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Computing the frequency response of linear systems with parametric perturbation

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Abstract: The problem of computing the frequency response of linear systems with parametric perturbation is addressed. Under the assumption that the coefficients of the system transfer function depend on the perturbation parameters in a linear manner, we provide results for plotting the frequency response of the perturbed transfer function. These results are useful in determining the H_∞ norm, gain margin and phase margin and in improving the diagonal dominance of multi-input multi-output uncertain systems. An illustrative example is provided to demonstrate the results.

Keywords: Frequency response; parametric perturbation; uncertain system; robustness.

1. Introduction and problem formulation

The classical frequency response methods developed by Nyquist [1], Bode [2] and Nichols [3] have been extensively used in the synthesis and analysis of feedback control systems. They provide a good graphical means for investigating a system's stability and performance. They have also been proven to be very useful in the control design of uncertain systems. For example, Horowitz's Quantitative Feedback Theory (see, e.g., [4,5,6]) is based on the value sets (templates) of the uncertain transfer function on the Nichol's chart at various frequencies of interest. Once these value sets are determined, a systematic approach can be applied to design a feedback compensator and a prefilter such that the closed loop system satisfies the desired stability margin, robustness and performance. Another approach for control design of systems with structured perturbation is proposed by Sideris and Safonov [7] which uses conformal mapping to convert the value sets onto the unit disk so that the structured perturbation problem becomes an unstructured one. The resulting unstructured perturbation problem can be solved using the H^∞ theory or Nevanlinna–Pick interpolation methods. In contrast to [7], Iftar and Ozguner [8] show that a class of unstructured uncertainties can be modelled by parametric variations. In every approach mentioned above, the determination of the frequency response of an uncertain system is crucial. Unfortunately, the computation of the value sets is very tedious in general, and little attention has been paid to this issue.

The aim of this paper is to address the computation of the frequency response of a transfer function with parametric perturbation described by

$$P(s, q) = \frac{N(s, q)}{D(s, q)} = \frac{N_0(s) + \sum_{i=1}^{l} q_i N_i(s)}{D_0(s) + \sum_{i=0}^{l} q_i D_i(s)}, \quad q \in Q, \tag{1}$$

where $N_0(s)$ and $D_0(s)$ are *nominal polynomials* or *nominal quasipolynomials* in the case when the system contains delay terms; $N_i(s)$, $D_i(s)$, $i = 1\ 2, \ldots, l$ are *perturbation polynomials* or *perturbation quasipolynomials*; $q = [q_1, q_2, \ldots, q_l]^{\mathrm{T}}$ is the perturbation parameter vector belonging to a bounding set $Q \subset R^l$. We assume that Q is a hyperrectangle; i.e.,

$$Q = \{q\colon q_i^- \le q \le q_i^+,\ i = 1, 2, \ldots, l\}. \tag{2}$$

For further engineering motivation of this type of parametric perturbation, the reader is referred to, among numerous papers and books, [9,10,11] and the references thereof.

The problem of plotting the frequency response of the perturbed transfer function (1) is as follows: Given a finite sequence of (radian) frequencies $\{\omega_k\}$, determine the *value sets*

$$P(\mathrm{j}\omega_k, Q) := \{P(\mathrm{j}\omega_k, q)\colon q \in Q\} \tag{3}$$

in the complex plane.

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The most related work to this paper are the recent results by Bailey and Panzer [12], Bailey, Panzer and Gu [13], and Bailey and Hui [14] where two special cases of $P(s, q)$ in (1) are considered. In [12] and [13], $P(s, q)$ is required to satisfy an *interval* structure, i.e.,

$$P(s, q) = \sum_{i=0}^{m} a_i s^i / \sum_{i=0}^{n} b_i s^i$$

where a_i and b_i are bounded by intervals $[a_i^-, a_i^+]$ and $[b_i^-, b_i^+]$, respectively. An algorithm is given in [12] and [13] using the fact that the value sets of the numerator and the denominator of $P(j\omega, Q)$ at each fixed ω are independent rectangles. In [14], it is assumed that the perturbation parameters for the numerator of $P(s, q)$ and those for the denominator are *independent*. Another algorithm is established by these authors using the fact that the value sets mentioned above are independent convex polygons. However, these approaches are difficult to extend to the case where the numerator and the denominator of $P(s, q)$ are correlated to each other.

