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Worst-Case Properties of the Uniform Distribution and Randomized Algorithms for Robustness Analysis†

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Abstract

Motivated by the current limitations of the existing algorithms for robustness analysis, in this paper we take a different direction which follows the so-called *probabilistic approach.* That is, we aim to estimate the probability that a control system with uncertain parameters q restricted to a box Q attains a certain level of performance γ. Since this probability depends on the underlying density function $f(q)$, we study the following question: What is a "reasonable" density function so that the estimated probability makes sense? To answer this question, we define two new worst-case criteria and prove that the uniform density function is optimal in both cases. In the second part of the paper, we turn our attention to a subsequent problem. That is, taking $f(q)$ as the uniform density function, we estimate the size of the so-called "good" and "bad" sets. Roughly speaking, the good set contains the parameters $q \in Q$ that have performance level better than or equal to γ while the bad set is the set of parameters $q \in Q$ that have performance level worse than γ. To estimate the size of both sets, sampling is required. Then, we give bounds on the minimum sample size to attain a given accuracy and confidence.

1. Introduction and Preliminaries

Consider a (Lebesgue) measurable function $u(q) : \mathbf{R}^n \to \mathbf{R}$, where $q = [q_1, q_2, \ldots, q_n]'$ and each q_i is restricted to a bounded interval.

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Without loss of generality, we normalize each q_i into the interval $[-1/2,\ 1/2]$ and define $Q = [-1/2,\ 1/2]^n \subset \mathbf{R}^n$. In addition, for any measurable subset $\underline{Q} \subseteq Q$, we define the volume of $\underline{Q}$, denoted by $\mathrm{vol}(\underline{Q})$, as

$$\mathrm{vol}(\underline{Q}) = \int_{\underline{Q}} dq.$$

By this notion, $u(\cdot) : Q \subset \mathbf{R}^n \to \mathbf{R}$ and $\mathrm{vol}(Q) = 1$. The following two problems are of most interest in robustness analysis:

Problem 1: To find $q_{\max} \in Q$ such that

$$u(q_{\max}) \doteq \max_{q \in Q} u(q)$$

or, for given error bound $\epsilon > 0$, to find $\bar{q} \in Q$ such that

$$|u(q_{\max}) - u(\bar{q})| \leq \epsilon.$$

Problem 2: For given performance level $\gamma > 0$, to check whether

$$u(q) \leq \gamma$$

for all $q \in Q$.

Note that by proper formulation, if the vector q represents the uncertain parameters entering into a control system, many robustness analysis questions belong to one of the above two problems. For instance, if $u(q)$ is set to be equal to the maximum real part of the roots of the closed loop polynomial, then $u(q_{\max})$ determines whether the system is robustly stable. If $u(q)$ is set to be equal to the H_∞ norm of the sensitivity function and if $u(q)$ is smaller than γ for all $q \in Q$, then robust performance is attained. Several robustness problems which can be formulated in either one of the two cases described above are listed in Section 4. Additional problems of this kind are also studied in the work [8] in the classical $M - \Delta$ setting.

Given these motivations, we now introduce the notion of *good set* and *bad set*. For given $\gamma > 0$, define the good set $Q_g(\gamma) \subseteq Q$ and the bad set $Q_b(\gamma) \subseteq Q$ as

$$\begin{aligned} Q_b(\gamma) &\doteq \{q \in Q : u(q) > \gamma\}; \\ Q_g(\gamma) &\doteq \{q \in Q : u(q) \leq \gamma\}. \end{aligned}$$

Roughly speaking, the good set $Q_g(\gamma)$ contains the parameters $q \in Q$ that have performance level better than or equal to γ and the bad set $Q_b(\gamma)$ is the set of parameters $q \in Q$ that have performance level worse than γ. Obviuosly, the union of these two sets coincides with Q.

