q \in Q \tag{7}$$

(see [6] for definition of quadratic stabilizability). □

Remark 2.1 Although the uncertain parameter vector q is restricted to be constant in (2) and (3), the result in this section applies even when q is an arbitrary time varying function provided $q(t) \in Q$, $\forall t \in [0, \infty)$. This is the nature of quadratic stabilization and this comment applies to all the results in this paper. □

Remark 2.2 The continuity assumption on $A(\cdot)$, $b(\cdot)$ and $a(\cdot)$ is for convenience only and can be relaxed. □

Lyapunov Function and Controller Design

The key to quadratic stabilization of the up-augmented system (3) is to construct a suitable Lyapunov function and stabilizer. This is where our approach differs from Wei [6] where only *stabilizability* is addressed and no Lyapunov function is directly provided. Consider the following Lyapunov function candidate for (3):

$$V^+(x_0, x) = [x_0 - (\gamma \ \ 0)Px]^2 + x^TPx \tag{8}$$

where the scalar $\gamma < 0$ is to be specified.

Defining $x^+ = (x_0, \ x^T)^T$ and

$$P^+ = \begin{bmatrix} 1 & -(\gamma \ \ 0)P \\ -P\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & P + P\begin{pmatrix} \gamma \\ 0 \end{pmatrix}(\gamma \ \ 0)P \end{bmatrix} \tag{9}$$

then the Lyapunov function candidate in (8) becomes

$$V^+(x^+) = (x^+)^TP^+x^+. \tag{10}$$

Before discussing the stabilization of the system (3) with our new approach, we shall give a lemma.

Lemma 2.1 Suppose $s \neq 0$, γ be a constant and $P = P^T > 0$. Denote

$$P^+ = \begin{bmatrix} 1 & -(\gamma \ \ 0)P \\ -P\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & sP + P\begin{pmatrix} \gamma \\ 0 \end{pmatrix}(\gamma \ \ 0)P \end{bmatrix}.$$

Then the inverse of P^+ is

$$S^+ = \begin{bmatrix} 1 + s^{-1}(\gamma \ \ 0)P\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & s^{-1}(\gamma \ \ 0) \\ s^{-1}\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & s^{-1}P^{-1} \end{bmatrix}.$$

The proof is straightforward and thus omitted.

Theorem 2.1. For the up-augmented system (3) satisfying Assumptions 2.1-2.2, there exist $\gamma < 0$, $\alpha > 0$ and $\varepsilon^+ > 0$ such that the linear controller

$$u(t) = -\alpha(z^+)^Tb^+ = -\alpha z^Tb \tag{11}$$

will render

$$\dot{V}^+(x^+) \leq -\varepsilon^+V^+(x^+), \quad \forall x^+ \in \Re^{n+1}. \tag{12}$$

Moreover, the following choice of γ, α and ε^+ will suffice :

$$\begin{aligned} & 0 < \bar{\varepsilon} < \bar{\varepsilon}_{max} \\ & = \min_{q\in Q} \lambda_{min}\left[-P^{-1}\left(A^T(q)P + PA(q)\right)P^{-1}\right], \quad (13) \\ & \gamma < \gamma_{max} = \min_{q\in Q} \frac{1}{2\theta_0(q)}\left[a(q)\left(A^T(q)P + PA(q)\right.\right. \\ & \qquad \left.\left.+\bar{\varepsilon}P^2\right)^{-1}a^T(q) - \bar{\varepsilon}\right], \quad (14) \end{aligned}$$

$$\varepsilon^+ = \frac{1}{2}\bar{\varepsilon}\lambda_{min}(P^+) \quad and \quad \alpha = \frac{\delta^2}{\bar{\varepsilon}} \tag{15}$$

where

$$\delta = \max_{q\in Q}\left\|[d(q) \ \ 0](P^+)^{-1} + \begin{bmatrix} \frac{\gamma d_{n1}(q)}{\theta_n(q)} & 0 \end{bmatrix}\right\| \tag{16}$$

and

$$z^+ = (z_0, \ z^T)^T = P^+x^+. \tag{17}$$

Proof : The derivative of $V^+(x^+)$ along the trajectory of the system (3) is

$$\begin{aligned} \dot{V}^+ &= 2[x_0 - (\gamma \ \ 0)Px][\dot{x}_0 - (\gamma \ \ 0)P\dot{x}] + 2x^TP\dot{x} \\ &= 2(x^+)^TP^+\dot{x}^+. \end{aligned} \tag{18}$$