In this paper, we consider the computation of the frequency response of the perturbed transfer function $P(s, q)$ in (1) and show the following results:

(1) At each fixed frequency ω, the boundary of the value set $P(j\omega, Q)$ on the Nyquist plot is composed of a number of arcs or line segments mapped from the edges of Q.

(2) Each arc or line segment above can be simply determined by using the values of $P(j\omega, q)$ at the two end points and the midpoint of the corresponding edge of Q.

Consequently, a simple algorithm is created for plotting the frequency response of the perturbed transfer function. An illustrative example is provided to demonstrate the algorithm. Some applications of the results are displayed which include the computation of the maximal H_∞-norm, gain and phase margins and diagonal dominance improvement of uncertain transfer matrices for multi-input multi-output systems.

In order to guarantee that the value sets (3) are bounded, we assume $D(j\omega_k, q) \neq 0$ for all $q \in Q$ and every member of $\{\omega_k\}$. Note that this condition can be simply tested. Indeed, it can be shown that for any fixed complex number s, $D(s, Q)$ contains zero if and only if the following conditions hold [15]: for each $1 \le k \le l$,

$$[\mathrm{Re}(d_0(s))\,\mathrm{Im}(d_k(s)) - \mathrm{Im}(d_0(s))\,\mathrm{Re}(d_k(s))] + \sum_{i=1}^{l} q_i^0 [\mathrm{Re}(d_i(s))\,\mathrm{Im}(d_k(s)) - \mathrm{Im}(d_i(s))\,\mathrm{Re}(d_k(s))] \le 0, \tag{4}$$

$$[\mathrm{Re}(d_0(s))\,\mathrm{Im}(d_k(s)) - \mathrm{Im}(d_0(s))\,\mathrm{Re}(d_k(s))] + \sum_{i=1}^{l} q_i^1 [\mathrm{Re}(d_i(s))\,\mathrm{Im}(d_k(s)) - \mathrm{Im}(d_i(s))\,\mathrm{Re}(d_k(s))] \ge 0, \tag{5}$$

where

$$(q_i^0, q_i^1) := \begin{cases} (q_i^-, q_i^+) & \text{if } \mathrm{Re}(d_i(s))\,\mathrm{Im}(d_k(s)) - \mathrm{Im}(d_i(s))\,\mathrm{Re}(d_k(s)) \ge 0, \\ (q_i^+, q_i^-) & \text{otherwise.} \end{cases}$$

It is easily examined that this test requires at most $(4l + 1)l$ multiplications and $(3l + 1)l$ additions.

The bounding set Q has $l2^{l-1}$ edges given by

$$\{(q_1^*, \ldots, q_{i-1}^*, q_i, q_{i+1}^*, \ldots, q_l^*):\ q_i \in [q_i^-, q_i^+]\},\quad i = 1, 2, \ldots, l, \tag{6}$$

where q_j^* is equal to either q_j^- or q_j^+, $j \neq i$.

2. Main results

In this section, we show that the boundary of the value set $P(j\omega, Q)$ in (3) (at each fixed ω) is mapped from the edges of Q (Theorem 2.1) and that the value set corresponding to each edge of Q is an arc or a line segment which can be easily determined (Theorem 2.2). Based on these discoveries, a simple algorithm for plotting the frequency response of $P(s, Q)$ is created. In the sequel, $E(Q)$ denotes the edges of Q.

Theorem 2.1. *Let ω be an arbitrary frequency and suppose $P(j\omega, Q)$ is bounded. Then,*

$$\partial P(j\omega, Q) \subset P(j\omega, E(Q)),$$

where ∂ denotes the boundary.

Proof. By the continuity of $P(j\omega, q)$ with respect to q and the closedness of Q, we know that $\partial P(j\omega, Q) \subset P(j\omega, Q)$. Given an arbitrary $\zeta \in \partial P(j\omega, Q)$, we need to show that $\zeta \in P(j\omega, E(Q))$. For any given complex number z, we define