The problems described above are in general very complex as far as computations are concerned. In particular, it has been shown recently (see e.g., [4], [10] and [12]) that several key problems in robustness, including μ calculation and stability of interval matrices, are NP-hard. Motivated by this gloomy picture, a number of researchers have recently taken a different direction which leads to a probabilistic-based approach; e.g., see the earlier works [13] and [14] and the subsequent papers [2], [8] and [15]. The key idea in this framework is to solve both Problems 1 and 2 in terms of probability. That is, instead of a guaranteed answer as in the classical setting, we now look for a probabilistic answer. For instance, we can say that the probability that $u(q) \leq \gamma$ is at least $1-\delta$, where $\delta \in (0,1)$. Similarly, for $\epsilon \in (0,1)$, given $\overline{q} \in Q$, we estimate

$$|u(q_{\max}) - u(\overline{q})| \leq \epsilon$$

with probability $1 - \delta$. Following the terminology in [16], we call ϵ the *accuracy* and $1 - \delta$ the *confidence* parameter. One interesting feature of this probabilistic setting is that, unlike its deterministic counterpart, the complexity of randomized algorithms may not increase exponentially with the number of uncertain parameters entering into the control system; see Lemma 3.1, the discussion in Section 5 and the references [8], [13] and [15]. A drawback of this setting, however, is that the results obtained depend on the specific choice of the density function f and the results can be totally different for different density functions. Define $p(f)$ as

$$p(f) \doteq \int_{Q_b(\gamma)} f(q)dq$$

which is the probability of the "size" of the bad set $Q_b(\gamma)$ when the density function is taken to be f. Then, for given $u(\cdot)$ and performance level γ, we may ask the following question: How do we calculate $p(f)$? This probability can be easily estimated by using some classical results such as the Bernoulli [11] or Chernoff bounds [5]. In particular, let $q^1, q^2, \ldots, q^N$ be i.i.d. random samples in Q generated according to the given density function f. Define

$$z_i = \begin{cases} 1 & \text{if } q^i \in Q_b(\gamma); \\ 0 & \text{otherwise.} \end{cases}$$

Then, invoking the Chernoff bound [5], we conclude that if

$$N \geq \frac{1}{2\epsilon^2} \ln \frac{2}{\delta}$$

then,

$$\text{Prob}\{|\frac{1}{N}\sum_{i=1}^{N} z_i - p(f)| \leq \epsilon\} \geq 1 - \delta$$

for any $\epsilon \in (0,1)$ and $\delta \in (0,1)$. The interpretation of this result is the following: If ϵ and δ are "small," the estimated probability $\hat{p}(f) = \frac{1}{N}\sum_{i=1}^{N} z_i$ is an accurate estimate of the true probability $p(f)$. We also observe that the number of samples required to compute this estimate is *linear* in both $\frac{1}{\epsilon^2}$ and $\ln \frac{1}{\delta}$. However, without some reasoning attached to the chosen density function f, this $p(f)$ is meaningless. To argue this, consider two extreme cases. First, let f be chosen such that $f(q) = 0$ if $q \in Q_b(\gamma)$. Then, $p(f) = \text{Prob}\{q \in Q_b(\gamma)\} = 0$. On the other hand, if f is chosen such that $f(q) = 0$ if $q \in Q_g(\gamma)$, then, $p(f) = \text{Prob}\{q \in Q_b(\gamma)\} = 1 - \text{Prob}\{q \in Q_g(\gamma)\} = 1$. In other words, for an arbitrarily chosen f, the probability of q being in the bad set does not mean too much. This brings a key question of the randomized approach in robustness analysis: What is a "reasonable" density function so that the obtained results based on this f make sense? In this direction, a pioneer study is carried out in [2] showing that the uniform density function $f_{uni} = 1$ in Q minimizes the probability $\text{Prob}\{q \in \mathcal{T}\}$ among all symmetric and non-increasing density functions and where the target $\mathcal{T}$ is a convex and centrally symmetric set. In the same paper, this result is then applied to robustness analysis of an affine polynomial family, taking as target the so-called value set. However, the fact that $\mathcal{T}$ needs to be convex and centrally symmetric seems a critical requirement which is generally not satisfied for the sets $Q_g(\gamma)$ and $Q_b(\gamma)$. Given these motivations, the first objective in this paper is to show that the uniform density function f_{uni} has several additional interesting properties among all density functions f that may not be symmetric and non-increasing. In addition, in this paper, the mapping $u(\cdot)$ is nonlinear, which in turn means that the target set is not necessarily convex and centrally symmetric.