Thus, from system (3), equation (18) can be rewritten as

$$\begin{aligned} \dot{V}^+ &= (x^+)^T\left[P^+A^+(q) + A^+(q)^TP^+\right]x^+ \\ &\quad +2(x^+)^TP^+b^+(q)[u + d(q)x_0]. \end{aligned} \tag{19}$$

where

$$A^+ = \begin{bmatrix} 0 & a(q) \\ 0 & A(q) \end{bmatrix}; \quad b^+ = \begin{pmatrix} 0 \\ b(q) \end{pmatrix}. \tag{20}$$

Denoting $S^+ = (P^+)^{-1}$ and applying the Lemma 2.1, it holds

$$\begin{aligned} & A^+(q)S^+ + S^+A^+(q)^T \\ & = \begin{bmatrix} 2a(q)\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & a(q)P^{-1} + (\gamma \ \ 0)A^T(q) \\ P^{-1}a^T(q) + A(q)\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & P^{-1}A^T(q) + A(q)P^{-1} \end{bmatrix} \end{aligned} \tag{21}$$

With the transformation (17), (19) and (21) lead to

$$\begin{aligned}\dot{V}^+ &= (z^+)^T\left[A^+(q)S^+ + S^+A^+(q)\right]z^+ \\ &\quad +2(z^+)^T b^+(q)[u+d(q)x_0] \\ &= (z^+)^T\begin{bmatrix} 2\theta(q)\gamma & a(q)P^{-1} \\ P^{-1}a^T(q) & P^{-1}A^T(q)+A(q)P^{-1}\end{bmatrix} z^+ \\ &\quad +2z^TA(q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} z_0 + 2(z^+)^Tb^+(q)[u+d(q)x_0].\end{aligned} \tag{22}$$

The definition of $\bar{\varepsilon}$ and γ in (13) and (14) guarantees

$$\begin{bmatrix} 2\theta_0(q)\gamma & a(q)P^{-1} \\ P^{-1}a^T(q) & P^{-1}A^T(q)+A(q)P^{-1}\end{bmatrix} < -\bar{\varepsilon}I \tag{23}$$

because by Schur complement, (23) is equivalent to

$$P^{-1}A^T(q)+A(q)P^{-1}+\bar{\varepsilon}I < 0$$

and

$$\begin{aligned}2\theta_0(q)\gamma + \bar{\varepsilon} - a(q)P^{-1}[P^{-1}A^T(q)+A(q)P^{-1} \\ +\bar{\varepsilon}I]^{-1}P^{-1}a^T(q) < 0\end{aligned}$$

which hold when (13)-(14) do.

On the other hand, with the structure of the matrix $A(q)$ and the vector $b(q)$ in Assumption 2.1, we can easily get

$$A(q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} = \gamma\frac{d_{n1}(q)}{\theta_n(q)}b(q).$$

Also note that $(z^+)^Tb^+ = z^Tb$. Therefore, from the above augments and (22), we have

$$\dot{V}^+ \le -\bar{\varepsilon}^+(z^+)^T z^+ + 2(z^+)^Tb^+(q)\left[u+\bar{d}(q)z^+\right] \tag{24}$$

where

$$\bar{d}(q)z^+ = d(q)x_0 + \gamma\frac{d_{n1}(q)}{\theta_n(q)}z_0$$

and

$$\bar{d}(q) = [d(q) \quad 0](P^+)^{-1} + \left[\gamma\frac{d_{n1}(q)}{\theta_n(q)} \quad 0\right].$$

Using (16) and triangular inequality, we have

$$2(z^+)^Tb^+(q)\bar{d}(q)z^+ \le \frac{2\delta^2}{\bar{\varepsilon}}((z^+)^Tb^+(q))^2 + \frac{\bar{\varepsilon}}{2}(z^+)^T z^+. \tag{25}$$

