# Capacity of MIMO Wireless Systems, with Spatially Correlated Receive Elements

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***Abstract*— We present a general channel capacity model for a random A.W.G.N. channel in the presence of spatially correlated interference. We consider the case where the channel is random, and full rank and the number of receive elements is allowed to increase indefinitely within a given volume. We show that capacity of the channel does not increase indefinitely for densely packed elements. Moreover, the limit is governed by the correlation properties of the noise and the channel and as such, a power constraint on the receive elements is unnecessary. We examine two cases: the i.i.d. noise case and the spatially correlated noise case and show that bounds exist on the capacity in each case. Monte Carlo simulations are used to verify the results.**

## I. Introduction

The work of [1], [2] has lead to the concept of a high capacity MIMO wireless channel, where capacity increases linearly in proportion to the minimum number of transmit and receive elements. The assumptions required for this remarkable capacity growth are that the MIMO channel is independent.

More recently, there has been some work suggesting that the linear growth is bounded. The work of [3] and [4] has shown that for specific scattering geometries, the linear growth diminishes as the channel becomes correlated. This has placed limits on point-to-point MIMO channel capacity when the channel is over a large distance.

Further work [5] has shown that an intuitive limit on the channel capacity growth must exist, even in the presence of so-called "rich scattering" as depicted in [6]. The intuitive limit is developed from the concept that for a (small) fixed volume, the total power received by an antenna array cannot increase beyond some finite limit. This observation seems to contradict normal array signal processing theory where the received power for a linear array is allowed to increase proportionally with the number of receive elements, even for a finite transmit power [7]. Other authors [8], [9] have also addressed the problem of "dense" receiver elements, however the results suffer from an assumption that as receiver elements become close, the noise signals received will remain independent.

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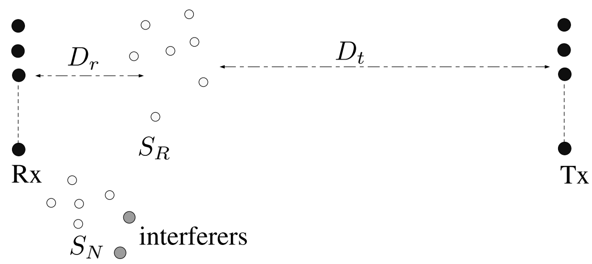


Fig. 1. Physical arrangement

We present an antenna model which includes the *effective receive area* of the antenna. Following classic electromagnetic theory we show that the received power is proportional to this effective area [10] The area model is piece-wise linear. We also conjecture that as many receivers are added, interference sources will have a spatial arrangement. As such, a component of noise will be passed through a transfer function dependent on the array.

This paper is organized as follows: Section II presents an antenna model based on the effective area of the receiver. This provides a piecewise linear model of total power received. Section III develops the capacity model of the channel, given the power limit and the correlation of the received signal and interference. Section IV provides Monte-Carlo simulation results, and V summarizes our results.

## II. Scattering model

Consider the physical arrangement described in figure 1. A group of $N_R$ receivers, arranged in a linear array receive signals from a group of $N_T$ transmitters, passed through a set of randomly arranged local scatterers $S_R$.

Additionally, the receivers experience interference from a set of distant white sources $e$. This model reflects the case for densely packed receivers, where the continuous nature of the electromagnetic field means that noise sources will begin to exhibit a spatial correlation. The interference signals pass through a set of local scatterers $S_N$, where $S_N$ may be different from $S_R$. Both sets of scatterers are assumed to be randomly placed, near the receive elements, within a radius $D_r$. This is a "local scatter" model, and corresponds to scatterers having small reflection coefficients. It is assumed that the transmit signals will have passed through other (non-local) scattering before reaching $S_R$. In this case we may assume that the transfer matrix between transmitters and receivers is Rayleigh distributed.

If we denote the received signal vector over all receivers as $y$, the transmit signal vector as $x$ and the interference signal as $e$ we may write:

$$y = H_T x + H_N e + w \tag{1}$$

where $w$ represents the independent noise at each receiver element. Since both $e$ and $w$ are i.i.d. random gaussian noise, we shall use a scale $k$ to adjust the dominant noise between independent gaussian, and spatially correlated noise. We will use the channel model:

$$y = H_T x + [(1-k)H_N + kI_{N_R}]\, e \tag{2}$$

Where $0 \leq k \leq 1$.