We now briefly summarize the main results of this paper. In Section 2, we study two new worst-case optimality criteria for the uniform distribution. First, we prove that the uniform distribution f_{uni} maximizes the smallest under-estimate of $p(f)$ and, simultaneously, minimizes the largest over-estimate of $p(f)$ over all subsets Q_s of Q having the same volume of the bad set. Secondly, we prove that the uniform density function f_{uni} is "optimal" in the sense that it requires the minimum number of samples to attain a certain confidence $1 - \delta$ for all functions $u(\cdot)$ in the class $\mathcal{U}_L$ of Lipschitz continuous functions. In Section 3, for the case when f is a uniform distribution, we compute the minimum sample size N required to estimate the probability that the volume of the bad set is smaller than a certain percentage of the volume of the set Q. As in the case of the Chernoff Bound, N is independent of the number of uncertain parameters. In Section 4, we then apply these results to uncertain control systems. In particular, we show how a number of applications in robustness can be reformulated in this setting. In Section 5, we discuss some issues and drawbacks of the existing results and, in particular, we study cases when the bound N grows with the problem size. Finally, in Section 6 we provide conclusions. The proofs are given in [3].

2. Worst-Case Properties of the Uniform Distribution

First, we define the set of allowable probability density functions on Q.

Definition 2.1 *Let $\mathcal{F}$ be the set of all bounded and (Lebesgue) measurable time-invariant probability density functions $f(\cdot)$ on Q, i.e., the distribution function $F_u(\gamma)$ of $u(q)$*

$$F_u(\gamma) = \mathrm{Prob}\{u(q) \leq \gamma\} = \int_{q \in Q, u(q) \leq \gamma} f(q)dq$$

is well-defined for any γ.

For given $u(\cdot)$ and performance level γ, our goal is to describe the bad set $Q_b(\gamma)$ or, at least, its size. As previously discussed, although this bad set is fixed for given $u(\cdot)$ and γ, it is unknown and its measure seems very hard to determine exactly. Thus, the idea is to use randomized algorithms to estimate this measure. That is, for each density function f, we can estimate $p(f) = \mathrm{Prob}\{q \in Q_b(\gamma)\}$ and use this $p(f)$ to evaluate the approximate size of $Q_b(\gamma)$. As discussed in the previous section, however, this probability can assume the extreme values zero or one, depending on the specific choice of f. Of course, these two cases are extreme and in a more realistic setting, since $Q_b(\gamma)$ is fixed but unknown, $p(f)$ always lies between

$$\inf_{Q_s \in \mathcal{Q}(\gamma)} p(f) \quad \text{and} \quad \sup_{Q_s \in \mathcal{Q}(\gamma)} p(f)$$

where

$$\mathcal{Q}(\gamma) =$$

$$\{\text{all subset(s) } Q_s \text{ of } Q \text{ such that } \mathrm{vol}(Q_s) = \mathrm{vol}(Q_b(\gamma))\}.$$

Conceptually, $\sup p(f)$ is an over-estimate of $p(f)$ and $\inf p(f)$ is an under-estimate; they both depend on f and neither is a good estimate of $p(f)$. However, these two bounds can be always achieved in a worst-case scenario. Thus, to de-emphasize the dependence on f, a better choice would be

$$\sup_f \inf_{Q_s \in \mathcal{Q}(\gamma)} p(f) \quad \text{or} \quad \inf_f \sup_{Q_s \in \mathcal{Q}(\gamma)} p(f).$$

The interpretation is that to approximate $p(f)$ and, accordingly, its size, we use either the largest under-estimate, which is a lower bound, or the smallest over-estimate, which is an upper bound. In fact, the next result shows that the largest under-estimate coincides with the smallest over-estimate. Moreover, both these estimates are achieved when the density function is taken as the uniform density function f_{uni}.