Applying (11), (15), (25) and the fact $1 \ge \theta_n(q) \ge \underline{\theta}_n > 0$, (24) becomes

$$\begin{aligned}\dot{V}^+ &\le -\frac{\bar{\varepsilon}}{2}(z^+)^T z^+ + \frac{2\delta^2}{\bar{\varepsilon}}((z^+)^Tb^+(q))^2 + 2((z^+)^Tb^+(q))u \\ &= -\frac{\bar{\varepsilon}}{2}\left((P^+)^{1/2}x^+\right)^T P^+\left((P^+)^{1/2}x^+\right) \\ &\quad +\frac{2\delta^2}{\bar{\varepsilon}}\theta_n^2(q)\left(z^Tb\right)^2 - \frac{2\delta^2}{\bar{\varepsilon}}\theta_n(q)\left(z^Tb\right)^2 \\ &\le -\varepsilon^+V^+(x^+).\end{aligned} \tag{26}$$

Hence, the system (3) is quadratically stabilized. ∇∇∇

3 Nonlinear Systems

In this section we generalize Theorem 2.1 to nonlinear systems up-augmneted from a base system. The key mechanism involved is a two-step control law. In the first step, a nonlinear controller is applied to the base system so that its state converges to a "small" bounded set Ω while the state of the up-augmentation part is not regulated. In the second step, another nonlinear controller is designed to maintain the state of the base system within Ω while reducing the state of the up-augmented part. When the state of the up-augmented system becomes sufficiently small, the second controller will then be able to drive the combined state to zero. Overall, this two-step control law achieves robust global asymptotic stabilization (RGAS) and robust local exponential stabilization (RLES).

Up-Augmentation

As in the linear case, we start with a base system

$$\dot{x}(t) = f(x(t),q) + b(q)u(t) \tag{27}$$

where $x(t)$, $u(t)$, q and $b(q)$ are the same as before, $f(x,q)$ is continuous in q and smooth in x with $f(0,q)=0$.

The up-augmented system is given by

$$\begin{aligned}\dot{x}_0 &= f_0(x,q) \\ \dot{x} &= f(x,q) + b(q)[u + d(x,x_0,\eta,q)]\end{aligned} \tag{28}$$

where $x_0 \in \Re$, $f_0(x,q)$ is continuous in q and smooth in x with $f_0(0,q)=0$, $d(x,x_0,\eta,q)$ is continuous in q and smooth in (x,x_0,η) with $d(0,0,0,q)=0$. The parameter η reprensents the state variables other than x_0, x if (28) is a subsystem of a large system. We denote $x^+ = [x_0, x^T]^T$ and $d(x^+,\eta,q) = d(x,x_0,\eta,q)$.

Assumption 3.1 (*Local Smoothness Properties*): There is a local region Ω_n such that, for any $x \in \Omega_n$, we can express

$$\begin{aligned}&f_0(x,q) = a(x,q)x; \quad f(x,q) = A(x,q)x \\ &b(q) = \theta_n(q)\begin{pmatrix}0 \\ 1\end{pmatrix} = \theta_n(q)b\end{aligned} \tag{29}$$

with

$$\begin{aligned}A(x,q) &= \begin{bmatrix} 0 & A^-(x,q) \\ d_{n1}(x,q) & \star\end{bmatrix} \\ a(x,q) &= [\theta_0(x,q) \quad \star]\end{aligned} \tag{30}$$

and $1 \ge \theta_0(x,q) \ge \underline{\theta}_0 > 0$; $1 \ge \theta_n(x,q) > \underline{\theta}_n$, $\forall x \in \Omega_n$, $\forall q \in Q$. □

Assumption 3.2 There exists a constant matrix $P = P^T > 0$ and $\varepsilon > 0$ such that

$$PA(x,q) + A^T(x,q)P \le -\varepsilon I, \quad \forall x \in \Omega_n, q \in Q. \tag{31}$$

Remark 3.1 Assumptions 3.1-3.2 imply that, when $u \equiv 0$, the system of (27) is RLES, i.e., there exist $\mu > 0$ and some (possibly different) $\varepsilon > 0$ such that the local Lyapunov function $V(x) = x^T P x$ satisfies the following property:

$$\frac{\partial V}{\partial x} f(x,q) \leq -\varepsilon V(x), \qquad \forall x \in \Omega. \tag{32}$$

where

$$\Omega = \{x : \; x^T P x < \mu\} \subset \Omega_n. \tag{33}$$

Assumption 3.3 (*Global Properties*): The following system, driven from (28),

$$\dot{x} = f(x,q) + b(q)[u + d(x^+, \eta, q)] \tag{34}$$

has the following property: Given any continuous function $\eta(t)$ and $0 < \rho < 1$, there exists a smooth controller $u_n(x^+, \eta)$ such that, with

$$u(t) = u_n(x^+(t), \eta(t)), \tag{35}$$

the state of the system (34) will be driven into $\rho\Omega$ in a finite time for some $0 < \rho < 1$, where Ω is given in (33), and $\rho\Omega = \{\rho x : \; x \in \Omega\}$. □