$$f(j\omega, q, z) := N(j\omega, q) - zD(j\omega, q)$$

and

$$F(j\omega, Q, z) := \{ f(j\omega, q, z) : q \in Q \}.$$

It is obvious that $z \in P(j\omega, Q)$ if and only if $0 \in F(j\omega, Q, z)$. In particular, $0 \in F(j\omega, Q, \zeta)$. Let $\{z_i\}$ be a convergent sequence in the complement of $P(j\omega, Q)$ such that $z_i \to \zeta$ as $i \to \infty$. This implies that $0 \notin F(j\omega, Q, z_i)$ for all z_i. Since $f(j\omega, q, z)$ is a continuous function with respect to z, the distance between $F(j\omega, Q, z)$ and the origin varies continuously with respect to z. Therefore, $0 \in \partial F(j\omega, Q, \zeta)$. Note that $f(j\omega, q, z)$ is an affine function with respect to q and that Q is a polytope. Consequently, every point on the edges (boundary) of $F(j\omega, Q, \zeta)$ has at least one preimage on the edges of Q (see, e.g., [16]). That is,

$$\partial F(j\omega, Q, \zeta) \subset F(j\omega, E(Q), \zeta).$$

Now we conclude that $0 \in F(j\omega, E(Q), \zeta)$ or equivalently,

$$\zeta \in P(j\omega, E(Q)). \quad \square$$

Theorem 2.2. *Let ω be a fixed frequency and E be a line segment in R^l with the end points denoted by q^L and q^R and the midpoint by q^M. Suppose $P(j\omega, q)$ is bounded for all $q \in E$. Then, the value set $P(j\omega, E)$ is an arc or a line segment* [1] *in the complex plane which starts from $P_L := P(j\omega, q^L)$, passes through $P_M := P(j\omega, q^M)$, and ends at $P_R := P(j\omega, q^R)$. More specifically, the center of the circle is given by the intersection of the following two linear lines:*

$$L_1(\alpha) := \frac{P_M + P_L}{2} + j\alpha(P_M - P_L), \quad \alpha \in R, \quad (7)$$

$$L_2(\beta) := \frac{P_R + P_M}{2} + j\beta(P_R - P_M), \quad \beta \in R, \quad (8)$$

and the radius is equal to the distance from the center to any one of P_L, P_M or P_R. In the case when the two lines above are in parallel, $P(j\omega, E)$ becomes a line segment from P_L to P_R.

Proof. Let $q \in E$. We need to show that $P(j\omega, q)$ is an arc. Since E is a line segment, we can write

$$q = q^L + \lambda[q^R - q^L]$$

for some $\lambda \in [0, 1]$. Using the fact that the coefficients of $N(s, q)$ and $D(s, q)$ are affine functions of q, we have

$$P(j\omega, q) = \frac{N(j\omega, q^L) + \lambda[N(j\omega, q^R) - N(j\omega, q^L)]}{D(j\omega, q^L) + \lambda[D(j\omega, q^R) - D(j\omega, q^L)]}. \quad (9)$$

That is, the mapping from E to $P(j\omega, E)$ is bilinear. Hence, $P(j\omega, E)$ at each fixed ω is either an arc or a line segment with its endpoints and some internal point corresponding to the endpoints and the midpoint of E.

Given three points P_L, P_M and P_R on the arc, the determination of the center and the radius is straightforward. $\square$

Remark 2.3. In Theorem 2.2, once the values of $P(j\omega, q^L)$, $P(j\omega, q^M)$ and $P(j\omega, q^R)$ are calculated, the arc $P(j\omega, E)$ can be determined geometrically. This is illustrated in Figure 1.

Based on Theorem 2.1 and 2.2, we establish the following algorithm for plotting the frequency response of $P(s, Q)$:

Algorithm 2.4. For each ω_k in $\{\omega_k\}$:
Step 1: Use (4)–(5) to confirm that $P(j\omega_k, Q)$ is bounded.

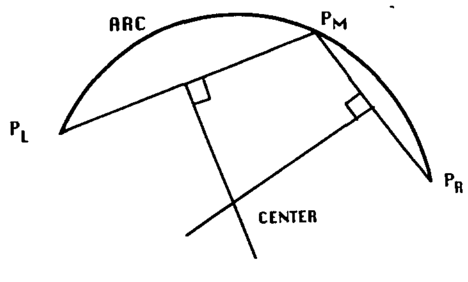

Fig. 1. Determining an arc geometrically.

[1] We may consider a line segment an arc with its radius equal to infinity.

