MIMO Wireless Systems: Capacity Limits for Sparse Scattering

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Abstract — **We present a limiting function for the capacity of a sparsely scattered, outdoor MIMO channel. We use a result from random matrices to show that the capacity of the channel will diminish logarithmically with a parameter η. The parameter is inversely dependent on the distance between the transmitter array and the receiver array. This loss is in addition to any free-space losses. We show that below a given value of this parameter, the capacity of the channel is significantly reduced and that the benefits of additional receive or transmit elements do not eventuate. Monte Carlo simulations are used to compare the theoretical results with the physical model.**

I. Introduction

The work of [1] and [2] predicted a remarkable capacity increase for multiple transmit, multiple receive wireless systems in the presence of multi-path scattering. In [2] a *linear* growth in capacity is predicted, proportional to the minimum number of transmit and receive antennas in the MIMO system. The predicted increase was shown for a practical indoor environment, where scatterers are dense and no line-of-sight component is present, in [3].

Fundamental to this work is the assumption that the wireless channel may be modelled by an independent random $N \times M$ transfer matrix, where N and M are the numbers of the transmit and receive elements respectively. That is, the entries of the transfer matrix are assumed to be independent, complex random variables. The assumption of independence guarantees a (statistically) well-conditioned transfer matrix and prevents a loss of capacity due to correlations between the channels.

The work of [4] suggested that a limit to the linear capacity growth of [2] will exist in outdoor environments where the scattering is sparse. In [5] a non-line of sight (NLOS) MIMO channel was developed and was shown to exhibit a loss in capacity dependent on wavelength λ, scatter cloud radius R and distance between transmitters and receivers D. It was suggested that the capacity of an outdoor, sparsely scattered MIMO channel is dependent on a parameter η:

$$\eta = \frac{2\pi R^2}{D\lambda} \tag{1}$$

However, the authors were unable to give a closed form of the channel capacity.

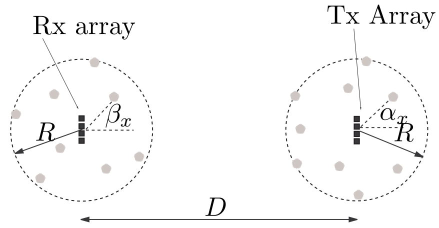

Figure 1: Scatter arrangement

This paper provides a bound for the capacity of a MIMO channel, which allows us to show a direct dependence on the parameter η for the case where η is sufficiently small. As seen from (1) this is when the separation of the transmitters from the receivers is large, in comparison with the extent of local scattering. We show that there is a transition from the "rich scattering environment" which models indoor NLOS channels [3] to a highly correlated channel "pin-hole" channel, and that this transition is dependent on η.

This paper is arranged as follows: The configuration of the MIMO channel is developed in section II. The channel configuration is used to expose the underlying structure of the transfer matrices in section III. Using results from the theory of random matrices, we derive a formula for the capacity of the MIMO channel in section IV. Monte Carlo simulations of the channel model are presented in section V and compared with the theoretical results. Section VI provides a summary of our work.

II. Configuration

Consider the two ring scattering model shown in figure 1. Scatterers are modelled as small, spherical, memoryless reflectors. We allow the scatterers to be lossy - resulting in random gains. We consider only scatterers which are close to either the transmit or receive array. All signals are assumed to be narrowband, high frequency so that plane wave assumptions hold and the channel is assumed to exhibit frequency-flat fading.