Remark 3.3 Assumption 3.3 may appear to be strong. However, we note that it is automatically satisfied for first order systems because a "high-gain" $u(t)$ can be designed to "overcome" both $f(x,q)$ and $d(x^+, \eta, q)$, forcing the state to converge to $\rho\Omega$. In next section, we will show that this property can be preserved in the process of up-augmentation. □

Lyapunov Function and Controller Design

Now we pay attention to controller design for (28). First, we utilize Assumption 3.3 and apply (35) to drive $x(t)$ into $\rho\Omega$ in a finite time T. In this step, $x_0(t)$ is not regulated. Once $x(t) \in \rho\Omega$, we switch to a local mode where a different controller $u^+(x^+, \eta)$ is applied. This controller will maintain $x(t)$ in Ω while driving $x^+(t)$ to zero. The design of $u^+(x^+, \eta)$ relies on a local Lyapunov function for (28)

$$V^+(x^+) = (x_0 - (\gamma \;\; 0)Px)^2 + \ln \frac{\mu}{\mu - x^T P x} > 0, \qquad \forall x \in \Omega \tag{36}$$

where $\gamma < 0$ is a constant to be specified. Note that this Lyapunov function is non-quadratic. However, as $x \to 0$, $V^+(x^+)$ becomes quadratic in x^+. We also note that the $\ln(\cdot)$ resembles a "potential barrier" and this Lyapunov function is valid only for $x \in \Omega$, i.e.,

$$V^+(x^+) \to \infty \quad as \quad x^T P x \to \mu \tag{37}$$

This implies that future $x \in \Omega$ as long as that $V^+(x^+)$ remains bounded.

Differentiating $V^+(x^+(t))$ along the trajectory of (28), we have

$$\dot{V}^+ = 2(x_0 - (\gamma \;\; 0)Px)(\dot{x}_0 - (\gamma \;\; 0)P\dot{x}) + \frac{2x^T P \dot{x}}{\mu - x^T P x}.$$

Defining $s(x) := \frac{1}{\mu - x^T P x}$, above equation can be rewritten as

$$\dot{V}^+ = 2(x^+)^T P^+ \dot{x}^+ \tag{38}$$

where

$$P^+ = \begin{bmatrix} 1 & -(\gamma \;\; 0)P \\ -P\begin{pmatrix} \gamma \\ 0 \end{pmatrix} & s(x)P + P\begin{pmatrix} \gamma \\ 0 \end{pmatrix}(\gamma \;\; 0)P \end{bmatrix}. \tag{39}$$

Further define

$$z^+ = (z_0, \; z^T)^T = P^+ x^+. \tag{40}$$

To simplify the analysis, we start with the case in which the system (28) does not involve with the function η, i.e., $d(x^+, \eta, q) = d_1(x^+, q)$. Since $d_1(x^+, q)$ is smooth in x^+ and $d_1(0, q) = 0$, we can rewrite

$$d_1(x^+, q) = D^+(x, q)x^+ = D^+(x^+, q)S^+ z^+ \tag{41}$$

for some smooth $D^+(x, q)$.

Theorem 3.1 For the up-augmented system (28) satisfying Assumptions 3.1-3.3 and $d(x^+, \eta, q) = d_1(x^+, q)$, there exist $\gamma < 0$ and $\alpha(x^+) > 0$ such that the nonlinear controller

$$u(x^+) = -\alpha(x^+)(z^+)^T b^+ = -\alpha(x^+) z^T b \tag{42}$$

will render

$$\dot{V}^+(x^+) \leq -\varepsilon^+(x) V^+(x^+), \quad \forall x \in \Omega, x_0 \in \Re \tag{43}$$

for some constant $\varepsilon^+(x) > 0$, $x \in \Omega$. Moreover, the following choice of γ, $\alpha^+(x^+)$ and ε^+ will suffice :