Theorem 2.1 *For any measurable function $u(\cdot)$, performance level γ and $Q_b(\gamma)$, we have*

$$\sup_{f \in \mathcal{F}} \inf_{Q_s \in \mathcal{Q}(\gamma)} p(f) = p(f_{uni}) = \inf_{f \in \mathcal{F}} \sup_{Q_s \in \mathcal{Q}(\gamma)} p(f)$$

Next, we turn our attention to the following question: Given δ and ϵ, what is an "optimal distribution" in terms of requiring the minimum number of samples to meet a prescribed probability for all $u(\cdot)$ in the class of Lipschitz continuous functions? Interestingly, this optimal distribution turns out to be the uniform distribution. To state this result precisely, we need a definition.

Definition 2.2 *Let $\mathcal{U}_L$ be the set of Lipschitz continuous functions $u(\cdot)$ with Lipschitz constant L and let $\mathcal{F}_L$ be the set of all Lipschitz continuous density functions $f \in \mathcal{F}$ with Lipschitz constant L such that $f(\overline{q}) \leq 1 - \xi$ for some $\overline{q} \in Q$ and $\xi > 0$.*

We remark that for any non-uniform distribution, there always exists $\overline{q} \in Q$ such that $f(\overline{q}) \leq 1 - \xi$ for some $\xi > 0$. Thus, in practice, $\mathcal{F}_L$ is the set of all Lipschitz continuous functions besides the uniform density function f_{uni}. We take $q^1, q^2, \ldots, q^N$ i.i.d. random samples in Q according to $f \in \mathcal{F}_L$ and denote the largest $u(q^i)$ as

$$u(q^N_{\max}) = \max_{i=1,2,\ldots,N} u(q^i).$$

Finally, for given ϵ and δ, we denote by $\rho(f)$ the minimum number of samples required to satisfy

$$\rho(f) = \arg\min\{\mathrm{Prob}\{\sup_{u \in \mathcal{U}_L} |u(q_{\max}) - u(q^N_{\max})| \leq \epsilon\} \geq 1 - \delta\}. \tag{2.1}$$

On the contrary of the criterion used to state the Chernoff bound, we observe that here there is only one level of probability. In this context, the result below shows that the uniform distribution gives the minimum sample size. However, this sample size is an exponential function of the number of parameters; see the comments in Section 5.

We are now ready to state the second result of this section.

Theorem 2.2 *Consider the sets $\mathcal{U}_L$ and $\mathcal{F}_L$ previously defined. For any $\delta \in (0,1)$ and $\epsilon \in (0, \xi]$, we have*

$$\rho(f_{uni}) \leq \min_{f \in \mathcal{F}_L} \rho(f).$$

3. On the Minimum Sample Size for the Uniform Distribution

Motivated by the latter result of the previous section, we now elaborate on the issue of the minimum sample size. We define the sample complexity as

$$N_0 \doteq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\epsilon}}$$

and, for completeness, we recall that the minimum sample size for the problem of estimating $u_{\max}$ with sampling is given by the result below.

Lemma 3.1 *Let $u(q)$ be (Lebesgue) measurable. For any $f \in \mathcal{F}$, if*

$$N \geq N_0$$

then,

$$\text{Prob}\{\text{Prob}\{u(q) > u(q_{\max}^N)\} \leq \epsilon\} \geq 1 - \delta$$

for any ϵ and $\delta \in (0,1)$.