Step 2: For each edge E of Q,

Step 2.1: calculate $P(\mathrm{j}\omega_k, q)$ for q equal to the two end points and the midpoint of E;

Step 2.2: determine the center and radius of the arc $P(\mathrm{j}\omega_k, E_r)$ either numerically or geometrically and plot the arc (or a line segment in the case when these three points are lined up).

Step 3: Determine the outer bound of the arcs. This outer bound is the boundary of the value set $P(\mathrm{j}\omega_k, Q)$.

To conclude this section, we now examine the computational complexity of the algorithm above. For simplicity, we assume $N_i(s)$ and $D_i(s)$ are polynomials in the form of

$$f(s) = \sum_{k=0}^{n} a_k s^k$$

and denote their degrees by n_i and d_i, respectively, $i = 0, 1, \ldots, l$. As pointed out before, Step 1 requires $(4l+1)l$ multiplications and $(3l+1)l$ additions. The evaluation of the polynomial $f(s)$ above at $s = \mathrm{j}\omega$ can be shown to require only $2n-1$ (real) multiplications and $n-1$ (real) additions. This implies that the evaluation of $P(\mathrm{j}\omega, q)$ requires $2l - 2 + 2\Sigma_{i=0}^{l}(n_i + d_i)$ multiplications, $2l - 2 + \Sigma_{i=0}^{l}(n_i + d_i)$ additions and a complex division which is equal to 6 multiplications, 3 additions and 2 divisions. Since there are 2^l vertices and $l2^{l-1}$ edges for Q, Step 2.1 takes $(2^l + l2^{l-1})(2l + 4 + 2\Sigma_{i=0}^{l}(n_i + d_i))$ multiplications, $(2^l + l2^{l-1})(2l + 1 + \Sigma_{i=0}^{l}(n_i + d_i))$ additions, and $2(2^l + l2^{l-1})$ divisions. It is estimated that computing the center and radius of an arc by using the method given in Theorem 2.2 takes 14 multiplications, 18 additions, 2 divisions and 1 square-root operation. Multiplying these numbers by $l2^{l-1}$ gives the amount of computation needed for finding all the centers and radii in Step 2.2. Overall, for each value set it takes about

$$(2^l + l2^{l-1})\left(2l + 4 + 2\sum_{i=1}^{l}(n_i + d_i)\right) + 14l2^{l-1} + (4l+1)l$$

multiplications,

$$(2^l + l2^{l-1})\left(2l + 1 + \sum_{i=0}^{l}(n_i + d_i)\right) + 18l2^{l-1} + (3l+1)l$$

additions, $2(2^l + l2^{l-1}) + 2l2^{l-1}$ divisions, $l2^{l-1}$ square-root operations, and plotting $l2^{l-1}$ arcs.

3. An illustrative example

Consider the following transfer function of an uncertain system:

$$P(s, q) = \{s^2 + (4 + 0.4q_1 + 0.2q_2)s + (20 + q_1 - q_3)\} \times \{s^4 + (9.5 + 0.5q_1 - 0.5q_2 + 0.5q_3)s^3 + (27 + 2q_1 + q_2)s^2 + (22.5 - q_1 + q_3)s\}^{-1}$$

where

$$(q_1, q_2, q_3) \in Q = \{(q_1, q_2, q_3): -3 \le q_i \le 3,\ i = 1, 2, 3\}.$$

The sequence of radian frequencies $\{\omega_k\}$ is chosen to be $\{0.2, 1, 3\}$. It is straightforward to verify (by using (4)–(5)) that all $P(\mathrm{j}\omega_k, Q)$ are bounded. Now for each ω_k and each of the 12 edges of Q, we evaluate $P(\mathrm{j}\omega_k, q)$ for q being the two end points and the midpoint of the edge. The values for $\omega_k = 1$ are given in Table 1. The resulting value sets are given in Figures 2–4. Figure 5 hooks these value sets together with the frequency response of the nominal transfer function.