$$0 < \bar{\varepsilon} < \bar{\varepsilon}_{max} = \min_{q \in Q; x \in \Omega} \lambda_{min} \left[-P^{-1} \left(A^T(x,q)P + PA(x,q) \right) P^{-1} \right], \tag{44}$$

$$\gamma < \gamma_{max} = \min_{q \in Q; \; x \in \Omega} \frac{1}{2\theta_0(q)} \left[a(x,q) \left(A^T(x,q)P + PA(x,q) + \bar{\varepsilon} P^2 \right)^{-1} a^T(x,q) - \bar{\varepsilon} \right], \tag{45}$$

$$\varepsilon^+(x) = \frac{\bar{\varepsilon}}{2} \lambda_{min} \left(s^{-1}(x) P^+(x) \right) > 0, \tag{46}$$

$$\alpha(x^+) = s^{-1}(x) \frac{\delta^2(x^+)}{\bar{\varepsilon}} \tag{47}$$

where $\delta(x^+)$ is any smooth function satisfying

$$\delta(x^+) \le \max_{q\in Q; x\in\Omega} \left\| D(x^+,q)S^+(s) + \gamma \frac{d_{n1}(x,q)}{\theta_n(q)} \right\|. \tag{48}$$

Proof : Using Assumptions 3.1-3.2 (see Remark 3.1), we obtain

$$\begin{aligned}\dot{V}^+ &= (x^+)^T \left[P^+A^+(x,q) + A^+(x,q)^T P^+\right] x^+ \\ &\quad +2(x^+)^T P^+ b^+(q)[u + d_1(x^+,q)] \end{aligned}\tag{49}$$

where $A^+(x,q)$ is defined as

$$A^+(x,q) = \begin{bmatrix} 0 & a(x,q) \\ 0 & A(x,q) \end{bmatrix}.$$

With the inverse matrix S^+ of P^+ in Lemma 2.1, it holds

$$\begin{aligned} & A^+(x,q)S^+ + S^+A^+(x,q)^T \\ &= s^{-1}(x) \begin{bmatrix} 2a(x,q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} & a(x,q)P^{-1} + (\gamma\ \ 0)A^T(x,q) \\ P^{-1}a^T(x,q) + A(x,q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} & P^{-1}A^T(x,q) + A(x,q)P^{-1} \end{bmatrix}. \end{aligned}\tag{50}$$

It follows from (49) and (50) that

$$\begin{aligned}\dot{V}^+ &= s^{-1}(x)\left(z^+\right)^T \begin{bmatrix} 2\theta_0(x,q)\gamma & a(x,q)P^{-1} \\ P^{-1}a^T(x,q) & P^{-1}A^T(x,q) + A(x,q)P^{-1} \end{bmatrix} z^+ \\ &\quad + 2s^{-1}(x)z^T A(x,q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} z_0 \\ &\quad + 2\left(z^+\right)^T b^+(q)[u + d_1(x^+,q)]. \end{aligned}\tag{51}$$

As in the linear case, the choice of $\bar{\varepsilon}$ and γ in (44) and (45) assures, for $\forall q \in Q$ and $x \in \Omega$,

$$\begin{bmatrix} 2\theta_0(x,q)\gamma & a(x,q)P^{-1} \\ P^{-1}a^T(x,q) & P^{-1}A^T(x,q) + A(x,q)P^{-1} \end{bmatrix} \le -\bar{\varepsilon} I.$$

Also, with Assumption 3.1 and $(z^+)^T b^+ = z^T b$, we have

$$\begin{aligned} z^T A(x,q)\begin{pmatrix}\gamma \\ 0\end{pmatrix} &= z^T \gamma d_{n1}(x,q) b \\ &= \gamma \frac{d_{n1}(x,q)}{\theta_n(q)} (z^+)^T b^+(q). \end{aligned}$$

Therefore, from the above discussion and (51), we have

$$\begin{aligned}\dot{V}^+ &\le -s^{-1}(x)\bar{\varepsilon}\left(z^+\right)^T z^+ + 2s^{-1}(x)(z^+)^T b^+(q) \\ &\quad \times \left[s(x)u + \gamma\frac{d_{n1}(x,q)}{\theta_n(q)} z_0 + d_1(x^+,q)\right]. \end{aligned}\tag{52}$$

Using (48), we have

$$\left\| D(x^+,q)S^+(s) + \gamma \frac{d_{n1}(q)}{\theta_n(q)} \right\| \le \delta(x^+)\|z^+\|. \tag{53}$$