With the above assumptions on the scattering channel, we may consider $H_T$ in terms two components: $H_{T1}$ which models the random effects of the scatterers and $H_{T2}$ which models the correlation caused by the closely spaced receive elements. We can see that $H_{T1}$ and $H_{T2}$ are independent. Similarly, we may decompose the interference transfer matrix $H_N$ into components $H_{N1}$ and $H_{N2}$. It remains to show how the elements of these four matrices are formed.

An antenna receives power proportional to its effective area $A_{\text{eff}}$, and the distance between itself and the source $D$. [10]

$$\frac{P_{\text{receive}}}{P_{\text{transmit}}} = \frac{A_{\text{eff}}}{4\pi D^2} \tag{3}$$

If we assume the receiver is a half-wave antenna then we may write the effective area (for spatially independent elements) in terms of its wavelength $\lambda$ as:

$$A_0 = \frac{\alpha}{2}\lambda^2 \quad \alpha \approx 0.26 \tag{4}$$

For some scalar constant $\alpha$. We may use the concept of effective area for both the scatterers and the receive elements. Each scatterer will have an effective area which is a fraction of the total area covered by the scatter "cloud" [3], [4]. We may assume that each scatterer occupies an equal area within the cloud (receives equal power).

If we denote the area of the cloud as $A_{\text{cloud}}$ then for each scatter $s$ within a particular scattering cloud, we may write a transfer function of the received signal, from a transmit element $\tau$ to scatterer $s$ as $H_s$. We will denote $(\cdot)^*$ as the Hermitian (conjugate transpose) of a matrix.

$$\left[H_{S1}\right]_{s,\tau} = \sqrt{\frac{A_{\text{cloud}}}{N_S 4\pi D_{s,\tau}^2}}\, g_{s,t} \exp\left(-\iota\phi_{s,t}\right) \tag{5}$$

$$\mathcal{E}\left\{H_{S1}H_{S1}^*\right\} = \alpha(D_{s,\tau})^2 I \tag{6}$$

$$\alpha(D_{s,\tau}) \propto \sqrt{\frac{A_{\text{cloud}}}{4\pi D_{s,\tau}^2}}$$

$$g_{s,t} \in [0,1]\,,\ \phi_{s,\tau} \in [0, 2\pi)$$

where we have used $\iota = \sqrt{-1}$. The variables $g$ and $\phi$ are chosen uniformly at random, and $D_{s,\tau} \approx D_s$ is the mean distance from scatterer $s$ to transmitter $\tau$. We may now interpret each interferer $n$ as a transmitter sending white signals. We may then apply (6) to write $H_{T1}$ and $H_{N1}$:

$$\left[H_{T1}\right]_{s,t} \approx \alpha(D_{s,t})\frac{1}{\sqrt{N_S}} g_{s,t} \exp\left(-\iota\phi_{s,t}\right) \tag{7}$$

$$g_{s,t} \in [0,1] \quad \phi_{s,t} \in [0, 2\pi]$$

Similarly for $H_{N1}$:

$$\left[H_{N1}\right]_{s,n} \approx \alpha(D_{s,n})\frac{1}{\sqrt{N_S}} g_{s,n} exp\left(-\iota\phi_{s,n}\right) \tag{8}$$

$$g_{s,n} \in [0,1] \quad \phi_{s,n} \in [0, 2\pi]$$

We wish to consider the case $N_R \to \infty$. We assume that the receiver array is constrained to fit within a given length $L$. Given the assumed geometry of the receive array, the inter-element spacing $\mu_R$ will be a function of the number of receive antennae $N_R$ and the length of the array $L$. In this case $\mu_R = \frac{\mu_0}{N_R}$.