4. Some applications

A number of applications of the results in Section 2 can be simply observed. For example,

Table 1
Values of $P(\mathrm{j}\omega, q)$ in Section 3 for $\omega = 1$

Edge	$P(\mathrm{j}\omega, q)$		
	$q_i = q_i^{\mathrm{H}} = 3$	$q_i = q_i^{\mathrm{M}} = 0$	$q_i = q_i^{\mathrm{L}} = -3$
$(3,3,q_3)$	−0.44 − 0.31j	−0.54 − 0.32j	−0.64 − 0.32j
$(-3,3,q_3)$	−0.24 − 0.36j	−0.34 − 0.43j	−0.45 − 0.49j
$(-3,-3,q_3)$	−0.31 − 0.45j	−0.43 − 0.54j	−0.58 − 0.63j
$(3,-3,q_3)$	−0.56 − 0.32j	−0.68 − 0.32j	−0.80 − 0.31j
$(3,q_2,3)$	−0.44 − 0.31j	−0.49 − 0.32j	−0.56 − 0.32j
$(-3,q_2,3)$	−0.24 − 0.36j	−0.27 − 0.40j	−0.31 − 0.45j
$(-3,q_2,-3)$	−0.45 − 0.49j	−0.51 − 0.55j	−0.58 − 0.63j
$(3,q_2,-3)$	−0.64 − 0.32j	−0.71 − 0.32j	−0.80 − 0.31j
$(q_1,3,3)$	−0.44 − 0.31j	−0.36 − 0.36j	−0.24 − 0.36j
$(q_1,-3,3)$	−0.56 − 0.32j	−0.46 − 0.41j	−0.31 − 0.45j
$(q_1,-3,-3)$	−0.80 − 0.31j	−0.75 − 0.47j	−0.58 − 0.63j
$(q_1,3,-3)$	−0.64 − 0.32j	−0.57 − 0.42j	−0.45 − 0.49j

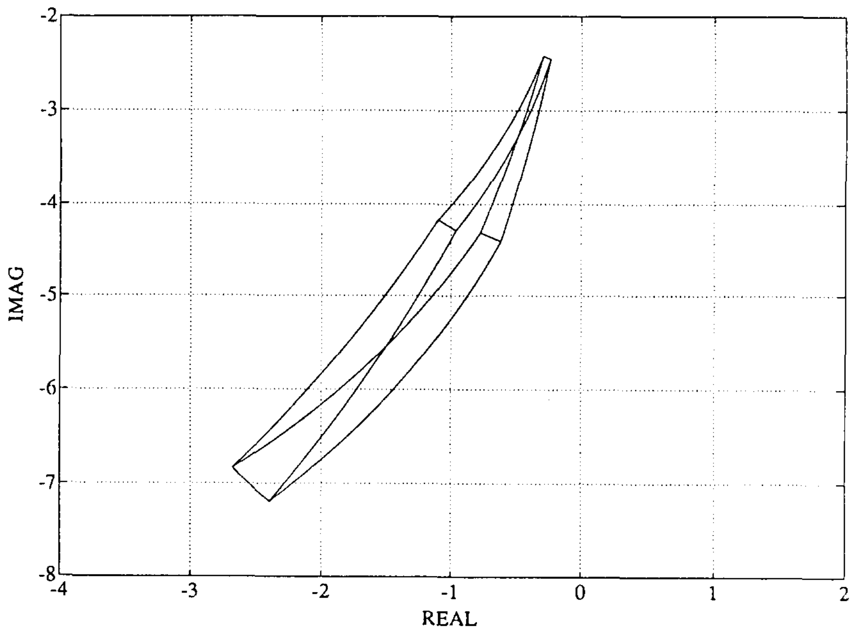

Fig. 2. The value set of $P(j\omega, Q)$ in Section 3 for $\omega = 0.2$.

the (worst) gain margin and phase margin can be trivially obtained from the Nyquist plot; the Nyquist plot can be easily converted to the Bode plot or magnitude vs. phase plot on the Nichol's chart by mapping the boundary of value sets at various frequencies. Below, we address the following two issues: (1) computation of maximal H_∞-norm, and (2) diagonal dominance improvement