Then substituting (53), (47), (48) and the controller (42) into (52) results in

$$\begin{aligned}\dot{V}^+ &\le -s^{-1}(x)\frac{\bar{\varepsilon}}{2}(z^+)^T z^+ \\ &\le -s^{-1}(x)\frac{\bar{\varepsilon}}{2}\left(P^{+1/2}x^+\right)^T \left(s^{-1}(x)P^+\right)\left(P^{+1/2}x^+\right) \\ &\le -\varepsilon^+(x^+)V(x^+). \end{aligned}\tag{54}$$

In particular, $\varepsilon^+(0) > 0$. Hence, we have RGAS and RLES for the closed-loop system (28). ∇∇∇

4 Nested Up-Augmentation

When an up-augmented system (28) is a subsystem of a large system such as (1), the up-augmented system may involve state variables, denoted by η, from other parts of the large system. We assume that η is available for control and we note that it appears in Assumptions 3.1-3.3. In this section, we will show that, under the Assumptions 3.1-3.3, there exists a robust controller such that the up-augmentation process preserves these assumptions. Thus, robust stabilization of (1) can be done recursively.

Since $d(x^+,\eta,q) = 0$ as $(x^+,\eta) = 0$, the function $d(x^+,\eta,q)$ can be decomposed as

$$d(x^+,\eta,q) = d_1(x^+,q) + d_2(x^+,\eta,q)\eta \tag{55}$$

where $d_1(x^+,q) = 0$ as $x^+ = 0$.

Consider the following controller

$$u(x^+,\eta) = u_1(x^+) + u_2(x^+,\eta). \tag{56}$$

Applying Theorem 3.1, design the controller $u_1(x^+) = u(x^+)$ in (42). Then, the system

$$\begin{aligned} \dot{x}_0 &= f_0(x,q) \\ \dot{x} &= f(x,q) + b(q)[u_1(x^+) + d_1(x^+,q)] \end{aligned}\tag{57}$$

is RGAS and RLES.

Since the $f_0(x,q)$, $f(x,q)$ and $b(q)[u_1(x^+) + d_1(x^+,q)]$ are smooth functions, the system (57) will satisfy the Assumption 3.1. Denote $A_c^+(x^+,q)$ such that

$$\begin{bmatrix} f_0(x,q) \\ f(x,q) + b(q)[u_1(x^+) + d_1(x^+,q)] \end{bmatrix} = A_c^+(x^+,q)x^+ \tag{58}$$

where the matrix $A_c^+(x^+,q)$ has following structure

$$A_c^+(x^+,q) = \begin{bmatrix} 0 & a^+(x,q) \\ \begin{pmatrix} 0 \\ \star \end{pmatrix} & A_c(x^+,q) \end{bmatrix}.$$

Theorem 4.1 For the system (57), there exist a constant matrix P_0^+ and positive constants ε_0^+ and μ^+

$$P_0^+ = \begin{bmatrix} 1 & -(\gamma\ 0)P \\ -P\begin{pmatrix}\gamma \\ 0\end{pmatrix} & s_0P + P\begin{pmatrix}\gamma \\ 0\end{pmatrix}(\gamma\ 0)P \end{bmatrix} \quad (59)$$

where $s_0 > s(0) > 0$, such that

$$P_0^+ A_c^+(x^+, q) + \left(A_c^+(x^+, q)\right)^T P_0^+ \le -\varepsilon_0^+ I \quad (60)$$

for $x^+ \in \Omega^+ = \{x^+ : (x^+)^T P_0^+ x^+ < \mu^+\}$.

Proof: Assuming that the variable x^+ in the matrix $A_c^+(\cdot, q)$ is independent from the system state x^+, from the process of proof in Theorem 3.1, we have

$$P^+(s)A_c^+(x^+, q) + A_c^{+T}(x^+, q)P^+(s) \le -\varepsilon^+(x)I. \quad (61)$$