The inter-element spacing will also dictate the effective area $A_{\text{eff}}$ of each antenna. As the elements of the receive array become closer, their effective areas will tend to shadow one another:

$$A_r \leq \frac{\lambda}{2}\mu_R = \frac{\lambda\mu_0}{2N_R}\,; N_R \to \infty \tag{9}$$

Clearly, the effective area will be constrained by the upper limits of (4) and (9). We may make the following piecewise linear simplification:

$$A_{\text{eff}} = \begin{cases} \frac{\alpha}{2}\lambda^2 & \text{if } \frac{L}{N_R} \geq \alpha\lambda \\ \frac{\lambda\mu_0}{2N_R} & \text{otherwise} \end{cases} \tag{10}$$

Combining (3) and (10) we have:

$$P_{\text{rec}}(\lambda, N_R) = \begin{cases} \frac{\alpha}{2}\lambda^2 \frac{P}{4\pi D^2} & \text{if } \frac{L}{N_R} \geq \alpha\lambda \\ \frac{\lambda\mu_0}{2N_R}\frac{P}{4\pi D^2} & \text{otherwise} \end{cases} \tag{11}$$

Where $D$ is the distance from source to receiver, and $P$ is the transmit power.

Consider the transfer from the each scatter $sr$ in the scattering cloud $S_R$ to the each receiver element $r$. We can see that there are no multi-path components in this transfer, as such the signal at $r$ will be given by the power received, the phase of the signal at the first element in the array and a phase shift due to the position of $r$ in the array. We may therefore write $H_{T2}$, as :

$$\begin{aligned} \left[H_{T2}\right]_{r,sr} \approx & P_{\text{rec}}(\lambda, N_R)^{\frac{1}{2}} \exp\left(-\iota D_{sr}\frac{2\pi}{\lambda}\right) \\ & \cdot \exp\left(-\iota \mu_R \frac{2\pi}{\lambda} r \sin\phi_{sr}\right) \end{aligned} \tag{12}$$

where $D_{sr}$ is the mean distance from the local scatterers to the first receiver in the array and $\phi_{sr}$ is the broad-side angle from the array to the scatterer $sr$. The values of $\phi_{sr}$ are selected from $[0, \alpha]$ uniformly at random. The value $\alpha < 2\pi$ denotes the angular spread of the scatterers as seen from the receive array. If we write

$$\beta_{sr} = \exp\left(-\iota \mu_R \frac{2\pi}{\lambda}\sin\phi_{sr}\right) \tag{13}$$

we may write (12) as:

$$\begin{aligned} \left[H_{T2}\right]_{r,sr} = & P_{\text{rec}}(\lambda, N_R)^{\frac{1}{2}} \\ & \cdot \exp\left(-\iota D_{sr}\frac{2\pi}{\lambda}\right)\left(\beta_{sr}\right)^r \end{aligned} \tag{14}$$

Likewise for $H_{N2}$ we may write:

$$\begin{aligned} \left[H_{N2}\right]_{r,sn} = & P_{\text{rec}}(\lambda, N_R)^{\frac{1}{2}} \\ & \cdot \exp\left(-\iota D_{sn}\frac{2\pi}{\lambda}\right)\left(\beta_{sn}\right)^r \end{aligned} \tag{15}$$

where the definitions in (15) have the same meaning as (14) with scatterer $sr$ in $S_R$ replaced by scatterer $sn$ in $S_N$.

Explicitly we now have:

$$\begin{aligned} y = & H_{T2}H_{T1}x \\ & + \left[(1-k)H_{N2}H_{N1} + kI_{N_R}\right]e \end{aligned} \tag{16}$$

We may note that the definition of (14) suggests a decomposition of $H_{T2}$ into two sub-components:

$$H_{T2} = V_T\Lambda_T \tag{17}$$

where $\Lambda_T$ is a diagonal matrix, giving the signal gain from each scatterer $sr$ and $V_T$ is a Vandermonde matrix accounting for the term $(\beta_{sr})^r$. We can see that the correlation of the signal will be entirely determined by the properties of the matrix $V_T$

$$\left[V_T\right]_{r,sr} = \exp\left(-\iota\frac{2\pi}{\lambda}\frac{L}{N_R} r\sin\phi_{sr}\right) \tag{18}$$

which is valid for the approximation that $\frac{1}{N_R} \to 0$.