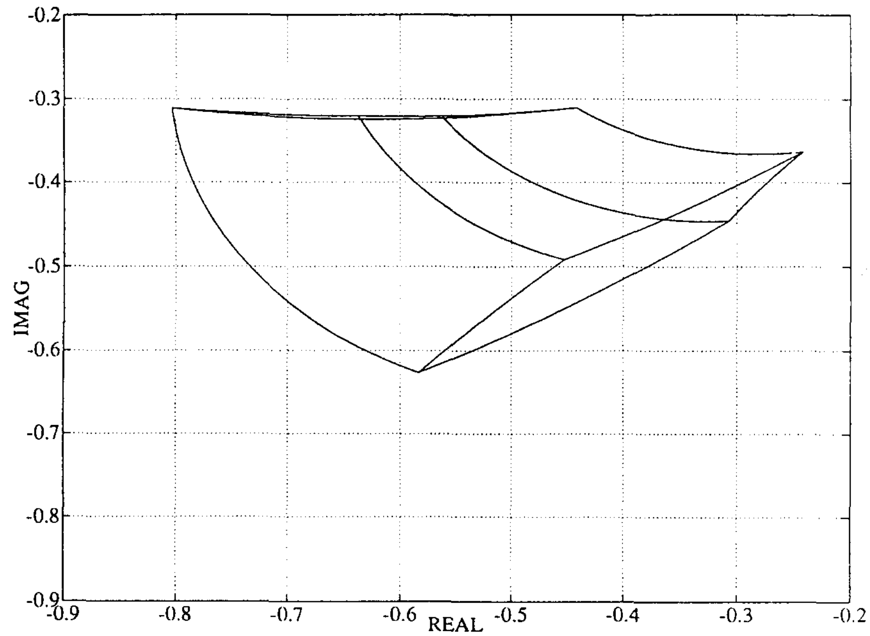

Fig. 3. The value set of $P(j\omega, Q)$ in Section 3 for $\omega = 1$.

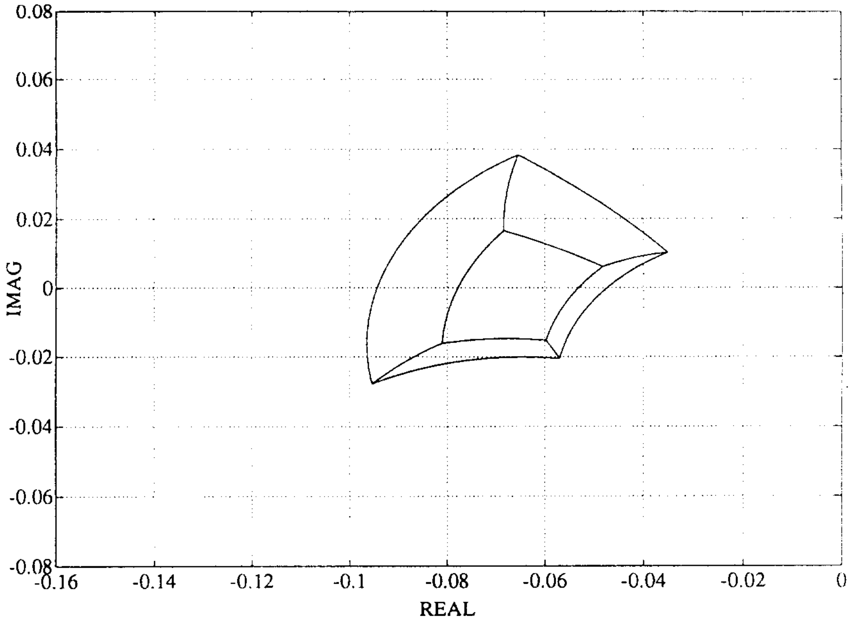

Fig. 4. The value set of $P(j\omega, Q)$ in Section 3 for $\omega = 3$.

of multi-input and multi-output uncertain systems.

Maximal H_∞-norm of uncertain transfer functions. Given the family of transfer functions (1), (2) which are stable, the maximal H_∞-norm of the transfer functions can be calculated in two different ways. Using Theorem 2.1, we have

$$\max\{\|P(s, q)\|_\infty: q \in Q\}$$
$$= \max\{\|P(s, q)\|_\infty: q \in E(Q)\}, \qquad (10)$$

or,

$$\max\{\|P(s, Q)\|_\infty: q \in Q\}$$
$$= \max\{|P(j\omega, E(Q))|: \omega \geq 0\}. \qquad (11)$$

Equation (10) is effective when the number of edges of Q is relatively small. Note that the bisection algorithm given in [17] can be called for calculating the H_∞-norm of each fixed transfer function. Reference [18] provides an algorithm for computing the the maximal H_∞-norm of a one-dimensional set of transfer functions. Equation (11) will require less computation when the number of edges of Q becomes large.

The minimal H_∞-norm can be calculated in a similar manner.