This result was independently derived in our previous work [15] and in the paper [8]. We notice that the bound given in Lemma 3.1 greatly improves upon classical results such as the Chernoff bound. For example, if $\epsilon = \delta = 0.01$, by Lemma 3.1 we compute $N = 460$ while with the Chernoff bound we obtain $N = 26,592$. Finally, we observe that this result is independent of the density function $f \in \mathcal{F}$. We now use Lemma 3.1 to establish a connection with the volume of the bad set $Q_b(\gamma)$ for the specific case of uniform distribution. A similar result has been established in [8]; the difference is that here we consider the specific case of uniform distribution thus providing some valuable intuition for the result of Lemma 3.1.

Corollary 3.1 *Let f_{uni} be a uniform density function and let $q^1, q^2 \ldots, q^N$ be i.i.d. random samples generated according to f_{uni}. For any ϵ and $\delta \in (0,1)$, if*

$$N \geq N_0$$

then,

$$\text{Prob}\{\frac{\text{vol}(Q_b(u(q_{\max}^N)))}{\text{vol}(Q)} \leq \epsilon\} \geq 1 - \delta.$$

The proof follows directly from Lemma 3.1 by setting f to be equal to the uniform distribution and taking $\gamma = u(q_{\max}^N)$. This result can be interpreted in terms of the "amount of badness" of the set $Q_b(\gamma)$ as follows: If

$$\text{vol}(Q_b(\gamma)) \leq \epsilon \ \text{vol}(Q)$$

we can say that $Q_b(\gamma)$ is ϵ-bad. Then, from Corollary 3.1 we conclude that if

$$N \geq N_0 = \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\epsilon}},$$

then, with a probability at least $1 - \delta$, $Q_b(u(q_{\max}^N))$ is ϵ-bad.

4. Applications to Probabilistic Robustness Analysis of Control Systems

The results derived in the previous sections can be immediately applied to several problems in robustness analysis. We now list a number of them; see the paper [8] for a similar discussion.

Application 1: Let $u(q)$ be the maximum real part of the eigenvalues, where $q \in Q$ denotes the uncertain parameters. Let q^i, $i = 1, 2 \ldots, N$, be i.i.d. random samples in Q generated according to a uniform distribution. If $u(q^i) < 0$ for all $i = 1, \ldots, N$ and $N \geq N_0$, then, with a probability at least $1 - \delta$, the volume of the unstable set $\{q \in Q : u(q) \geq 0\}$ is smaller than the volume of $Q_b(u(q_{\max}^N))$ which is no greater than ϵ vol(Q). Thus, we conclude that the volume of the unstable set it at most ϵ-bad. The same argument clearly holds for discrete time systems. In this case, it suffices to take $u(q)$ as the maximum magnitude of the eigenvalues and $|u(q^i)| < 1$ for all $i = 1, \ldots, N$.

Application 2: Let $u(q) = ||S(s,q)||_\infty \doteq \sup_\omega |S(j\omega, q)|$ be the H_∞ norm of the sensitivity function $S(s,q)$. As in the first example of this section, let q^i, $i = 1, 2 \ldots, N$, be i.i.d random samples in Q generated according to a uniform distribution in Q. If $u(q^i) < \gamma$ for all $i = 1, \ldots, N$ and $N \geq N_0$, then, with a probability at least $1 - \delta$, the volume of the set of "bad" plants with performance level greater than γ is smaller than the volume of $Q_b(u(q_{\max}^N))$ which is no greater than ϵ vol(Q). We conclude again that the volume of the set of bad plants is at most ϵ-bad. For discrete time systems, the same argument holds taking $u(q) = ||S(z,q)||_\infty \doteq \sup_{\theta \in [0,2\pi]} |S(e^{j\theta}, q)|$.