If we assume that the $D^2$ losses due to distance from scatterer $sr$ to receiver element $r$ are approximately constant for all scatterers in the cloud $S_R$ we may write $\Lambda_T$ as:

$$\left[\Lambda_T\right]_{sr,sr} = P_{\text{rec}}(\lambda, N_R)^{\frac{1}{2}} \exp\left(-\iota D_{sr}\frac{2\pi}{\lambda}\right) \tag{19}$$

By similar argument:

$$H_{N2} = V_S\Lambda_S \tag{20}$$

For completeness we write:

$$\left[V_T\right]_{r,sn} = \exp\left(-\iota\frac{2\pi}{\lambda}\frac{L}{N_R} r\sin\phi_{sn}\right)$$

$$\left[\Lambda_T\right]_{sn,sn} = P_{\text{rec}}(\lambda, N_R)^{\frac{1}{2}} \exp\left(-\iota D_{sn}\frac{2\pi}{\lambda}\right)$$

We may rewrite (2) using (16), (17) and (20):

$$\begin{aligned} y = & V_T\Lambda_T H_{T1}x \\ & + \left[(1-k)V_N\Lambda_N H_{N1} + kI_{N_R}\right]e \end{aligned} \tag{21}$$

Equation (21) provides a general formula for the received signal at antenna elements, given both spatially correlated and independent noise. The correlation of the signal and spatial correlation of the noise is given by the matrices $V_T$ and $V_N$ respectively.

We may assume that the receiver applies automatic gain to the signal, such that the power received at each element in the array remains constant. In this case, we may normalise the distance losses and set

$$\Lambda_T = \Lambda_N = I \tag{22}$$

This allows us to simplify (21) to:

$$y = V_T H_{T1}x + \left[(1-k)V_N H_{N1} + kI_{N_R}\right]e \tag{23}$$

## III. Capacity

We shall consider the case where the channel is unknown at the transmitter. We assume that the transmit signal has constant power output independent of $N_T$, such that:

$$\text{tr}\left(\mathcal{E}\left\{xx^*\right\}\right) = \mathcal{E}\left\{x^*x\right\} = P \tag{24}$$

We shall assume that $N_T \geq N_R$ so that any capacity loss will be entirely due to receiver elements. For the unknown channel case, a white transmit signal maximises the capacity. We may write:

$$Q_x = \mathcal{E}\left\{xx^*\right\} = \frac{P}{N_T}I_{N_T} \tag{25}$$

Let $\Psi$ be the correlation matrix of the complete transfer function for the noise signals $e$. Then we write $\Psi$ as:

$$\Psi = \mathcal{E}\left\{\left[(1-k)H_N + kI\right]\left[(1-k)H_N + kI\right]^*\right\} \tag{26}$$

From (8) we can see that in the limit of large $N_R$:

$$H_{N1}H_{N1}^* \rightarrow I_{N_R} \quad (27)$$

We may use this result and the fact that $H_{N1}$ and $V_N$ are independent to simplify (26)

$$\begin{aligned}\Psi =&(1-k)^2\mathcal{E}\left\{V_N V_N^*\right\} + k^2\mathcal{E}\left\{I_{N_R}\right\} \\ &+ k(1-k)\mathcal{E}\left\{V_N H_N + H_N^* V_N^*\right\}\end{aligned} \quad (28)$$

We may interpret the third term in (28) as a cross-correlation between the spatial transfer function and the i.i.d. direct transfer function. We may view (28) as an interpolation between the two extremes of i.i.d. noise $k = 1$ and correlated noise $k = 0$.

Using $\Psi$ we may "whiten" the signal $y$ to give:

$$\begin{aligned}\hat{y} &= \Psi^{-\frac{1}{2}}y \\ &= \Psi^{-\frac{1}{2}}H_T x + \widehat{H_N}e\end{aligned} \quad (29)$$

where $\widehat{H_N}$ is white:

$$\mathcal{E}\left\{\widehat{H_N}\widehat{H_N}^*\right\} = I_{N_R}$$

If we define $C$ as the capacity of the channel, then

$$C = \mathcal{E}_{\widehat{H_N},\Psi^{-\frac{1}{2}}H_T}\left\{\mathcal{E}_{x,e}\left[\log\det\left(\hat{y}\hat{y}^*\right)\right]\right\}$$

Since $x$ and $e$ are independent, we may use the standard form of [1]:

$$C = \mathcal{E}\left\{\log_2\det\left(\frac{P}{N_T}H_T H_T^*\Psi^{-1} + I_{N_R}\right)\right\} \quad (30)$$

We shall now consider the two extreme cases $k = 0$ and $k = 1$ for the noise sources. For other situations where $0 < k < 1$ we may evaluate (30) numerically.