Diagonal dominance improvement of uncertain transfer matrices: Many control design methods for multi-input multi-output systems require diagonal dominance of the system transfer matrix. If the given system transfer matrix is not diagonally dominant, a post-compensating matrix is often applied to improve the diagonal dominance. This is described as follows [19]. Consider an uncertain $n \times n$ transfer matrix $G(s, q) = \{g_{ij}(s, q)\}$, $q \in Q$, where each element $g_{ij}(s, q)$ is in the form of (1). The objective is to choose a constant diagonal post-compensating matrix $K = \mathrm{diag}\{k_i\}$ such that at some given complex number s_0 (usually at certain frequency), the matrix $KG(s_0, q)$ is diagonally dominant for all $q \in Q$, i.e.,

$$\sum_{j \neq i} |g_{ij}(s_0, q)k_j| < |k_i g_{ii}(s_0, q)|,$$
$$i = 1, 2, \ldots, n; \quad q \in Q. \qquad (12)$$

Define

$$T(s, q) = \{|g_{ij}(s, q)/g_{ii}(s, q)|\} \qquad (13)$$

and assume that $T(s, q)$ is primitive [2] for every

[2] A non-negative matrix M is called primitive if M^r is a positive matrix for some integer $r > 0$. This condition is rather technical and is generically satisfied.

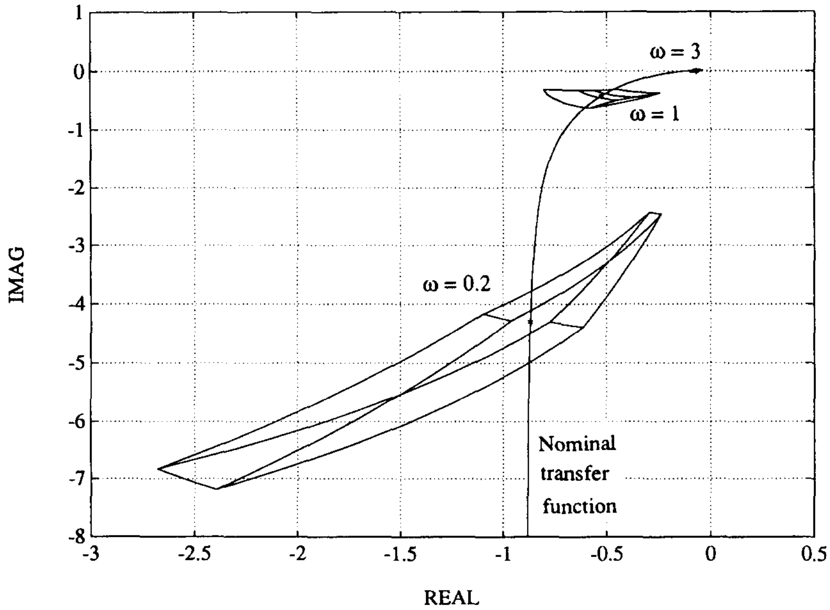

Fig. 5. The hooked value sets of $P(j\omega, Q)$ in Section 3.

$q \in Q$. For each fixed $q \in Q$, it is shown by Mees [19] that $KG(s_0, q)$ can be made diagonally dominant if and only if the Perron–Frobenius eigenvalue of $T(s_0, q)$ is less than 2. In addition, the post-compensating matrix $K(q) = \mathrm{diag}\{u_i(q)\}$, where $[u_1(q), \ldots, u_n(q)]^{\mathrm{T}}$ is the left Perron–Frobenius eigenvector. Since the post-compensating matrix needs to be independent of q, we now define

$$T(s) = \left\{ \max_{q \in Q} | g_{ij}(s, q)/g_{ii}(s, q) | \right\} \tag{14}$$

and assume that $T(s_0)$ is primitive. Note that the Perron–Frobenius eigenvalue of $T(s_0)$ is greater than or equal to that of any $T(s_0, q)$. We conclude that the diagonal dominance of $G(s_0, Q)$ is possible if the Perron–Frobenius eigenvalue of $T(s_0)$ is less than 2 and the associated K can be chosen using the left Perron–Frobenius eigenvector of $T(s_0)$ as mentioned above.