This is the well known *non-line of sight, local scattering* model for outdoor transmission. Scatterers are placed randomly within the local rings. The arrangement of scatterers is assumed to be quasi-static - therefore the random arrangement will change at certain intervals. Each transmit local scatterer x is placed at location $(R \times t_x, \alpha_x)$ where R is the local ring radius, $t_x \in (0, 1]$

and $\alpha_x \in [0, 2\pi)$ selected uniformly at random. Similarly each receive scatterer x is placed at $(R \times r_x, \beta_x)$ with appropriate translation of origin. We assume that there are S_r scatterers in the receive ring, and S_t scatterers in the transmit ring, with S_r and S_t both being large: $\frac{S_t}{S_r} \to c \approx 1$, $S_r, S_t \to \infty$

For transmit signal vector x and receive signal vector y we have a MIMO transfer function of the form:

$$y = Hx = H_r H_s H_t x \tag{2}$$

Where H_r is the MIMO transfer function from the receive local scatterers to the receiver elements. H_s is the MIMO transfer function between the two rings. H_t is the transfer function from the transmitter elements to the transmit local scatterers.

For a uniform linear array with array element spacing of μ_r, we may write H_r as follows:

$$H_r = \frac{1}{\sqrt{S_r}} H_{rv} H_{rd} \tag{3}$$

$$H_{rd} \stackrel{\text{def}}{=} \operatorname{diag}\left\{ \frac{1}{R_{x,r0}} e^{-j\frac{2\pi}{\lambda}R_{x,r0}} \right\} \tag{4}$$

$$\left[H_{rv}\right]_{y,x} = e^{-j\frac{2\pi}{\lambda}\mu_r \sin(\beta_x)} \tag{5}$$

Where $R_{x,r0}$ is the distance from scatterer x to the first receive element $r0$. For the case where the array elements are placed at a half-wavelength spacing or greater $\mu_r \geq \frac{\lambda}{2}$, we may assume the array elements are uncorrelated. Consequently the entries in H_{rv} will be independent identically distributed random variables:

$$\left[H_{rv}\right]_{y,x} \approx e^{-j\phi_{x,y}} \quad \phi_{x,y} \in [0, 2\pi) \tag{6}$$

Likewise we may write H_t as a product of a diagonal (gain) matrix and a random (phase) matrix:

$$H_t = \frac{1}{\sqrt{S_t}} H_{td} H_{tv} \tag{7}$$

$$H_{rd} \stackrel{\text{def}}{=} \operatorname{diag}\left\{ \frac{1}{R_{x,t0}} e^{-j\frac{2\pi}{\lambda}R_{x,t0}} \right\} \tag{8}$$

$$\left[H_{tv}\right]_{y,x} = e^{-j\frac{2\pi}{\lambda}\mu_t \sin(\alpha_x)} \approx e^{-j\theta_{x,y}} \quad \theta_{x,y} \in [0, 2\pi) \tag{9}$$

As both H_r and H_t are iid matrices, we may consider that Rayleigh fading is present within both local scattering rings. This may be interpreted as having dense local scattering at the transmit and receive arrays. Such a situation may result when transmitting from a room in one building to a room in another building, a large (greater than the size of the room) distance away. We may also see such an arrangement in the case of a mobile (hand-held) transmitter where the holder moves between buildings.

The matrix H_s is the transfer between the two clouds of scatterers. Each element of H_s will consist of a gain $g(y, x)$ which incorporates free-space losses, and a phase due to the distance between transmit scatterer x and receive scatterer y. This distance is given by $R_2(y, x)$. We write H_s below:

$$\left[H_s\right]_{y,x} = g(y, x) e^{-j\frac{2\pi}{\lambda}R_2(y,x)} \tag{10}$$

III. Analysis

H_s is dominated by the phase offset due to the distance between the two scattering clouds. This can be seen through a Taylor series on on R_2 using $\frac{R}{D} \to 0$

$$R_2(y, x) = \sqrt{\begin{array}{l} \left[D + R(t_x \cos\alpha_x - r_y \cos\beta_y)\right]^2 \\ + R^2 \left(t_x \sin\alpha_x - r_y \sin\beta_y\right)^2 \end{array}}$$

$$R_2(y, x) \approx D + Ra(y, x) + \frac{R^2}{2D} b(y, x) \tag{12}$$

The functions $a(y, x)$ and $b(y, x)$ are defined by:

$$a(y, x) \stackrel{\text{def}}{=} t_x \cos\alpha_x - r_x \cos\beta_y \tag{13}$$

$$b(y, x) \stackrel{\text{def}}{=} \left(t_x \sin\alpha_x - r_y \sin\beta_y\right)^2 \tag{14}$$