### A. Independent Identically Distributed Noise: $k = 1$

From (28) we may see that $\Psi = I_{N_R}$ and we may write (30) as:

$$C = \mathcal{E}\left\{\log_2\det\left(\frac{P}{N_T}H_T H_T^* + I_{N_R}\right)\right\} \quad (31)$$

which is the standard result from [1]. However, the matrix $H_T$ is not i.i.d. However, from (7) and the weak law of large numbers, we can see that:

$$\frac{1}{N_T}H_T H_T^* \rightarrow \mathcal{E}\left\{V_T|\Lambda_T|^2 V_T^*\right\} \quad (32)$$

as $N_T$ becomes large (see appendix). Using the above assumptions (22) we may then write:

$$\frac{1}{N_T}H_T H_T^* \rightarrow \mathcal{E}\left\{V_T V_T^*\right\} \quad (33)$$

We now wish to evaluate the expression $\mathcal{E}\left\{V_T V_T^*\right\}$. We may firstly note that by definition, any Vandermonde matrix $V_T$ may be written as:

$$V_T = \begin{bmatrix} 1 \\ v \\ v \otimes v \\ \vdots \\ v^{\otimes N_R - 1}\end{bmatrix}$$

where $v$ is a row vector of independent elements, $\otimes$ denotes the element-wise product and $(\cdot)^{\otimes k}$ denotes repeated application of $\otimes$. As the ordering of the scatterers in (18) is not important, without loss of generality we may assume that the values $v_i = \beta_{sr}$ are arranged in ascending order. Further, as the values of $\phi_{sr}$ are uniformly distributed, we may assume that as the $N_R$ becomes large, the element values will exactly match their distribution - and become uniform over the interval $[0, 2\pi)$

If we approximate the eigenvectors of the matrix $V_T$ by the vectors $v^{\otimes k}$ then the eigenvalues $\xi_k\left\{V_T\right\}$ will be given by:

$$\xi_k\left\{V_T\right\} \leq \mathcal{E}\left\{v^{\otimes k}v^{*\otimes k}\right\}$$

where the inequality becomes equality when the approximation of the eigenvectors becomes exact.

We may now write a limit for the eigenvalues of the matrix $V_T V_T^*$:

$$\xi_k\left\{V_T V_T^*\right\} \leq \left[\mathcal{E}\left\{v^{\otimes k}v^{*\otimes k}\right\}\right]^2 \quad (34)$$

From [4] we note that as $N_R$ becomes large we may approximate (34) by:

$$0 < \xi_k\left\{V_T V_T^*\right\} < 2^{-k} \quad (35)$$

With the above simplifications we may remove the expectation of (31) and write:

$$\begin{aligned}C &< 2K\log_2(N_R) + \sum_{k=1}^{N_R}\log_2\left(1 + P2^{-k}\right) \\ &< 2K\log_2(N_R) + P\left(\frac{1}{2}\right)^{N_R - 2}\end{aligned}$$

for a constant $K$. We can see that as $N_R$ increases, the benefit of additional receivers will diminish *without requiring an artificial scaling factor.* Any shadowing effects will be in addition to the roll-off in capacity due to correlation of input signals.

### B. Correlated Noise: $k = 0$

Referring to (28) we have $\Psi = \mathcal{E}\left\{V_N V_N^*\right\}$. For the case of $N_S \gg N_R$ we may approximate both $H_T$ and $\Psi$ as follows:

$$\begin{aligned}\frac{1}{N_T}H_T H_T^* &\rightarrow T\left[f(\alpha)\right] \\ \Psi &\rightarrow T\left[f(\beta)\right]\end{aligned} \quad (36)$$

where $T\left[f(\cdot)\right]$ is a Toeplitz matrix. The values $\alpha$ and $\beta$ are the respective angular spread of the scattering of the transmit signal and the angular spread of the scattering of the correlated noise respectively. We may therefore write the term

$$H_T H_T^*\Psi^{-1} \approx \frac{T\left[f(\alpha)\right]}{T\left[f(\beta)\right]} = T\left[f(\alpha) - f(\beta)\right] \quad (37)$$