From the computational point of view, now the stumbling block is the calculation of $T(s)$, or equivalently,

$$t_{ij}(s) = \max\{ | g_{ij}(s, q)/g_{ii}(s, q) | \colon q \in Q \}.$$

This problem can be solved by directly applying Theorem 2.1. Consequently,

$$t_{ij}(s) = \max\{ | g_{ij}(s, q)/g_{ii}(s, q) | \colon q \in E(Q) \}.$$

As a final remark, we note that if the diagonal dominance is required on a range of frequencies S instead of a single point s_0, we can simply replace the matrix $T(s_0)$ by

$$T = \left\{ \max_{s \in S} t_{ij}(s) \right\},$$

as suggested in [19].

5. Conclusion

In this paper, we have considered the problem of plotting the frequency response of the transfer function with parametric perturbation in (1). Useful results have been obtained (Theorems 2.1 and 2.2). Some applications of the results are displayed which include the computation of the maximal H_∞-norm, gain and phase margins and diagonal dominance improvement of uncertain transfer matrices for multi-input multi-output systems. Note that our results apply not only to continuous systems but also to time-delay systems and discrete systems.

References

[1] H. Nyquist, Regeneration theory, *Bell System Technical Journal* **11** (1) (1932) 126–147.

[2] H.W. Bode, *Network Analysis and Feedback Amplifier Design* (Van Nostrand, Princeton, NJ, 1945).
[3] H.M. James, N.B. Nichols and R.S. Phillips, *Theory of Servo-Mechanisms* (McGraw-Hill, New York, 1947).
[4] I. Horowitz, *Synthesis of Feedback Systems* (Academic Press, New York, 1963).
[5] I. Horowitz, Quantitative feedback theory, *Proc. IEE-D* **129** (1982) 215–226.
[6] I. Horowitz and M. Sidi, Synthesis of feedback systems with large plant ignorance for prescribed time-domain tolerance, *Internat. J. Control* **6** (2) (1972) 287–309.
[7] A. Sideris and M.G. Safonov, A design algorithm for the robust synthesis of SISO feedback control systems using conformal maps and H^∞-theory, *Proceedings of American Control Conference*, Seattle, WA (1986).
[8] A. Iftar and U. Ozguner, Structured modeling of unstructured uncertainties and robust controller design in state space, presented at *Winter Annual Meeting of the American Society of Mechanical Engineers*, Chicago, IL (1988).
[9] A.C. Bartlett, C.V. Hollot and H. Lin, Root locations of an entire polytope of polynomials: It suffices to check the edges, *Math. Control Signals Systems* **1** (1988) 61–71.
[10] M. Fu and B.R. Barmish, Polytopes of polynomials with zeros in a prescribed set, *IEEE Trans. Automat. Control* **34** (5) (1989) 544–546.
[11] B.R. Barmish, A generalization of Kharitonov's four polynomial concept for robust stability problems with linearly dependent coefficient perturbations, *IEEE Trans. Automat. Control* **34** (2) (1989) 157–165.
[12] F.N. Bailey and D. Panzer, A fast algorithm for computing interval rational functions, *Proceedings of American Control Conference*, Atlanta, GA (1988).
[13] F.N. Bailey, D. Panzer and G. Gu, Two algorithms for frequency domain design of robust control systems, *Internat. J. Control* **48** (5) (1988) 1787–1806.
[14] F.N. Bailey and C.H. Hui, A fast algorithm for computing parametric rational functions, to appear.
[15] M. Fu, Polytopes of polynomials with zeros in a prescribed region: New criteria and algorithms, in: M. Milanese, R. Tempo and A. Vicino, Eds., *Proceedings of the International Workshop on Robustness in Identification and Control*, Turin, Italy (1988).
[16] M. Fu, A.W. Olbrot and M.P. Polis, Robust stability for time-delay systems: The edge theorem and graphical tests, *IEEE Trans. Automat. Control* **34** (8) (1989) 813–820.
[17] S. Boyd, V. Balakrishnan and P. Kabamba, On computing the H^∞ norm of a transfer matrix, *Proceedings of American Control Conference*, Atlanta, GA (1988).
[18] P.T. Kamamba and S.P. Boyd, On parametric H-infinity optimization, *Proceedings of the 27th IEEE Conference on Decision and Control*, Austin, TX (1988).
[19] A.I. Mees, Achieving diagonal dominance, *Systems Control Lett.* **1** (3) (1981) 155–158.