Using (12) we write:

$$H_s = D_3 \Omega D_2 D_1 \tag{15}$$

D_3, D_1 and D_2 are diagonal matrices. Only D_3 has elements a non-unitary magnitude. Ω is then defined as:

$$\left[\Omega\right]_{y,x} \stackrel{\text{def}}{=} e^{-j\eta\omega(y,x)} \tag{16}$$

$$\omega(y, x) \stackrel{\text{def}}{=} t_x r_y \sin\alpha_x \sin\beta_y \tag{17}$$

From (17) we may interpret the function $\omega(y, x)$ as a random variable, chosen over the range $\omega \in [-1, 1]$, with density function [7]:

$$f(\omega) = \frac{1}{4\pi^2} \frac{1}{(1 - \omega^2)} \tag{18}$$

We may see that that the elements of Ω are correlated only by the value of η. We may expand (16) via a Taylor series, to write:

$$\Omega \approx u_{S_r} u_{S_t}^* + \eta G \tag{19}$$

$$u_{S_r}^* \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

Where u_k is a vector of length k and G is an $S_r \times S_t$ Gaussian random matrix, with i.i.d. entries. Equation (19) allows us to re-write the original transfer function H:

$$\begin{aligned} HH^* &= H_r H_s H_t H_t^* H_s^* H_r^* \\ &= H_r u_{S_r} u_{sr}^* H_t H_t^* u_{S_r} u_{sr}^* H_r^* + \eta^2 H_r G H_t H_t^* G^* H_r^* \\ &= \frac{\Theta}{S_r S_t} + \eta^2 H_r G H_t H_t^* G^* H_r^* \end{aligned} \tag{20}$$

Θ is a rank one matrix.

IV. Capacity

We wish to determine the capacity of the MIMO channel. For fixed transmit power we may use the result of [2] for a SNR of ρ dB:

$$\begin{aligned} C &= \mathcal{E}\left[\log_2 \det\left(I_M + \frac{\rho}{N} H H^*\right)\right] \\ &= \mathcal{E}\left[\log_2 \det\left(I_M + \frac{\rho}{N}\frac{\Theta}{S_r S_t} + \frac{\rho\eta^2}{N} H_r G H_t H_t^* G^* H_r^*\right)\right] \end{aligned} \tag{21}$$

$$C = \mathcal{E}\left[\sum_{k=1}^{M} \log_2\left(1 + \frac{\rho}{N}\xi_k\left\{\frac{\Theta}{S_r S_t} + H_r G H_t H_t^* G^* H_r^*\right\}\right)\right] \tag{22}$$

Where ξ_k gives the k^{th} largest eigenvalue. We may use the following result from eg. [8]: For Hermitian matrices A and B, where A has rank r

$$\xi_k(A+B) \leq \begin{cases} \xi_1(A) & k \leq 2r \\ \xi_{k-r}(B) & k > 2r \end{cases}$$

We may then write (22) as:

$$\begin{aligned} C &\leq \begin{array}{c} 2\log_2(1+\hat{\rho}M) \\ + \\ \mathcal{E}\left[\sum_{k=3}^{M} \log_2\left[1 + \xi_k\left\{\frac{\eta^2\rho}{N} H_r G H_t H_t^* G^* H_r^*\right]\right\}\right] \end{array} \\ &< \begin{array}{c} 2\log_2(1+\hat{\rho}M) \\ + \\ \mathcal{E}\left[\sum_{k=1}^{M} \log_2\left[1 + \xi_k\left\{\frac{\eta^2\rho}{N} H_r G H_t H_t^* G^* H_r^*\right]\right\}\right] \end{array} \end{aligned} \tag{24}$$

$$C < 2\log_2(1+\hat{\rho}M) + C_{\log} \tag{25}$$

We may approximate $C_{\log}$ with a strong law of large numbers argument as follows:

We use the following identities:

$$\begin{aligned} \xi_k\{MM^*\} &= \xi_k\{M^*M\} \quad k \leq \operatorname{rank}\{M\} \\ &\leq \xi_k\{M^*\}\xi_k\{M\} \\ \det(I+AB) &= \det(I+BA) \end{aligned}$$

This allows us to re-arrange $C_{\log}$:

$$C_{\log} < \mathcal{E}\sum_{k=1}^{M} \log_2\left[1 + \frac{\eta^2\rho}{N}\xi_k\{H_r H_r^*\}\xi_k\{GG^*\}\xi_k\{H_t H_t^*\}\right] \tag{26}$$

By the *strong law of large numbers* a matrix $X \in M_{n,m}$ with elements $[X]_{m,n} = e^{-j\phi_{m,n}}$, $\phi_{m,n}$ chosen at random with identical distribution, exhibits the following property:

$$\lim_{m\to\infty} \frac{1}{m} XX^* \to \frac{1}{m}\mathcal{E}[XX^*] \approx I_n$$

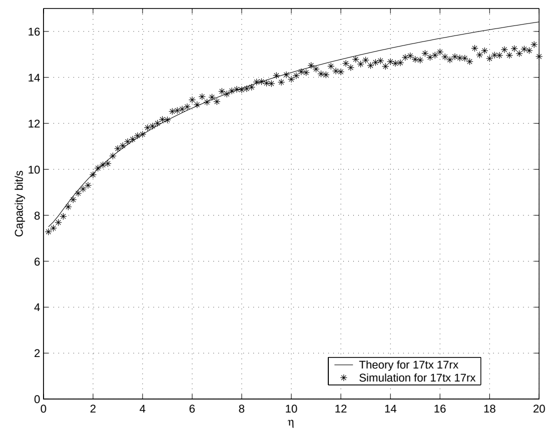

Figure 2: Capacity vs η for 15 transmit and 15 receive elements

As S_t becomes large, we have $\frac{1}{S_t}GG^* \to I_{S_r}$. Similarly, given (3) and (6) we may write:

$$H_r H_r^* \to h_r I_M \tag{28}$$

$$h_r \overset{\text{def}}{=} H_{rd} H_{rd}^* = \mathcal{E}\left[\frac{1}{R_{x,r0}}\right]$$

Using these approximations we may write:

$$\begin{aligned} C_{\log} &< \mathcal{E}\left[\sum_{k=1}^{M} \log_2\left[1 + \eta^2 \frac{\rho}{N}\xi_k\{H_t H_t^*\}\right]\right] \\ C &< \log_2(1+\hat{\rho}M) + \mathcal{E}\left[\log_2 \det\left(I_N + \eta^2\frac{\rho}{N} H_t H_t^*\right)\right] \end{aligned} \tag{29}$$

We may remove the expectation in (24) by noting that each matrix in $H_r G H_t$ is random. H_r and H_t have elements chosen i.i.d from the distribution $\mathcal{N}(0,1)$. Therefore, we apply the modified distribution function of (18) (for H_s and adjust the variable of integration ν by $\frac{1}{2\pi}$ to accommodate for the increased variance due to the use of the matrices H_r and H_t. This allows us to immitate the Gaussian matrix requirements of [9]. We may use equation (13) of [2] to calculate the capacity of the channel, substituting the value η. Define $m = \min\{N, M\}$. For the maximum transfer case of $m = M = N$ we have $\nu_- = 0$ $\nu_+ = 4$ from [9]:

$$C_{\log} \to M\int_0^4 \frac{\log_2\left(1+\eta^2\rho\frac{\nu}{2\pi}\right)}{\left(1-\frac{\nu}{4}\right)^2}\frac{1}{4\pi^2}\sqrt{\left(\frac{1}{\nu}-\frac{1}{4}\right)}\,d\nu \tag{30}$$

Equation (30) has two significant factors. It is clearly *logarithmic* in terms of η, with a linear term in M. We may combine (25) and (30) to give a general case bound on the capacity, for outdoor MIMO channels:

$$C < \log_2(1+\rho M) + M\kappa\log_2\left(1+\rho\eta^2\right) \tag{31}$$

Where κ is a constant due to the integration.