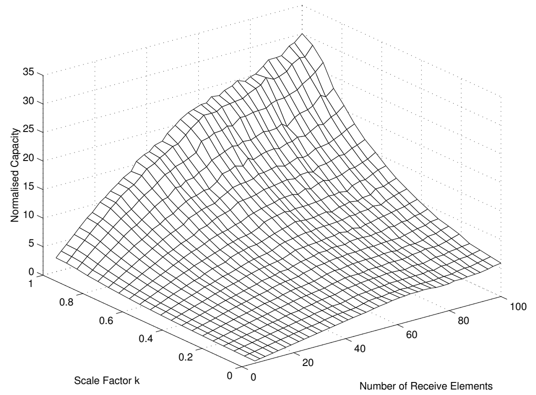


Fig. 2. Capacity of channel with increasing numbers of receivers and different values of $k$

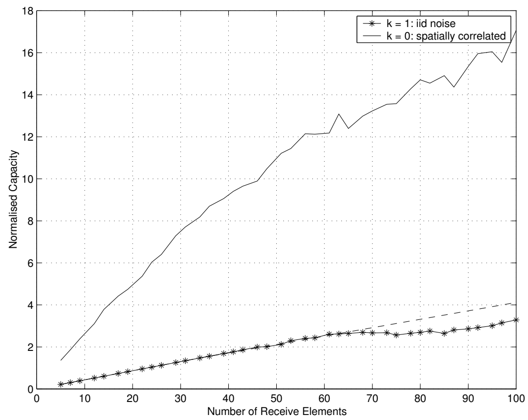


Fig. 3. Capacity of channel with increasing numbers of receivers for $k = 0$ and $k = 1$

where we have used a well known identity for Toeplitz matrices [11]. For $\alpha = \beta$ we have equal scattering angles for both noise and signal and we may write:

$$H_T H_T^* \Psi^{-1} = T\left[f(\alpha) - f(\beta)\right] = T[0] = I$$

and so (30) simplifies to:

$$C = N_R \log_2 \left(1 + K \frac{P}{\sigma_w^2}\right) \quad (38)$$

for a constant $K$. This provides linear growth in terms of the number of receive elements. For the case where the angular spread of the scattering regions is different, we may bound the eigenvalues of (30) by the eigenvalues of the Toeplitz matrix (37).

However, as $N_R$ becomes large, the approximation (36) loses accuracy and so the inverse of $\Psi$ becomes highly dependent on the distribution of the scattering placement - we may note the eigenvalue distribution from (34). In this case the capacity growth will diminish.

This may be interpreted as follows:

When we have 'densely" arranged receive elements the channel transmitters and receivers become highly correlated. If we assume that the MIMO channel $H$ was i.i.d. *before* we added additional receive elements, then it is reasonable to expect that for independent transmitting elements, the channel correlation is entirely governed by the spacing of the receive elements. Likewise, we may consider spatial correlation of the noise (interference) as a product of the correlation of receive elements.

It is well known [1] that the ideal MIMO channel is one with i.i.d. entries, and due to the convexity of the capacity formula (30) any process which moves $H$ toward $\hat{H}$ (an i.i.d. channel) will improve capacity. Since the interference is correlated by the same process as the signal, and we expect the original interference to be white, whitening the interference consequently de-correlate the incoming signal. The whitening matrix $\Psi$ therefore ensures that the entries of $H$ becomes i.i.d.

As $N_R$ increases, however, small errors in the estimation of $\Psi$ begin to have a significant impact on $\hat{H}$. Eventually $V_T$ and hence $\Psi$ becomes very badly conditioned, where the benefits of "whitening" are lost.

## IV. Simulation

We have tested the channel capacity (30) using Monte-Carlo simulations of a channel, for increasing numbers of receivers. The frequency chosen was 3.0GHz, giving $\lambda = 0.1m$. We modelled the receivers as being equally spaced along a line $L = 5m$ - becoming progressively closer as the number $N_R$ increased.

Scattering was modelled as being placed uniformly at random over the interval $[0, 2\pi)$ with different instantiations used for the interference and signal paths.