Application 3: Let $u(q)$ be equal to the inverse of the structured singular value μ; see e.g., [6] and [7]. If the samples q^i, $i = 1, 2, \ldots, N$, are randomly generated in Q according to a uniform distribution and if $u(q^i) < 1/\mu$ for all $i = 1, 2, \ldots, N$ and $N \geq N_0$, then, with a probability at least $1 - \delta$, the volume of the set of plants with robustness margin no greater than $1/\mu$ is at most ϵ-bad.

5. Discussions and Remarks

The results given in this paper may have applications broader than robustness analysis. The fact that the results are independent of the problem dimension seems very powerful. This is not surprizing because the same feature is well-known in the literature on Monte Carlo simulations. This is a consequence of the fact that the minimum sample size in Corollary 3.1 is stated in terms of the ratio

$$\frac{\text{vol}(Q_b(u(q_{\max}^N)))}{\text{vol}(Q)}.$$

If the size and/or the dimension of Q increases, the size of vol$(Q_b(u(q_{\max}^N)))$ increases as well. On a negative side, we remark that the fact that $Q_b(u(q_{\max}^N))$ is ϵ-bad does not necessarily imply that $u(q_{\max}^N)$ is "close" to $u(q_{\max})$. In other words, except for some simple cases, it is not possible to estimate accurately the difference between $u(q^N_{\max})$ and $u(q_{\max})$ or the difference between q^i and $q_{\max}$ taking only N_0 samples in Q. To elaborate, we study the two cases $|u(q_{\max}) - u(q^N_{\max})| \leq \epsilon$ and $||q^i - q_{\max}|| \leq \epsilon$ separately.

For the case $||q^i - q_{\max}|| \leq \epsilon$, let $u(\cdot)$ achieve the maximum $q_{\max} \in Q$ and take the norm $||\cdot||$ as ℓ_∞; the same conclusion holds if a different norm is used. Then, $||q^i - q_{\max}|| \leq \epsilon$ if q^i is in the box of center $q_{\max}$ and radius ϵ

$$B(q_{\max}, \epsilon) = \{q : ||q - q_{\max}|| \leq \epsilon\}.$$

The volume of this box is

$$\text{vol } B(q_{\max}, \epsilon) = (2\epsilon)^n.$$

For small ϵ, $\text{vol}(B(q_{\max}, \epsilon) \bigcap Q)$ converges to zero exponentially as the dimension of q increases. In order to have at least one q^i in the box $B(q_{\max}, \epsilon) \bigcap Q$, the number of samples required *has* to increase exponentially, except for pathological cases described below. In general, the minimum number of samples required is no longer N_0 but it is an exponential function of the dimension of q. One exception to this exponential growth is when $u(\cdot)$ achieves more and more maximum points $q_{\max}$ in Q at a rate faster than the decreasing rate of $\text{vol}(B(q_{\max}, \epsilon) \bigcap Q)$. Only in such pathological cases, the number of samples required to meet $||q^i - q_{\max}|| \leq \epsilon$ with probability $1 - \delta$ does not depend on the dimension of q.

For the second case, when $|u(q_{\max}) - u(q^N_{\max})| \leq \epsilon$, let $u(\cdot)$ be a Lipschitz continuous function with Lipschitz constant L. Note that $|u(q_{\max}) - u(q^N_{\max})| \leq \epsilon$ if some samples q^i are in the box $B(q_{\max}, \epsilon/L) \bigcap Q$. As previously discussed, for ϵ/L small, the volume of $B(q_{\max}, \epsilon/L) \bigcap Q$ converges to zero exponentially. In turn, this implies that the number of samples needed to satisfy $|u(q_{\max}) - u(q^N_{\max})| \leq \epsilon$ grows exponentially as the dimension of q increases and it is no longer given by N_0.

6. Conclusion

In this paper, we have shown some new results for the so-called probabilistic approach for robustness of uncertain systems. A subsequent and promising line of research will focus on adaptive instead of passive randomized algorithms [17] with the specific goal to quantify the size of the "bad" set.

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