The second factor of (31) provides the "linear growth" as predicted by [1] and [2]. It can be seen that for large η,

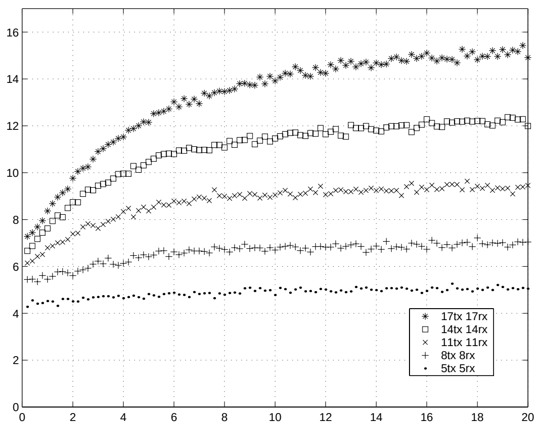

Figure 3: Capacity vs η for differing numbers of transmit and receive antennae

the growth in capacity is dominated by M. In this case the effect of sparse scatter is negligible, and the model returns to the nominal random fading model of [1] and [2]. However, for small η equation (31) is dominated by η - consequently the growth in capacity will be significantly lower.

In figure 2 we have plotted equation (30) for the case where $M = N = 15$ and $\rho = 10dB$. This is compared to a simulated case. It can be seen that for $\eta \leq 20$ we have close correlation between the approximate (solid) curve and the simulated curve (asterisks). For large η the upper bound is not tight. This is due to inaccuracies in the original Taylor expansion, and also over-simplification of the random matrices.

V. Simulation

We have generated Monte-Carlo simulations of the channel model, at various array separations D, with increasing numbers of transmit and receive antennae. A transmit frequency of 2.0GHz ($\lambda = 0.15m$) was used, with a ring radius of $R = 10m$. A fixed value of $S_r = S_t = 100$ scatterers were used, in each ring. A signal to noise ratio of $\hat{\rho} = 10dB$ was assumed. For each Monte Carlo iteration a new (random) arrangement of scatterers was produced, with the given parameters, and the channel was assumed known to the receiver. This provided an approximation to the expectation of the capacity.

Figure 3 shows the capacity of the channel for increasing η and for different numbers of transmit and receive antennae. In each case, $N = M$ to ensure a maximum capacity of the channel. It can be seen that in all cases the MIMO channel capacity diminishes for small values of η. In addition, the capacity gained by adding additional transmit and receive elements is very small. It can also be seen that for large η increasing the numbers of transmitters and receivers linearly increases the capacity of the channel.

VI. Conclusion

In this paper we have analyzed the NLOS channel developed in [5]. We have shown that an upper bound exists on the capacity of the MIMO channel, which is dependent on a factor η. We have shown, using random matrix theory, that for small values of η, the channel capacity grows *logarithmically* with respect to η.

For large values of η we the channel becomes the nominal i.i.d. Rayleigh channel, as used in [2]. In this case we can expect a linear growth in capacity for increasing numbers of transmit and receive antennas. For small η this growth does not occur. Instead, the capacity is dominated by the factor η. It has been shown that the loss in capacity occurs even if both transmitter and receiver have local Rayleigh fading present.

The benefit of additional transmit and receive elements is significantly reduced once the scattering environment is no longer "dense" over the entire length of the multi-path channel.

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