Figure 2 shows the normalised capacity of the channel plotted as $N_R$ is increased for various values of $k$. Of particular interest are the edges of the surface, where $k = 0$ and $k = 1$. We note that the capacity of the correlated noise channel $k \approx 1$ is much larger than the capacity of the i.i.d. channel. This agrees with intuition, as the noise for the correlated channel case may be selectively "cancelled" by $\Psi$. So that the noise power present in the correlated case is effectively much lower than for the i.i.d. case.

In figure 3 we have plotted the extreme cases of $k = 0$ and $k = 1$ for comparison. The i.i.d. case is shown with asterisks. The predicted linear growth for independent elements is shown dashed. We can see that while the elements remain uncorrelated, the linear growth of the channel continues proportional to the number of elements. However, after the elements become correlated, the capacity of the channel begins to roll-off. This result is consistent with intuition and with previous results [8].

## V. Conclusion

We have presented a model for the MIMO channel which accommodates spatially correlated noise. We have shown that the capacity of the channel will roll-off as the receive elements become densely spaced, even if the transmitters are independent. This reduction is independent of the power model used for the receive elements. As such, it is not necessary to consider "shadowing" of receive antenna elements to find an upper limit to capacity.

We have also shown that if the dominant noise is spatially arranged - as is the case with interference, then the capacity of the channel is significantly improve through the use of the whitening matrix $\Psi$. However, eventually the correlation effects of the channel overcome the additional knowledge provided by spatially correlated noise.

## APPENDIX

Here we prove the identity for $\mathcal{E}\{VDV^*\}$ where $V$ is a Vandermonde matrix of a random vector $v$, and $D$ is a diagonal matrix of a random vector $d$. $V$ has $N_R$ rows and $N$ columns. We may argue that $N \geq N_R$ as for increasing numbers of receivers, we can expected to detect the presence of increasing numbers of scatterers. Therefore we may write

$$\lim_{Nr \to \infty} \frac{N}{N_r} = \tau \geq 1$$

Let $v$ and $d$ have $N$ elements, denoted $v_k$ and $\left(\frac{1}{d_k}\right)^2$ respectively. $V$ is of the form:

$$\begin{aligned} [V]_{m,k} &= (v_k)^m = e^{-\iota \frac{2\pi m \delta}{\lambda N_R} \sin \phi_k} \\ &\phi_k \in [0, \alpha] \\ [D]_{k,k} &= \left(\frac{1}{d_k}\right)^2 \quad d_k \in [1, R] \\ \frac{1}{N}[VDV^*]_{m,n} &= \frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{d_k}\right)^2 e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k} \end{aligned} \tag{39}$$

It is clear that the random variables $d_k$ and $v_k$ are independent. As such, the random variables $X_k = \left(\frac{1}{d_k}\right)^2 e^{i\left(\frac{m-n}{N_R}\right)\phi_k}$ are identically distributed. Therefore each element $X_k$ in the sum of (39) may be considered to be the result of an independent experiment. From the weak law of large numbers [12], we have:

$$\lim_{N \to \infty} \frac{1}{N}\sum_{k=1}^{N} X_k \quad \overrightarrow{\text{pr}} \quad \frac{1}{N}\sum_{k=1}^{N}\mathcal{E}\{X_k\} = \mathcal{E}\{X\}$$

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N}[VDV^*]_{m,n} \\ &\quad = \mathcal{E}\left\{\left(\frac{1}{d_k}\right)^2 e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k}\right\} \\ &\quad = \mathcal{E}\left\{\left(\frac{1}{d_k}\right)^2\right\}\mathcal{E}\left\{e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k}\right\} \end{aligned}$$

$$\begin{aligned} \mathcal{E}\left\{\left(\frac{1}{x}\right)^2\right\} &= \frac{1}{R}\int_1^R \left(\frac{1}{x}\right)^2 dx \\ &\approx \frac{1}{R} \quad R \gg 1 \end{aligned}$$

$$\mathcal{E}\left\{e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k}\right\} = \frac{1}{\alpha}\int_0^\alpha e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k} d\phi$$

Using an integration of products form we have:

$$\begin{aligned} \mathcal{E}\left\{e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\phi_k}\right\} &\approx e^{i\left(\frac{n-m}{N_R}\right)\frac{2\pi\delta}{\lambda}\sin\alpha} \\ &= T[f(\alpha)] \end{aligned} \tag{40}$$

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