Denoting $\Delta s = s_0 - s(x)$, $\Delta\varepsilon = \varepsilon(x) - \varepsilon_0$ and applying (61), it follows that

$$\begin{aligned} & P^+(s_0)A_c^+(x^+, q) + \left(A_c^+(x^+, q)\right)^T P^+(s_0) \\ &= P^+(s)A_c^+(x^+, q) + \left(A_c^+(x^+, q)\right)^T P^+(s) \\ &+ \Delta s \begin{bmatrix} 0 & a^+(x, q)P \\ P(a^+(x, q))^T & PA_c(x^+, q) + A_c^T(x^+, q)P \end{bmatrix} \\ &\le -\varepsilon_0^+ I + \Delta\varepsilon I \\ &+ \Delta s \begin{bmatrix} 0 & a^+(x, q)P \\ P(a^+(x, q))^T & PA_c(x^+, q) + A_c^T(x^+, q)P \end{bmatrix}. \end{aligned} \quad (62)$$

Noting that Δs, $\Delta\varepsilon$ and

$$\lambda_{max} \begin{bmatrix} 0 & a^+(x^+, q)P \\ P(a^+(x^+, q))^T & PA_c(x^+, q) + A_c^T(x^+, q)P \end{bmatrix}$$

are smooth functions of x^+ and $\varepsilon(x) < 0$, $\forall x \in \rho\Omega$ and $0 < \rho < 1$, there exist constants s_0, ε_0^+ and ν such that, when $\|x^+\| \le \nu$, Δs, $\Delta\varepsilon$ are small enough and the right side of (62) will be negative. Hence, choosing $\mu^+ = \nu / \lambda_{max}(P^+(s_0))$ result in (60). ∇∇∇

Theorem 4.2 Suppose that the up-augmented system (28) satisfies Assumptions 3.1-3.3. Given $\beta > 0$, the controller

$$u(t) = u_1(x^+) + u_2(x^+, \eta, \beta) \quad (63)$$

will locally render

$$\dot{V}^+(x^+) \le -\varepsilon^+ V^+(x^+) + \beta \quad (64)$$

for $\varepsilon^+ > 0$ where

$$u_2(x^+, \eta, \beta) = -\frac{1}{\beta} s^{-1}(x)(1 + \eta^2)(z^+)^T b^+ \delta_2^2(x^+, \eta) \quad (65)$$

and

$$\delta_2(x^+, \eta) > |d_2(x^+, \eta, q)|, \ q \in \Omega. \quad (66)$$

Proof: From (55), (52) and Theorem 3.1 , there holds

$$\begin{aligned} \dot{V}^+(x^+) \ &\le -\varepsilon^+ V^+(x^+) + 2s^{-1}(x) z^{+T} b^+ \\ &\times [s(x)u_2(\eta)(x^+, \eta, \beta) + d_2(x^+, \eta, q)\eta]. \end{aligned} \quad (67)$$

Applying the triangular inequality and the controller (65) leads (64). ∇∇∇

Remark 4.1 Theorems 4.1 and 4.2 imply that, under Assumptions 3.1-3.3, the controller (56) will preserve the properties in these assumptions for the up-augmented system. □

5 Conclusions

In this paper, we have generalized a quadratic stabilization result of Wei [6] to nonlinear systems. Our results allow us to robustly stabilize a class of uncertain nonlinear systems with an *up-triangular* structure, which are generated via a series of up-augmentations. The unique feature of our design method in comparison with methods given in [1]-[5] is that our controller is robust against uncertainties of large size. For uncertain linear systems, the result of Wei [6] is recovered.

References

[1] M. Jankovic, R. Sepulchre and P. V. Kokotovic, "Constructive Lyapunov stailization of nonlinear cascade systems". *IEEE Trans. Aut. Contr*, **AC-41**, (1996) 1723-1735.

[2] M. Jankovic, R. Sepulchre and P. V. Kokotovic, "Global adaptive stabilization of casade nonlinear systems". *Automatica*, **Vol. 33**, (1997) 263-268.

[3] F. Mazenc and L. Praly, "Adding integrations, saturated controls, and stabilization for feedforwarding systems". *IEEE Trans. Aut. Contr*, **AC-41**, (1996) 1559-1578.

[4] R. Sepulchre, M. Jankovic and P. V. Kokotovic, "Interlaced system and recursive designs for global stablization".(1996)

[5] A.Tell, "Feedback stabilization: Nonlinear solutions to inherently nonlinear problems," Memorandum no. UCB/ERL M92/65, June 12, 1992.

[6] K. Wei, "Quadratic stabilizability of linear systems with structural independent time-varying uncertainties". *IEEE Trans. Aut. Contr*, **AC-35**, (1990) 268-277.